The topology of singularities of flagged parallel manipulators

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1 Introduction

The spatial parallel manipulator can abstractly be described as two bodies, *base* and *platform*, joined by six segments (or *legs*) of variable lengths. The configuration space which describes all possible platform locations with respect to the base is $\mathbb{R}^3 \times SO_3(\mathbb{R})$ the *Euclidean motion group*. It is a differentiable manifold of dimension 6, and this is why the platform and the base are joined by six legs: to achieve the six degrees of freedom which enable the platform to reach every point in the configuration space.

The direct kinematics problem consists in finding the location of the platform with respect to the base from the lengths of the six legs, that is, in determining the pre-images of the (differentiable) map

$$\Phi : \mathbb{R}^3 \times SO_3(\mathbb{R}) \longrightarrow \mathbb{R}^6 \,. \tag{1}$$

The Jacobian of the map Φ is called the *singular locus*. Technically singular configurations cause problems: in a singularity small changes in the lengths of the legs cause large and uncontrolled movement of the platform. Therefore it is desirable to have a complete overview of the singular locus of the manipulator. Moreover, a complete characterization of the singular locus in the configuration space would permit identifying the non-singular regions separated by singularities, the restriction on manoeuverability occurring in each singular region, as well as the adjacency between all non-singular and singular regions. This would be most useful for manipulator design, including the use of redundant actuators to eliminate certain singularities, and also to plan trajectories away from singularities, or crossing them in a controlled way.

In this paper we derive stratifications of the Euclidean motion group, from classically known stratifications of the flag manifold, which provide a complete description of the singular locus in the configuration space of a family of parallel manipulators, called *flagged manipulators*. This class of robots was first identified in [8], and is worked in detail in [2], being expanded from a basic manipulator, and it is shown to contain large subfamilies of parallel and three-legged spatial manipulators (by substituting 2-leg groups by different serial chains). The stratifications of $\mathbb{R}^3 \times SO_3(\mathbb{R})$ derived in the present paper have become a valuable tool in the field of kinematics of robots and have been the source of inspiration of several works in that field: they were already applied in [8] to give explicitly all the singular strata and their connectivity, for all members in the class of flagged manipulators, irrespective of the metric of each particular robot design. Moreover, [8] also showed that these strata admit an easy control strategy to cross them, because it is possible to assign local coordinates to each stratum (in the configuration space of the manipulator) which correspond to uncoupled rotations and/or translations. The applicability of these stratifications to come up with designs which admit control strategies free of singularities has been exploited as well: in [1] redundant actuators have been inferred, obtained by adding an extra leg to any flagged manipulator, which admit a control strategy (by appropriately choosing which leg remains passive) that completely avoids singularities.

The organization of the paper is as follows. Section 2, after recalling some concepts and results concerning the manifold parameterizing the projective real flags, is devoted to derive stratifications (which are cell decompositions) of the space parameterizing the affine real flags, and to determine the adjacency between those strata. Section 3 shows how the stratifications for the affine real flags induce stratifications (which are cell decompositions) or the Euclidean motion group, and studies the adjacency between the higher dimensional cells.

2 From projective real flags to affine real flags

It is classically known that the flag manifold, which parameterizes the set of projective flags, admits well-behaved topological decompositions, namely stratifications (or cell decompositions), see [6], [4]. Moreover, the classical

Ehresmann-Bruhat order describes all the possible degenerations of a pair of flags in a linear space V under linear transformations of V. We shall deal with the flag manifold over $\mathbb{R}^n$, that is, the linear space $V = \mathbb{R}^n$, motivated, as explained in the introduction, by the issues of the results in the field of kinematics of robots. This same motivation leads us to consider affine flags. In this section we prove that some stratifications of the flag manifold restricted to the set of the affine flags are still a stratifications, and we introduce a refinement of the classical Ehresmann-Bruhat order that characterizes the adjacency between all the different strata, that is, describes all the possible degenerations of any configuration to more special ones. The main reference for this section is [3].

Given a positive integer n, the *flag manifold* $\mathcal{F}lag(n+1)$ over $\mathbb{R}^{n+1}$ is the set of flags in $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$, where a *flag* in $\mathbb{P}^n$ is a sequence $(V_0, \ldots, V_n)$ of projective subspaces with $V_0 \subset \cdots \subset V_n = \mathbb{P}^n$ and $\dim V_i = i$ for $1 \leq i \leq n$. The nested subspaces between the point V_0 and the hyperplane V_{n-1} defining a flag will be referred to as *flag features*. Once a distinguished hyperplane H_∞ in $\mathbb{P}^n$ is chosen, the *affine flags* are the flags $(V_0, \ldots, V_n)$ in $\mathbb{P}^n$ satisfying $V_0 \notin H_\infty$. Let $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ denote the open subset of the affine flags in $\mathcal{F}lag(n+1)$.

