Analyzing the Singularities of 6-SPS Parallel Robots Using Virtual Legs

JULIA BORRAS  FEDERICO THOMAS  CARME TORRAS
Institut de Robòtica i Informàtica Industrial (CSIC-UPC),
Llorens Artigas 4-6, 08028 Barcelona, Spain.
{jborras, fthomas, ctorras}@iri.upc.edu

Abstract: A virtual leg in a 6-SPS parallel robot is defined as a leg whose length is determined by the lengths of a subset of the actual legs of the robot. This necessarily implies that this subset of legs defines a rigid subassembly ...

1 Introduction

In general, substituting one leg in a 6-SPS parallel robot by another arbitrary leg modifies the location of the platform singularities in an rather unexpected way. Nevertheless, in those cases in which the considered platform contains rigid subassemblies, legs can be substituted in such a way as to the singularity locus is modified in a controlled way.

In this paper we will considered the three rigid subassemblies appearing in Fig. 1. They can be seen as rigid subassemblies involving a point and a line, two lines, and a line and a plane attached either to the base or the platform. In what follows, we will refer to them as PL, LL, and LPt subassemblies, respectively.

2 Substituting actual legs by virtual legs

Let us consider the 6-SPS parallel platform in Fig. ?? whose six linear actuators’ lengths are given by \( l_1, \ldots, l_6 \). It is well-known that the linear actuators’ velocities inputs, \( \dot{l}_1, \dot{l}_2, \ldots, \dot{l}_6 \), can be expressed in terms of the platform velocity vector \( \mathbf{W} = (\mathbf{v}, \Omega) \) as:

\[
\begin{bmatrix}
\dot{l}_1 \\
\dot{l}_2 \\
\vdots \\
\dot{l}_6
\end{bmatrix}
= J \begin{bmatrix}
\mathbf{v} \\
\Omega
\end{bmatrix},
\]

where

\[
J = \begin{bmatrix}
(a_1 \times n_1)^T & n_1^T \\
(a_2 \times n_2)^T & n_2^T \\
\vdots & \vdots \\
(a_6 \times n_6)^T & n_6^T
\end{bmatrix},
\]

that is, the matrix of Plücker coordinates of the six leg lines according to the notation used in Fig. ?? This matrix can be fac-
torized as follows:

\[
J = \begin{pmatrix}
\frac{1}{l_1} & 0 & \cdots & 0 \\
0 & \frac{1}{l_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{l_6}
\end{pmatrix}
\begin{pmatrix}
(a_1 \times (b_1 - a_1))^T \\
(a_2 \times (b_2 - a_2))^T \\
\vdots \\
(a_6 \times (b_6 - a_6))^T \\
(b_1 - a_1)^T \\
(b_2 - a_2)^T \\
\vdots \\
(b_6 - a_6)^T
\end{pmatrix}
= \text{diag}(1/l_1, 1/l_2, \ldots, 1/l_6) P.
\tag{3}
\]

Then, the singularities of the parallel robot are those configurations in which

\[
det(J) = \frac{1}{l_1l_2l_3l_4l_5l_6} \text{det}(P)
\tag{4}
\]

has a pole or a zero. The dividing term is usually neglected because it is assumed that, in practice, leg lengths cannot be null. Thus, only the term \(\text{det}(P)\), also known as the pure condition \([?]\), is considered. Nevertheless, we need to consider the dividing term as well, as it will be seen later.

Now, let us introduce a set of \(n\) new legs, \(n \leq 6\), whose lengths, say \(d_j, j = 1, \ldots, n\), are implicitly determined by scalar functions of the form:

\[
f_j(l_1^2, l_2^2, \ldots, l_6^2, d_j^2) = 0.
\tag{5}
\]

Then, the time derivative of \(d_j\) can be expressed as:

\[
\dot{d_j} = -\sum_{i=1}^{6} l_i \cdot \frac{\partial f_j}{\partial l_i^2} \cdot \dot{l}_i,
\tag{6}
\]

