A Concise Proof of Tienstra's Formula

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Abstract

The resection problem consists in finding the location of an observer by measuring the angles subtended by lines of sight from this observer to three known stations. Many researchers and practitioners recognize that Tienstra's formula provides the most compact and elegant solution to this problem. Unfortunately, all available proofs for this remarkable formula are intricate. This paper shows how, by using barycentric coordinates for the observer in terms of the locations of the stations, a neat and short proof is straightforwardly derived.

Keywords: Tienstra's formula, triangulation, resection, global localization, barycentric coordinates.

Subject Headings: Global positioning, Localization, Triangulation, Topographic surveys.

1 Introduction

The resection problem can be formally defined as follows (see Fig. 1). Given the coordinates of three stations ($\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$), the problem consists in determining the coordinates of a point, $\mathbf{p}_4$, from the angles, α_1 and α_2, between the lines connecting $\mathbf{p}_4$ and the three stations.

Bock (1959) reports more than 500 different procedures to solve the resection problem. Nevertheless, it is worth noting that these procedures were designed before the computer advent. Hence, most of them are graphical in nature, or numerically adapted to be applied with the aid of tables. Still nowadays new procedures for solving this problem appear from time to time (Font-Llagunes and Batlle, 2009). Descriptions of the most relevant ones are typically provided in surveying textbooks (Allan et al., 1982), where the procedure derived from the direct application of Tienstra's formula is usually recognized as the simplest one.

Tienstra's formula reads as follows

$$\mathbf{p}_4 = \frac{f_1\,\mathbf{p}_1 + f_2\,\mathbf{p}_2 + f_3\,\mathbf{p}_3}{f_1 + f_2 + f_3}, \tag{1}$$

where

$$f_i = \frac{1}{\cot(a_i) - \cot(\alpha_i)} = \frac{\sin(\alpha_i)\,\sin(a_i)}{\sin(\alpha_i - a_i)}. \tag{2}$$

J. M. Tienstra (1895-1951) was a professor of the Delft University of Technology where he taught the use of the barycentric coordinates in solving the resection problem. As pointed out by Greulich (1999), it seems most probable that his name became attached to the procedure for this reason. Nevertheless, precisely when and by whom this formula was first proposed is an open question. According to Greulich (1999), the earliest recorded occurrence is a 1889's paper by Neuberg and Gob (1889).

To the best of our knowledge, all the derivations of Tienstra's formula available in the literature are intricate. The most recent proof we are aware of is that by Hu and Kuang (1997, 1998) that involves 20 pages, divided in two papers, with algebraic manipulations that even requires the use of a computer algebra system at some steps. In this note, a straightforward and neat derivation is presented. The key point has been to express the observer location using barycentric coordinates in terms of oriented areas, as originally introduced by Möbius (1827).

2 A Concise Proof of Tienstra's Formula

In the n-dimensional Euclidean space, the barycentric coordinates of a point with Cartesian coordinates $\mathbf{p}_{n+1}$ are the weights $w_1, \ldots, w_n$ to be assigned to the Euclidean coordinates of the vertexes of a given reference simplex, $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$, so that their normalized weighted sum is $\mathbf{p}_{n+1}$ (Coxeter, 1969; Bradley, 2007). For the planar case (see Fig. 2), the Cartesian coordinates of the vertexes of a planar simplex (i.e., a triangle) can be denoted by $\mathbf{p}_i = (x_i, y_i)^\top$ with the indexing $i = 1, 3$ proceeding in a counterclockwise fashion around the triangle starting from an arbitrary vertex. Then, the Cartesian coordinates of an arbitrary point in the plane, $\mathbf{p}_4 = (x_4, y_4)^\top$, can be expressed as

$$x_4 = \frac{w_1\, x_1 + w_2\, x_2 + w_3\, x_3}{w_1 + w_2 + w_3}, \tag{3}$$

$$y_4 = \frac{w_1\, y_1 + w_2\, y_2 + w_3\, y_3}{w_1 + w_2 + w_3}, \tag{4}$$

where w_1, w_2, and w_3 correspond to masses that, when placed at the vertexes of the reference triangle $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, give $\mathbf{p}_4$ as the center of mass. Observe that barycentric coordinates are not unique since they can be arbitrarily scaled yielding the same result. For the special planar case, and when the weights are normalized, the barycentric coordinates are also called *areal coordinates*, since their values correspond to the areas of triangles $\{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$, $\{\mathbf{p}_1, \mathbf{p}_4, \mathbf{p}_3\}$, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$, normalized with respect to the area of the reference triangle. Observe that the weights are all positive only when $\mathbf{p}_4$ is inside the reference triangle and, thus, for the general case, oriented areas have to be considered.