The notion of well-behaved topological decomposition is formalized in the following definition. A *stratification* of subset S of a smooth manifold M is a partition $S = \cup_{i \in I} S_i$ such that: I is finite, S_i is a smooth submanifold of M for any $i \in I$, and if $S_i \cap \overline{S}_j \neq \emptyset$, then $S_i \subset \overline{S}_j$, where $\overline{S}_j$ stands for the closure of S_j. The S_i's are called *strata*. The third boundary condition guarantees that the boundary of a stratum is the union of the entire strata which are not disjoint from it, that is, there are not "exceptional" degenerations between strata: if a configuration of a stratum S_i is a degeneration of a configuration of a stratum S_i, then any configuration in S_i can be considered as a degeneration of some configuration of S_j.

Next, let us summarize the classical theory on stratifications of the flag manifold. Fix a *reference flag* $(V_0, \ldots, V_n)$. Let Σ_{n+1} be set of permutations of $n+1$ elements, and consider $w \in \Sigma_{n+1}$.

Definition 1 (Bruhat or Schubert cell). *The Bruhat or Schubert cell B^w associated to the permutation w is the set of all flags whose flag features have incidence relations with the reference flag determined by w in the following way:*

$$B^w = \{(V_0^*, \ldots, V_n^*) \in \mathcal{F}\mathrm{lag}(n+1) : \dim(V_p^* \cap V_q) = r_w(p, q) \text{ for } 1 \leq p, q \leq n\}$$

where $r_w(p,q) = \#\{i \leq p : w(i) \leq q\} - 1$.

Given a flag $(V_0, \ldots, V_n)$, we can choose a projective reference frame $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}; \mathbf{a}\}$ of $\mathbb{P}^n$ satisfying $\mathbf{a}_i \in V_{i-1}$ for $1 \leq i \leq n+1$, which will be called *a frame attached to the flag* $\mathcal{V}$. To each permutation w in Σ_{n+1}, we associate the distinguished flag

$$\mathcal{V}(w) = \left(\mathbf{a}_{w(1)}, \mathbf{a}_{w(1)} \vee \mathbf{a}_{w(2)}, \ldots, \mathbf{a}_{w(1)} \vee \ldots \vee \mathbf{a}_{w(n+1)}\right),$$

where $\mathbf{a}_{i_1} \vee \mathbf{a}_{i_2} \vee \ldots \vee \mathbf{a}_{i_s}$ denotes the projective linear variety spanned by the points $\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_s}$. Notice that then B^w is the set of the flags in $\mathcal{F}lag(n+1)$ whose flag features have the same incidence relations with $\mathcal{V}$ as the flag features of $\mathcal{V}(w)$.

It is a classical result that each choice of a reference flag gives a stratification or cell decomposition of the flag manifold:

Theorem 1 (Stratification of the flag manifold $\mathcal{F}lag(n+1)$). *(see [4] Ch. 13, Th. 4.3 or [3] Ch. 10): The disjoint union of all Bruhat cells B^w with $w \in \Sigma_{n+1}$ is a stratification for $\mathcal{F}\mathrm{lag}(n+1)$:*

$$\mathcal{F}\mathrm{lag}(n+1) = \cup_{w \in \Sigma_{n+1}} B^w \,. \tag{2}$$

The structure of each cell and the adjacency between them are also classically well established:

Proposition 1. *(see [4] Ch. 13, Prop. 4.7 or [3] Ch. 10, Prop. 7):*

1. *B^w is isomorphic to the affine space $\mathbb{R}^{\mathrm{length}(w)}$, where the* length *of a permutation w is defined as the number of inversions in w, that is,* $\mathrm{length}(w) = \#\{i < j : w(i) > w(j)\}$.

2. *If B^w and B^u are two cells of consecutive dimensions* $\mathrm{length}(w) = \mathrm{length}(u) + 1$, *then $\overline{B^w} \supset B^u$ if and only if there exists a transposition $t \in \Sigma_{n+1}$ such that $w = tu$.*

Example 1. *To illustrate the case $n = 3$, Fig. 1 shows the cells of dimensions* 6 *and* 5 *and their adjacency. The rectangle represents the 6D cell $B^{(4,3,2,1)}$, while the ellipses are the 5D cells: $B^{(4,3,1,2)}$, $B^{(3,4,2,1)}$ and $B^{(4,2,3,1)}$. Each 5D cell is labelled also with $v - p^*$, $p - v^*$ and $l \cdot l^*$, respectively, which characterize the incidence relations between the flag features of the flags $v^* \subset l^* \subset p^* \subset \mathbb{P}^3$ in each cell and those of the reference flag $v \subset l \subset p \subset \mathbb{P}^3$. A hyphen between two elements denotes that one is included in the other, and a dot means that they meet at a single point.*