\(j = 1, \ldots, n\). Now, let us suppose, without loss of generality, that the first \(n\) legs of the platform are substituted by the \(n\) new legs. Then,

\[
\begin{pmatrix}
\dot{d}_1 \\
\vdots \\
\dot{d}_n \\
\dot{l}_{n+1} \\
\vdots \\
\dot{l}_6
\end{pmatrix}
= \begin{pmatrix}
N & W
\end{pmatrix}
J
\begin{pmatrix}
\mathbf{v} \\
\mathbf{\Omega}
\end{pmatrix},
\]

where

\[
N = \begin{pmatrix}
\frac{l_1}{d_1} \frac{\partial f_1}{\partial l_1^2} & \cdots & \frac{l_6}{d_1} \frac{\partial f_1}{\partial l_6^2} \\
\vdots & \ddots & \vdots \\
\frac{l_1}{d_6} \frac{\partial f_6}{\partial l_1^2} & \cdots & \frac{l_6}{d_6} \frac{\partial f_6}{\partial l_6^2}
\end{pmatrix}
\tag{7}
\]

and

\[
W = \begin{pmatrix}
\frac{l_{n+1}}{d_1} \frac{\partial f_{n+1}}{\partial l_1^2} & \cdots & \frac{l_{n+6}}{d_1} \frac{\partial f_{n+6}}{\partial l_1^2} \\
\vdots & \ddots & \vdots \\
\frac{l_{n+1}}{d_{n+6}} \frac{\partial f_{n+6}}{\partial l_1^2} & \cdots & \frac{l_{n+6}}{d_{n+6}} \frac{\partial f_{n+6}}{\partial l_1^2}
\end{pmatrix}
\tag{8}
\]

We conclude that the singularities of the parallel robot after the above leg substitution are those configurations in which

\[
det(J) \text{det}(N) = \frac{\text{det}(P) \text{det}(N)}{l_1l_2l_3l_4l_5l_6}
\tag{9}
\]

has a pole or a zero.

This result has important consequences. For example, if in the working space of the robot \(\text{det}(N)\) is always different from zero, the introduced substitution leaves the singularities of the original robot invariant. On the contrary, if \(\text{det}(N)\) is identically zero, the substitution introduces an architectural singularity.

### 3 The simplest leg substitution

Let us consider the \(PL\) subassembly and the virtual leg shown in Fig. 2.

![Figure 2: A \(PL\) subassembly. The leg in light grey represents a virtual leg.](image)

Since the tetrahedron defined by points \(p_1, p_2, p_3\) and \(p_4\) has null volume, then

\[
D(1, 2, 3, 4) = \begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & (m+n)^2 & l_1^2 & m^2 \\
1 & (m+n)^2 & 0 & l_2^2 & n^2 \\
1 & m^2 & n^2 & d^2 & 0
\end{vmatrix} = 0.
\tag{10}
\]

In other words,

\[
m^2(l_1^2 + m^2) + (m+n)^2d^2 - mn(m+n) = 0.
\tag{11}
\]

Thus, using (6), the time derivative of the virtual leg length can be expressed as:

\[
\dot{d} = \frac{l_1 n}{d m + n} \frac{\dot{l}_1}{l_1} + \frac{l_2 m}{d m + n} \frac{\dot{l}_2}{l_1}.
\tag{12}
\]

Then,

\[
\begin{pmatrix}
\dot{l}_1 \\
\vdots \\
\dot{l}_n \\
\vdots \\
\dot{l}_6
\end{pmatrix}
= \begin{pmatrix}
\frac{l_1}{d m + n} & \frac{l_2}{d m + n} & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & 1 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
\dot{d} \\
\vdots \\
\dot{l}_n
\end{pmatrix}
\tag{13}
\]

The determinant of the above inverse matrix is equal to

\[
\frac{d m + n}{l_1}.
\tag{14}
\]

The above derivation can be greatly simplified by directly applying the condition (9) and the formulas for the required derivatives given by (9). In this case ...
4 Leg substitutions in \( PPt \) subassemblies

5 Leg substitutions in \( LL \) subassemblies

Let us consider the \( LL \) subassembly and the virtual leg appearing in Fig. 3.

These six points define a simplex in \( \mathbb{R}^5 \) but, since it is embedded in \( \mathbb{R}^3 \), its volume is null. Hence, \( D(1, 2, 3, 4, 5, 6) = 0 \). This defines a quadratic equation in \( s_{5,6} \), the length of the virtual leg. Nevertheless, this equation can be simplified by applying Jacobi’s theorem to the following partition of \( D(1, 2, 3, 4, 5, 6) \)