Barycentric coordinates provide elegant proofs of geometric theorems such as Routh's theorem, Ceva's theorem, and Menelaus' theorem (Coxeter, 1969) and we will exploit them here to provide a simple proof of Tienstra's formula.

According to the above, for three non-aligned stations with coordinates $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, as depicted in Fig. 2, the position of the observer, $\mathbf{p}_4$, can be expressed as

$$\mathbf{p}_4 = \frac{\Delta(2,3,4)\,\mathbf{p}_1 + \Delta(3,1,4)\,\mathbf{p}_2 + \Delta(1,2,4)\,\mathbf{p}_3}{\Delta(1,2,3)} \tag{5}$$

$$= \frac{\Delta(2,3,4)\,\mathbf{p}_1 + \Delta(3,1,4)\,\mathbf{p}_2 + \Delta(1,2,4)\,\mathbf{p}_3}{\Delta(2,3,4) + \Delta(3,1,4) + \Delta(1,2,4)} \tag{6}$$

$$= \frac{1}{1 + \frac{\Delta(3,1,4)}{\Delta(2,3,4)} + \frac{\Delta(1,2,4)}{\Delta(2,3,4)}}\,\mathbf{p}_1 + \frac{1}{\frac{\Delta(2,3,4)}{\Delta(3,1,4)} + 1 + \frac{\Delta(1,2,4)}{\Delta(3,1,4)}}\,\mathbf{p}_2 + \frac{1}{\frac{\Delta(2,3,4)}{\Delta(1,2,4)} + \frac{\Delta(3,1,4)}{\Delta(1,2,4)} + 1}\,\mathbf{p}_3 \tag{7}$$

where

$$\Delta(i,j,k) = \frac{1}{2}\begin{vmatrix} \mathbf{p}_i & \mathbf{p}_j & \mathbf{p}_k \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ 1 & 1 & 1 \end{vmatrix} \tag{8}$$

is the oriented area of the triangle defined by $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$.

Since the triangles $\{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_3\}$ and $\{\mathbf{p}_5, \mathbf{p}_2, \mathbf{p}_3\}$ have the same height, then

$$\frac{\Delta(1,5,3)}{\Delta(5,2,3)} = D(1,5,2), \tag{9}$$

where

$$D(i,j,k) = \zeta_{ijk} \frac{\|\mathbf{p}_j - \mathbf{p}_i\|}{\|\mathbf{p}_k - \mathbf{p}_j\|}, \tag{10}$$

ζ_{ijk} being the sign of the dot product $(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}_k - \mathbf{p}_j)$. Note that $D(i,j,k)$ is a signed ratio of distances. Likewise,

$$\frac{\Delta(1,5,4)}{\Delta(5,2,4)} = D(1,5,2), \tag{11}$$

and, as a consequence,

$$\frac{\Delta(3,1,4)}{\Delta(2,3,4)} = \frac{\Delta(1,5,3) - \Delta(1,5,4)}{\Delta(5,2,3) - \Delta(5,2,4)} = \frac{D(1,5,2)\,\Delta(5,2,3) - D(1,5,2)\,\Delta(5,2,4)}{\Delta(5,2,3) - \Delta(5,2,4)} = D(1,5,2). \tag{12}$$

Repeating the same reasoning for $\Delta(1,2,4)/\Delta(3,1,4)$ and $\Delta(2,3,4)/\Delta(1,2,4)$, we obtain

$$\mathbf{p}_4 = \frac{1}{1 + D(1,5,2) + \frac{1}{D(3,7,1)}}\,\mathbf{p}_1 + \frac{1}{\frac{1}{D(1,5,2)} + 1 + D(2,6,3)}\,\mathbf{p}_2 + \frac{1}{D(3,7,1) + \frac{1}{D(2,6,3)} + 1}\,\mathbf{p}_3. \tag{13}$$

Next, we show how all the signed distance ratios in the above equation can be expressed in function of the interior angles (a_1, a_2, and a_3) of the triangle defined by the three stations, and the angles measured from the observer (α_1, α_2, and $\alpha_3 = 2\pi - \alpha_1 - \alpha_2$).