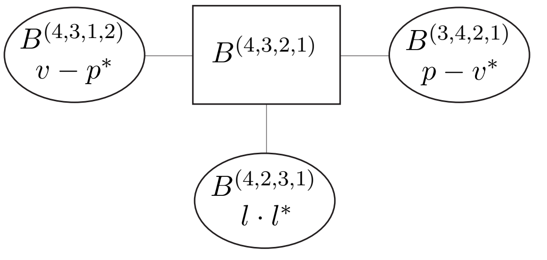

Figure 1: Adjacency between the higher dimensional cells of the flag manifold F$lag(4)$: the rectangle represents the 6D cell, and the ellipses are the 5D cells.

To describe for which pairs u and w of permutations the stratum B^u is contained in the closure of B^w, the *Ehresmann-Bruhat order* is defined in the set Σ_{n+1}: $u \leq w$ if and only if there is sequence of transpositions $(j_1, k_1), \ldots, (j_r, k_r)$ with $j_i < k_i$ for all i connecting w and u (i.e. if we set $w_0 = w$ and $w_i = w \cdot (j_1, k_1) \cdot \cdots \cdot (j_i, k_i)$ then $w_r = u$) and satisfying $w_{i-1}(j_i) > w_{i-1}(k_i)$ at each step $1 \leq i \leq r$.

Remark 1. *(see [3] 10.5): A canonical way to construct such a sequence from w to u is as follows: for $1 \leq i \leq r$ take j_i as the smallest integer such that $w_i(j_i) > u(j_i)$, and k_i as the smallest integer greater than j_i such that $w_i(j_i) > w_i(k_i) \geq u(j_i)$. This procedure can be carried out for any w and u, and if the chain does not arrive at u, then u is not less than w in the Ehresmann-Bruhat order. For example, if $n = 3$, $w = (4, 2, 3, 1)$ and $u = (2, 1, 4, 3)$, then the canonical sequence is*

$$w = (\underline{4}, 2, \underline{3}, 1) \geq (\underline{3}, \underline{2}, 4, 1) \geq (2, \underline{3}, 4, \underline{1}) \geq (2, 1, 4, 3) = u \ ,$$

where the pair switched at the next step is underlined. Besides, the first step shows, for example, that $w \not\geq (3, 4, 1, 2)$.

Proposition 2. *(see [3] 10.5): For u and v in Σ_{n+1}, the following are equivalent:*

1. *$u \leq v$,*
2. *$r_u(p, q) \geq r_v(p, q)$ for all p and q,*
3. *$B^u \subset \overline{B^v}$.*

Let us show that some stratifications of the flag manifold $\mathcal{F}lag(n+1)$ induce a stratification of the set of affine flags $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$. Fix from now on an affine reference flag, that is, $V_0 \subset \ldots \subset V_n = \mathbb{P}^n$ with $V_0 \notin H_\infty$. Consider the corresponding cell decomposition of $\mathcal{F}lag(n+1)$ as in (2). When restricted to the open subset of the affine flags $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ the partition (2) clearly induces a partition:

$$\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n) = \cup_{w \in \Sigma_{n+1}} (B^w \cap \mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)). \tag{3}$$

Since the reference flag is an affine flag, none of the above intersections is empty. However it might happen that some cell B^w would split off into two connected components: indeed, $B^w \cap \mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ is a unique connected component if and only if the permutation w starts with $w(1) = 1$. To see this, choose an affine reference frame $\{V_0; \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ attached to the reference flag, namely $\mathbf{e}_1$ is a vector representing the improper point $\mathbf{e}_1^\infty = V_1 \cap H_\infty$, $\mathbf{e}_2$ is a vector representing another point $\mathbf{e}_2^\infty$ on the improper line $V_2 \cap H_\infty$, and so on. Let $(x_1, \ldots, x_n)$ denote the projective coordinates in its associated projective reference $\{V_0, \mathbf{e}_1^\infty, \ldots, \mathbf{e}_n^\infty; \mathbf{a}\}$.

First, let us give a construction of the isomorphism of Proposition 1. Observe that each flag $\mathcal{V}^* = (V_0^*, \ldots, V_n^*) \in B^w$ is represented by a unique $(n+1) \times (n+1)$ matrix $\mathbf{M}$ whose first n rows span the flag features of $\mathcal{V}^*$, and where the p-th row has a 1 in the $w(p)$-th column, with all 0's at the right and below of this 1. $\mathbf{M}$ will be called the *canonical matrix* representing the flag $\mathcal{V}^*$. For example, for $n = 3$ and $w = (3, 4, 2, 1)$ the cell B^w is isomorphic to the set of matrices of the form

$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \\ * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where the stars denote arbitrary real numbers; in this case B^w is the set of all flags whose vertex lies on the plane $V_2 : \{x_4 = 0\}$. The number of stars appearing in the canonical matrices parameterizing the flags of B^w (for an arbitrary w) turns out to be the length of w (see [3] 10.2).