\[
\begin{align*}
| & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & l_1^2 & l_2^2 & l_3^2 & l_4^2 & l_5^2 & l_6^2 & l_7^2 & s_{1,6}^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & l_2^2 & 0 & l_3^2 & m_1^2 & l_5^2 & l_6^2 & l_7^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & m_1^2 & l_3^2 & 0 & l_4^2 & l_5^2 & l_6^2 & l_7^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & l_3^2 & m_1^2 & l_4^2 & 0 & s_{2,5}^2 & s_{3,6}^2 & s_{4,5}^2 & s_{5,6}^2 & s_{6,7}^2 \\
1 & p_{11}^2 & s_{2,5}^2 & n_1^2 & s_{5,4}^2 & 0 & d^2 & 0 & s_{2,5}^2 & s_{3,6}^2 \\
1 & s_{6,1}^2 & p_{2}^2 & s_{6,3}^2 & n_2^2 & d^2 & 0 & 0 & 0 & s_{2,5}^2 & s_{3,6}^2 \\
\end{align*}
\]

where \( p_1 = m_1 + n_1 \) and \( p_2 = m_2 + n_2 \). Then, we conclude that \( D(0, 1, 2, 3, 4, 5) = 0 \) yields

\[
\frac{D(0, 1, 2, 3, 4)D(0, 1, 2, 3, 5) - D^2(0, 1, 2, 3, 4; 0, 1, 2, 3, 5)}{D(0, 1, 2, 3)} = 0.
\]

Now, note that \( D(0, 1, 2, 3, 4) = 0 \) and \( D(0, 1, 2, 3, 5) = 0 \) because they correspond to volumes of simplices in \( \mathbb{R}^4 \). Thus, assuming that the tetrahedron defined by \( p_0, p_1, p_2 \) and \( p_3 \) is not degenerate,

\[
D(0, 1, 2, 3, 4; 0, 1, 2, 3, 5) = 0,
\]

which is linear in \( d \).

The unknown squared distances, \( s_{i,j} \), can be readily obtained using ...

6 Leg substitutions in \( LPt \) subassemblies

Let us consider the \( LPt \) subassembly and the virtual leg shown in Fig. 4.

In this case, let us consider points \( p_3, \ldots, p_7 \). These points can be seen as two pyramids sharing the same triangular base so that the distance between their apexes is the length of the length of the virtual leg. Clearly, there are two solutions for this length. These five points define a simplex in \( \mathbb{R}^4 \) but, since it is embedded in \( \mathbb{R}^3 \), its volume is null. Hence, \( D(3, 4, 5, 6, 7) = 0 \). This defines a quadratic equation in \( s_{6,7} \), that can be simplified by applying Jacobi’s theorem to the following partition of \( D(3, 4, 5, 6, 7) \)

\[
\begin{align*}
| & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & l_1^2 & l_2^2 & l_3^2 & l_4^2 & l_5^2 & l_6^2 & l_7^2 & s_{1,6}^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & l_2^2 & 0 & l_3^2 & m_1^2 & l_5^2 & l_6^2 & l_7^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & m_1^2 & l_3^2 & 0 & l_4^2 & l_5^2 & l_6^2 & l_7^2 & s_{2,5}^2 & s_{3,6}^2 \\
1 & l_3^2 & m_1^2 & l_4^2 & 0 & s_{2,5}^2 & s_{3,6}^2 & s_{4,5}^2 & s_{5,6}^2 & s_{6,7}^2 \\
1 & p_{11}^2 & s_{2,5}^2 & n_1^2 & s_{5,4}^2 & 0 & d^2 & 0 & s_{2,5}^2 & s_{3,6}^2 \\
1 & s_{6,1}^2 & p_{2}^2 & s_{6,3}^2 & n_2^2 & d^2 & 0 & 0 & 0 & s_{2,5}^2 & s_{3,6}^2 \\
\end{align*}
\]

concluding that \( D(3, 4, 5, 6, 7) = 0 \) yields

\[
\frac{D(3, 4, 5, 6)D(3, 4, 5, 7) - D^2(3, 4, 5; 3, 4, 5, 6)}{D(3, 4, 5)} = 0.
\]

Assuming that the triangle defined by \( p_3, p_4, \) and \( p_5 \) is not degenerate, then

\[
D(3, 4, 5, 6, 7)^2 - D(3, 4, 5, 7)D(3, 4, 5, 7) = 0.
\] (15)

Since this equation is linear in \( d^2 \), there exist two possible solutions for the length of the virtual leg ...

7 Example

8 Conclusions and further research

The roles of actual and virtual legs can be exchanged, so that in practice the proposed substitutions can be seen as transformations in the location of the legs so that the modification of the singularity locus is performed in a controlled way ....

Acknowledgment

This research has been partially supported by the Spanish Committee for Science and Technology (CICYT), project DPI2007-60858, and the Catalan Research Commission, through the Consolidated Robotics Group.
References