Consider the first ratio in Eq. (13). Applying the sinus rule to the triangles $\{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_3\}$ and $\{\mathbf{p}_5, \mathbf{p}_2, \mathbf{p}_3\}$ we get

$$D(1,5,2) = \zeta_{152} \frac{\|\mathbf{p}_5 - \mathbf{p}_1\|}{\|\mathbf{p}_2 - \mathbf{p}_5\|} = \frac{\|\mathbf{p}_3 - \mathbf{p}_5\| \frac{\sin(\delta)}{\sin(a_1)}}{\|\mathbf{p}_3 - \mathbf{p}_5\| \frac{\sin(\lambda)}{\sin(a_2)}} = \frac{\sin(\delta)\, \sin(a_2)}{\sin(\lambda)\, \sin(a_1)}. \tag{14}$$

Since $a_1 + (\pi - \theta) + \delta = \pi$ and $a_2 + \theta + \lambda = \pi$ then

$$\sin(\delta) = \sin(\theta - a_1), \tag{15}$$

$$\sin(\lambda) = \sin(\theta + a_2), \tag{16}$$

and taking into account that the three stations are not aligned (i.e., $\sin(\theta) \neq 0$) we have

$$D(1,5,2) = \frac{\sin(\theta - a_1)\ \sin(a_2)}{\sin(\theta + a_2)\ \sin(a_1)} = \frac{\frac{\sin(\theta - a_1)}{\sin(\theta)\ \sin(a_1)}}{\frac{\sin(\theta + a_2)}{\sin(\theta)\ \sin(a_2)}} = \frac{\cot(a_1) - \cot(\theta)}{\cot(a_2) + \cot(\theta)}. \tag{17}$$

On the other hand, applying the sinus rule to triangles $\{\mathbf{p}_1, \mathbf{p}_5, \mathbf{p}_4\}$, and $\{\mathbf{p}_5, \mathbf{p}_2, \mathbf{p}_4\}$ we get

$$D(1,5,2) = \zeta_{152}\, \frac{\|\mathbf{p}_5 - \mathbf{p}_1\|}{\|\mathbf{p}_2 - \mathbf{p}_5\|} = \frac{\|\mathbf{p}_4 - \mathbf{p}_5\|\ \frac{\sin(\delta')}{\sin(a_1')}}{\|\mathbf{p}_4 - \mathbf{p}_5\|\ \frac{\sin(\lambda')}{\sin(a_2')}} = \frac{\sin(\delta')\ \sin(a_2')}{\sin(\lambda')\ \sin(a_1')}. \tag{18}$$

Now, since $\delta' = \pi - \alpha_2$, $\lambda' = \pi - \alpha_1$, $a_1' + (\pi - \theta) + \delta' = \pi$, and $a_2' + \theta + \lambda' = \pi$ then

$$\sin(\delta') = \sin(\alpha_2), \tag{19}$$

$$\sin(\lambda') = \sin(\alpha_1), \tag{20}$$

$$\sin(a_1') = \sin(\theta - \delta') = -\sin(\theta + \alpha_2), \tag{21}$$

$$\sin(a_2') = \sin(\theta + \lambda') = -\sin(\theta - \alpha_1), \tag{22}$$

and we have,

$$D(1,5,2) = \frac{\sin(\alpha_2)\ \sin(\theta - \alpha_1)}{\sin(\alpha_1)\ \sin(\theta + \alpha_2)} = \frac{\frac{\sin(\theta - \alpha_1)}{\sin(\theta)\ \sin(\alpha_1)}}{\frac{\sin(\theta + \alpha_2)}{\sin(\theta)\ \sin(\alpha_2)}} = \frac{\cot(\alpha_1) - \cot(\theta)}{\cot(\alpha_2) + \cot(\theta)}. \tag{23}$$

Finally, eliminating $\cot(\theta)$ from Eqs. (17) and (23), we get

$$D(1,5,2) = \frac{\cot(a_1) - \cot(\alpha_1)}{\cot(a_2) - \cot(\alpha_2)}. \tag{24}$$

Repeating the same reasoning for $D(2,6,3)$ and $D(3,7,1)$, Tienstra's formula is straightforwardly obtained.

3 Conclusions

In this paper, we provided a concise proof of Tienstra's formula. The key point to derive it was to departure from the very definition of the barycentric coordinates of a point. Previous proofs that only rely on the use trigonometric identities get rapidly involved, to the point of requiring the use of computer-aided algebraic manipulation.

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