If we switch to affine flags and we take up again the example of the permutation $w = (3, 4, 2, 1)$, the affine flags of B^w are the disjoint union of two cells: one of them is isomorphic to the set of matrices of the form

$$\begin{pmatrix} a & * & 1 & 0 \\ * & * & 0 & 1 \\ * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

where the stars denote arbitrary real numbers and a denotes a positive real number; the other cell is isomorphic to the set of matrices of the same form (4) where the stars denote arbitrary real numbers and a denotes a negative real number. The matrices of the form (4) where a is zero correspond to flags which are not affine.

For a permutation w with $w(1) > 1$, let B^w_+ denote the connected component of $B^w \cap \mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ formed from the flags $\{(V_0^*, \ldots, V_n^*) \in B^w : V_0^* = (x_1, \ldots, x_{n+1})$ with $x_1 x_{w(1)} > 0\}$ and let B^w_- equal $\{(V_0^*, \ldots, V_n^*) \in B^w : V_0^* = (x_1, \ldots, x_{n+1})$ with $x_1 x_{w(1)} < 0\}$. Observe that the quotient $\frac{x_1}{x_{w(1)}}$ is the $(1,1)$ entry of the canonical matrix of any flag belonging to B^w. If $w(1) = 1$, set $B^w_+ = B^w_- = B^w$.

The interesting point of partition (3) is that it provides a stratification of $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ and that the adjacency between the cells may also be determined by considering a refinement of the Ehresmann-Bruhat order:

Theorem 2. *1. The partition*

$$\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n) = \bigcup_{\substack{w \in \Sigma_{n+1} \\ w(1) \neq 1}} (B^w_+ \cup B^w_-) \cup \bigcup_{\substack{w \in \Sigma_{n+1} \\ w(1) = 1}} B^w \tag{5}$$

is a stratification for the affine flags.

2. Let u and w be two permutations of Σ_{n+1}.

(a) If $u \leq w$, then $B^u_+ \subseteq \overline{B^w_+}$ and $B^u_- \subseteq \overline{B^w_-}$.

(b) If $u \leq w$ and $u(1) < w(1)$, then $B^u_+ \subseteq \overline{B^w_-}$ and $B^u_- \subseteq \overline{B^w_+}$.

Moreover, there are no other adjacency between cells than those in the two cases above.

Proof. Let $u, w \in \Sigma_{n+1}$ be two permutations such that $u \leq w$. In virtue of Remark 1, we may assume that $u = w \cdot (j, k)$, with $j < k$ and $w(j) = u(k) > w(k) = u(j)$. Let us consider first any flag $E \in B^u_+$ and suppose $\mathbf{M}$ is its canonical matrix; we will show that $E \in \overline{B}_+^w$ and, if $u(1) < w(1)$ (i.e., $j = 1$), then also $E \in \overline{B}_-^w$.

In the sequel, if $\mathbf{A}$ is a matrix, its i-th row will be denoted by $\mathbf{a}^i$, its j-th column by $\mathbf{a}_j$, and its element in the i-th row and the j-th column by a_j^i. Since $E \in B^u$, there is an unipotent lower triangular matrix, denoted by $\mathbf{L}$, such that $\mathbf{M} = \mathbf{M}_{\mathcal{V}(u)}\mathbf{L}$, where $\mathbf{M}_{\mathcal{V}(u)}$ is the canonical matrix representing the flag $\mathcal{V}(u)$. Moreover, as $m_1^1 = l_1^{u(1)}$ and $E \in B_+^u$, it follows that $l_1^{u(1)} > 0$.

Consider now, for any $t \in \mathbb{R}$, $t \neq 0$, the flag $G(t)$ whose canonical matrix $\mathbf{B}$ has the following rows:

$$\begin{aligned}
\mathbf{b}^i(t) &= \mathbf{e}_{w(i)} &&= \mathbf{e}_{u(i)} \ , &&\text{for } i \neq j, k, \\
\mathbf{b}^j(t) &= t(\tfrac{1}{t}\mathbf{e}_{w(k)} + \mathbf{e}_{w(j)}) &&= \mathbf{e}_{u(j)} + t\mathbf{e}_{u(k)} \ , && \\
\mathbf{b}^k(t) &= -\tfrac{1}{t}(\mathbf{e}_{w(k)} - \mathbf{b}^j(t)) &&= \mathbf{e}_{u(k)} \ , &&
\end{aligned} \tag{6}$$

where $\mathbf{e}_i \in \mathbb{R}^{n+1}$ is the vector all whose components are zero but for the i-th one, which equals one. On one hand, from the first expression, it follows that $G(t) \in B^w$ for any $t \in \mathbb{R}$, $t \neq 0$. On the other, the rightmost expression shows that the limit of $G(t)$ is the flag $\mathcal{V}(u)$ as $t \to 0$. Therefore the flag $F(t)$ represented by the matrix $\mathbf{A}(t) = \mathbf{B}(t)\mathbf{L}$ also belongs to B^w for all nonzero t, and the limit of $F(t)$ is the flag E as $t \to 0$. The last step that remains to be proved is whether $F(t) \in B_+^w$, i.e., $a_1^1(t) \cdot a_{w(1)}^1(t) > 0$. There are two possibilities:

- $w(1) = u(1)$. In this case, $j, k > 1$. Then we have, on one side,

  $$a_1^1(t) = \mathbf{b}^1(t)\mathbf{l}_1 = \mathbf{e}_{u(1)}\mathbf{l}_1 = l_1^{u(1)} \ ,$$

  which is positive by hypothesis. On the other side,

  $$a_{w(1)}^1(t) = \mathbf{b}^1(t)\mathbf{l}_{w(1)} = \mathbf{e}_{u(1)}\mathbf{l}_{u(1)} = l_{u(1)}^{u(1)} = 1 \ ,$$

  since $\mathbf{L}$ is unipotent. Therefore

  $$a_1^1(t)a_{w(1)}^1(t) = l_1^{u(1)} > 0 \ .$$

  This shows $F(t) \in B_+^w$ for any nonzero $t \in \mathbb{R}$, and thus $E \in \overline{B}_+^w$.

- $w(1) > u(1)$. This means $j = 1$, and then

$$a_1^1(t) = \mathbf{b}^1(t)\mathbf{l}_1 = (\mathbf{e}_{u(1)} + t\mathbf{e}_{w(1)})\mathbf{l}_1 = l_1^{u(1)} + tl_1^{w(1)} \ ,$$

$$a_{w(1)}^1(t) = \mathbf{b}^1(t)\mathbf{l}_{w(1)} = (\mathbf{e}_{u(1)} + t\mathbf{e}_{w(1)})\mathbf{l}_{w(1)} = l_{w(1)}^{u(1)} + tl_{w(1)}^{w(1)} = t \ ,$$

since $\mathbf{L}$ is lower triangular and unipotent. Hence

$$a_1^1(t)a_{w(1)}^1(t) = tl_1^{u(1)} + t^2 l_{u(1)}^{w(1)} \ .$$

Recall that $l_1^{u(1)}$ is positive by hypothesis. Thus, if $t \to 0^+$ the value of $a_1^1(t)a_{w(1)}^1(t)$ is positive, and then $F(t) \in B_+^w$; but if $t \to 0^-$ the value is negative and $F(t) \in B_-^w$. Therefore, in this case both inclusions $E \in \overline{B}_+^w$ and $E \in \overline{B}_-^w$ hold.

If we start the reasoning for any $E \in B_-^w$, the proof is analogous, since now we have $l_1^{u(1)} < 0$, but for the case $u(1) = 1$, for which $l_1^{u(1)} = 1 > 0$. This exceptional case is handled similarly, taking into account that, when $u(1) = 1$, $B_+^u = B_-^u$.

The first part of the theorem being proved, it remains to show that there are no more adjacency relations than those mentioned above. If $u \not\leq w$, from a well-known result on projective flags (see Appendix), we know that $B^u \not\subseteq \overline{B}^w$, and this implies that $B_a^u \not\subseteq \overline{B}_b^w$ for any $a, b \in \{+, -\}$. If $u \leq w$ and $u(1) = w(1)$, assume that there is a flag $E \in B_+^u \cap \overline{B}_-^w$. Since $E \in \overline{B}_-^w$, there is a sequence $\{F_m\}_m \subseteq B_-^w$ of flags whose limit is E as $m \to \infty$. Looking at the vertices of these flags, we have a sequence of projective coordinates of points $\mathbf{v}_m = (v_m^i)$, for which $v_m^1 v_m^{w(1)} < 0$, whose limit as $m \to \infty$ is $\mathbf{v} = (v^i)$, which satisfies $v^1 v^{u(1)} = v^1 v^{w(1)} > 0$, and this is impossible. Thus $B_+^u \cap \overline{B}_-^w = \emptyset$. Analogously $B_-^u \cap \overline{B}_+^w = \emptyset$, and this completes the proof. □

Example 2. *To illustrate the case* $n = 3$, *Fig. 2 shows the cells of dimensions* 6 *and* 5 *of* $\mathcal{F}_\mathcal{A}(\mathbb{P}^3)$ *and their adjacency. The rectangles represents the two 6D cells* $B_+^{(4,3,2,1)}$ *and* $B_-^{(4,3,2,1)}$, *while the ellipses are the six 5D cells:* $B_\varepsilon^{(4,3,1,2)}$, $B_\epsilon^{(3,4,2,1)}$ *and* $B_\varepsilon^{(4,2,3,1)}$, *with* $\varepsilon \in \{+, -\}$. *For the sake of clarity, each 5D cell is labelled also with* $(v{-}p^*)^\varepsilon$, $(p{-}v^*)^\varepsilon$ *and* $(l{\cdot}l^*)^\varepsilon$, *respectively, to make explicit the incidence relations between the flag features of the flags* $v^* \subset l^* \subset p^* \subset \mathbb{P}^3$ *in each cell and those of the reference flag* $v \subset l \subset p \subset \mathbb{P}^3$.

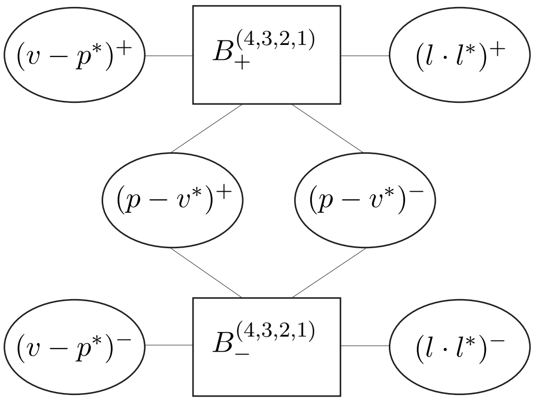

Figure 2: Adjacency between the higher dimensional cells of the set of affine flags $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^3)$: the rectangles represent the 6D cells, and the ellipses are the 5D cells.

3 Stratification of $\mathbb{R}^3 \times \mathbf{SO}(3)$ and adjacency between the higher dimensional cells

In the sequel we restrict to the case $n = 3$. Consider a positive-oriented orthonormal reference frame attached to a given flag $\mathcal{V}^*$. Observe that group of Euclidean transformations leaving this flag invariant is $\mathcal{H}_{\mathcal{V}^*} = \{\mathbf{I}, \mathbf{R}_x, \mathbf{R}_y, \mathbf{R}_z\}$, where $\mathbf{I}$ is the identity transformation, and $\mathbf{R}_k$ stands for a rotation of π radians about the k-axis (in the above mentioned reference frame attached to $\mathcal{V}^*$).

Fixed an affine reference flag $\mathcal{V}$, then any $\mathbf{q} \in \mathbb{R}^3 \times \mathrm{SO}(3)$ defines a unique flag $\mathbf{q}(\mathcal{V})$. On the other side, for any flag $\mathcal{V}^*$, there is some $\mathbf{q} \in \mathbb{R}^3 \times \mathrm{SO}(3)$ such that $\mathbf{q}(\mathcal{V}) = \mathcal{V}^*$. In fact, the set of 4 configurations yielding this same flag $\mathcal{V}^*$ is $\left\{\mathbf{Tq} \mid \mathbf{T} \in \mathcal{H}_{\mathcal{V}^*_{\mathbf{q}}}\right\}$. This gives a four-fold covering morphism $\pi : \mathbb{R}^3 \times \mathrm{SO}(3) \to \mathcal{F}_{\mathcal{A}}(\mathbb{P}^3)$[1] sending $\mathbf{q}$ to $\mathbf{q}(\mathcal{V})$ [7]. Via this 4-fold covering morphism π the stratification of $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^3)$ induces a stratification of $\mathbb{R}^3 \times \mathrm{SO}(3)$. For instance, via π, the two 6-dimensional disjoint cells of $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^3)$ correspond in $\mathbb{R}^3 \times \mathrm{SO}(3)$ to 8 6D cells, that is, 8 connected components. Analogously, the 6 5D cells of $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^3)$, correspond to 24 5D cells in $\mathbb{R}^3 \times \mathrm{SO}(3)$. We say that a 5D cell of $\mathbb{R}^3 \times \mathrm{SO}(3)$ is *of type* $v^* - p$, $v - p^*$ or $l \cdot l^*$ if it is one of

[1] This morphism corresponds to the restriction of what is known as the four-fold covering morphism between the partially oriented flag manifold $G(1,1 \mid 1,1)$ in $\mathbb{P}^3$ and $\mathcal{F}lag(4)$ [7].

the connected components of the inverse image of a cell $(v^* - p)^\varepsilon$, $(v - p^*)^\varepsilon$ or $(l \cdot l^*)^\varepsilon$, respectively, for some $\varepsilon \in \{+, -\}$ (see notation of Example 2). We shall focus on these cells of dimensions 5 and 6 and in determining their adjacency. To this aim we need to recall some concepts and results on paths and path lifting.

Definition 2. *A path in a manifold S is a continuous map γ from the unit real interval $[0, 1]$ to S; $\gamma(0)$ and $\gamma(1)$ are called the origin and end, respectively, of γ; γ is also called transition between $\gamma(0)$ and $\gamma(1)$. The path is closed if $\gamma(0) = \gamma(1)$. The inverse path of γ is defined as $\gamma^{-1}(t) = \gamma(1 - t)$.*

Given a covering morphism $\pi : \widetilde{S} \to S$, a lift of the path $\gamma : [0, 1] \to S$ is a path on $\widetilde{S}$, $\widetilde{\gamma} : [0, 1] \to \widetilde{S}$, so that $\pi \circ \widetilde{\gamma} = \gamma$.

Theorem 3 (Unicity of the lifting; see [5] 17.6). *Let $\pi : \widetilde{S} \to S$ be a covering morphism. Given a path $\gamma : [0, 1] \to S$ and a point $x \in \widetilde{S}$ so that $\pi(x) = \gamma(0)$, there is a unique lift $\widetilde{\gamma}$ of the path γ so that $\widetilde{\gamma}(0) = x$.*

To characterize each of the 8 6D cells of $\mathbb{R}^3 \times \mathrm{SO}(3)$ we use the triple of signs corresponding to the orientation of the three tetrahedra defined as follows. Suppose that the flag features of the affine reference flag $\mathcal{V}$ are spanned by three affine points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, that is, $\mathcal{V} = (\mathbf{a}_1, \mathbf{a}_1 \vee \mathbf{a}_2, \mathbf{a}_1 \vee \mathbf{a}_2 \vee \mathbf{a}_3, \mathbb{P}^3)$. Given $\mathbf{q} \in \mathbb{R}^3 \times \mathrm{SO}(3)$, consider the points $\mathbf{b}_i = \mathbf{q}(\mathbf{a}_i)$ for $1 \leq i \leq 3$. Observe that $\mathbf{q}$ lies on a 5D cell if, and only if, $\mathbf{b}_1$ lies on the plane $\mathbf{a}_1 \vee^a_2 \vee \mathbf{a}_3$, the lines $\mathbf{a}_1 \vee \mathbf{a}_2$ and $\mathbf{b}_1 \vee \mathbf{b}_2$ intersect, or $\mathbf{a}_1$ lies on the plane $\mathbf{b}_1 \vee \mathbf{b}_2 \vee \mathbf{b}_3$. Equivalently, $\mathbf{q}$ lies on a 6D cell if, and only if, the points in the sets

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1\},\ \{\mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_3\},\ \{\mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \tag{7}$$

form three non-degenerate tetrahedra. The orientation of a non-degenerate tetrahedron is given by the sign of the determinant of the matrix whose rows are the coordinates the vertices of the tetrahedron. Hence, by assigning to any $\mathbf{q} \in \mathbb{R}^3 \times \mathrm{SO}(3)$ the triple of signs corresponding to the orientation of its three associated tetrahedra, we obtain a partition of the 6D cells into 8 disjoint open sets which, by connectivity arguments, must correspond to the 8 6D cells. Notice that the four 6D cells $(\varepsilon, +, +)$, $(\varepsilon, +, -)$, $(\varepsilon, -, +)$ and $(\varepsilon, -, -)$ map by the covering π to $B_\varepsilon^{(4,3,2,1)}$ for $\varepsilon \in \{+, -\}$.

Theorem 4. *Each pair of 6D cells of $\mathbb{R}^3 \times SO(3)$ differing in only one sign are separated by two different 5D cells which are both of type $p - v^*$, $l \cdot l^*$ or $v - p^*$, if the differing sign occupies the first, second or third position, respectively.*

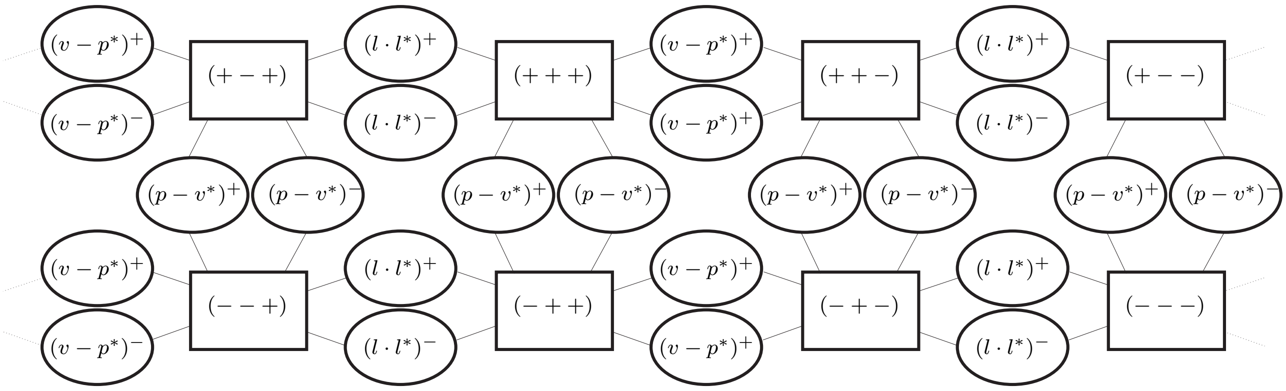

Figure 3: The graph shows the adjacency between the higher dimensional cells of the stratification of the Euclidean motion group. The rectangles represent the 6D cells, and the ellipses the 5D cells.

Proof. Directly due to the 4-fold covering π, there are two different 5D cells of type $p-v^*$ separating each pair of 6D cells $(+,\varepsilon_1,\varepsilon_2)$ and $(-,\varepsilon_1,\varepsilon_2)$ for any $\varepsilon_1,\varepsilon_2\in\{+,-\}$. Fix a flag $\mathcal{V}^*=\mathbf{q}(\mathcal{V})\in B_\varepsilon^{(4,3,2,1)}$, with $\varepsilon\in\{+,-\}$. We will consider in $\mathcal{F}_{\mathcal{A}}(\mathbb{P}^n)$ four different paths with origin $\mathcal{V}^*$ that will lie entirely in $B_\varepsilon^{(4,3,2,1)}$ except at a point, at which a 5D cell will be crossed. Namely, ρ_x and ρ_z are the rotations from 0 to π radians about the x-axis and z-axis, respectively, of an Euclidean reference frame attached to the flag $\mathcal{V}^*$; ρ_x^{-1} and ρ_z^{-1} are the respective inverse paths, i.e., rotations from 0 to $-\pi$ radians. Observe that the path $\rho_x(t)=(v^*,l^*,p^*(t))$ crosses the 5D cell $(v-p^*)^\varepsilon$ at the point $\rho_x(t_0)=(v^*,l^*,p^*(t_0))$ at which the platform plane $p^*(t_0)$ touches the vertex v of the base plane, and that $\rho_z(t)=(v^*,l^*(t),p^*)$ crosses the 5D cell $(l\cdot l^*)^\varepsilon$ at the point $\rho_z(t_1)=(v^*,l^*(t_1),p^*)$ at which the platform line $l(t_1)$ goes through the point $p^*\cap l$.

Let $\{\mathbf{q},\mathbf{R}_x\mathbf{q},\mathbf{R}_y\mathbf{q},\mathbf{R}_z\mathbf{q}\}$ be the 4 points in the fiber of $\mathcal{V}^*=\mathbf{q}(\mathcal{V})$. Consider the lifts of the paths ρ_x, ρ_x^{-1}, ρ_z and ρ_z^{-1} with origin $\mathbf{q}$ (cf. Theorem 3): $\widetilde{\rho}_x$, $\widetilde{\rho_x^{-1}}$, $\widetilde{\rho}_z$ and $\widetilde{\rho_z^{-1}}$. Notice that the transitions $\widetilde{\rho}_x$ and $\widetilde{\rho_x^{-1}}$ do not intersect except at the ends; the different configurations $\mathbf{q}_{t_0}$ and $\mathbf{q}'_{t_0}$ at which $\widetilde{\rho}_x$ and $\widetilde{\rho_x^{-1}}$, respectively, cross a 5D cell share the same flag $\rho_x(t_0)$, that is, $\pi(\mathbf{q}_{t_0})=\pi(\mathbf{q}'_{t_0})=\rho_x(t_0)$; at $\mathbf{q}_{t_0}$ and $\mathbf{q}'_{t_0}$ the volume of the last tetrahedra appearing in (7) becomes zero. Hence each transition crosses a different 5D cell in $\mathbb{R}^3\times\mathrm{SO}(3)$ of type $v-p^*$ and both transitions join two 6D cells whose differing sign occupies the third position. An analogous reasoning applies for transitions $\widetilde{\rho_z}$ and $\widetilde{\rho_z^{-1}}$: each of them crosses a different 5D cell in $\mathbb{R}^3 \times \mathrm{SO}(3)$ of type $l \cdot l^*$ and both transitions join two 6D cells whose differing sign occupies the second position. Finally a similar reasoning can be carried out with the lifts of the paths ρ_x and ρ_x^{-1} with origin $\mathbf{R}_z\mathbf{q}$, and with the lifts of the paths ρ_z and ρ_z^{-1} with origin $\mathbf{R}_x\mathbf{q}$ proving, thus completely, the statement of the Theorem and the adjacency displayed in Fig. 3. $\square$

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