Three Types of Parallel 6R Linkages

Zijia Li and Josef Schicho

Abstract In this paper, we consider a special kind of overconstrained 6R closed linkages which we call parallel 6R linkages. These are linkages with the property that they have three pairs of parallel joint-axes. We prove that there are three types of parallel 6R linkage. The first type is new, the other two also appear in a recent classification of linkages with angle equalities. We give constructions for each of the three types.

Key words: Dual quaternions, overconstrained 6R linkages, translation property, angle-symmetric 6R linkages

1 Introduction

Movable closed 6R linkages have been considered by many authors (see [1, 4, 5, 11, 12, 13]). In this paper, we study a certain class of such linkages, which we call parallel 6R linkages. By definition, they have three pairs of parallel joint-axes for all possible configurations, or at least for infinitely many configurations (it could be that a certain linkage has two components, where only one of them produces three pairs of parallel joint-axes). Two of the pairs of parallel joint-axes are adjacent, and the third one is a pair of opposite joint-axes. We came across this type of linkages when we investigated 6R linkages with coinciding angles being equal, so called angle-symmetric 6R linkages [10]. Also there, there exist three types of angle-symmetric linkages, and one of the three types consists of parallel linkages. But not all parallel linkages are angle-symmetric in the sense of [10]. A new type can be constructed

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by taking three arbitrary lines as axes and applying an arbitrary translation to get the other three rotation axes.[1]

This paper also contains the complete classification of parallel linkages. These parallel linkages would fit into [2, Section 3.8], a general investigation of 6H linkages; our case is labelled "get to be examined" there.

Our investigation uses Study's description of Euclidean displacements by dual quaternions (see [7, 8]).

The remaining part of the paper is set up as follows. In Section 2, we give the theorems for classifying parallel 6R linkages, defining three types. In Section 3, we give a construction for each type.

2 Classification

We recall some notations from [8]. The set of all possible motions of a closed 6R linkage is determined by the position of the six rotation axes in some fixed initial configuration. The choice of the initial configuration among all possible configurations is arbitrary.

The algebra $\mathbb{DH}$ of dual quaternions is the 8-dimensional real vector space generated by $1, \varepsilon, \mathbf{i}, \mathbf{j}, \mathbf{k}, \varepsilon\mathbf{i}, \varepsilon\mathbf{j}, \varepsilon\mathbf{k}$ (see [7, 8]). Following [7, 8], we can represent a rotation by a dual quaternion of the form $\left(\cot\left(\frac{\phi}{2}\right) - h\right)$, where ϕ is the rotation angle and h is a dual quaternion such that $h^2 = -1$ depending only on the rotation axis. We use projective representations, which means that two dual quaternions represent the same Euclidean displacement if only if one is a real scalar multiple of the other.

Let L be a 6R linkage given by 6 lines, represented by dual quaternions $h_1, \ldots, h_6$ such that $h_i^2 = -1$ for $i = 1, \ldots, 6$. A configuration (see [7, 8]) is a 6-tuple $(t_1, \ldots, t_6)$, such that the closure condition

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)(t_5 - h_5)(t_6 - h_6) \in \mathbb{R}\backslash\{0\} \qquad (1)$$

holds. The configuration parameters t_i – the cotangents of the rotation angles – may be real numbers or ∞, and in the second case we evaluate the expression $(t_i - h_i)$ to 1, the rotation with angle 0. The set of all configurations of L is denoted by K_L. We say L is movable when K_L is a one-dimensional set. Mostly, we will assume, slightly stronger, that there exists an irreducible one-dimensional set for which none of the t_i is fixed. Such a component is called a non-degenerate component. We also exclude the case $\dim_{\mathbb{C}} K_L \geq 2$. Linkages with mobility ≥ 2 do exist, for instance linkages with all axes parallel have mobility 3, but they are well understood.

If $L = [h_1, h_2, h_3, h_4, h_5, h_6]$ is a 6R linkage with mobility 1, then we say that L is a parallel linkage if the axes h_1, h_6 are parallel and the axes h_3, h_4 are parallel, and

[1] Just in the last moment, we learned that a special case of this linkage was discovered in *A. Gfrerrer and P.J. Zsombor-Murray*, Robotrac Mobile 6R Closed Chain, Proc. CSME Forum 2002, see also www.geometrie.tugraz.at/lehre/KinematikRobotik/CrankAxlePerspektive.gif.

the non-adjacent axes h_2, h_5 are parallel for infinitely many configurations in K_L. The parallelity conditions in the initial configuration can be expressed as:

$$\begin{aligned} h_1 = p_1 + \varepsilon q_1, \quad h_2 = p_2 + \varepsilon q_2, \quad h_3 = p_3 + \varepsilon q_3,\\ h_6 = -p_1 + \varepsilon q_6,\, h_5 = -p_2 + \varepsilon q_5,\, h_4 = -p_3 + \varepsilon q_4, \end{aligned} \tag{2}$$

where p_i are the primal part of h_i and h_{7-i} for $i = 1,2,3$, and q_j are the dual part of h_j for $j = 1,\dots,6$.

There is a subset of K_L, denoted by K_{qsym}, defined by the additional restrictions $t_1 = t_6, t_2 = t_5, t_3 = t_4$. For all configurations in $\tau \in K_{qsym}$, the transformed lines h_2^τ and h_5^τ are again parallel. Conversely, if $K_0 \subseteq K_L$ is an irreducible component of dimension 1 that contains the initial configuration ∞^6 and that preserves the parallelity of the second and the fifth axis, then $K_0 \subseteq K_{qsym}$.

Remark 1. There exist a 6R linkage L with a one dimensional $K_0 \subseteq K_{qsym}$, but L is not a parallel 6R linkage. A possible construction can be found in [7, 8]).

Before the following lemma, we recall the definition of *coupling space* and its dimension in [6, 9]. For a sequence $h_i, h_{i+1}, \dots, h_j$ of consecutive joints, we define the coupling space $L_{i,i+1,\dots,j}$ as the linear subspace of $\mathbb{R}^8$ generated by all products $h_{k_1} \cdots h_{k_s}, i \le k_1 < \cdots < k_s \le j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product is allowed, its value is 1. The *coupling dimension* $l_{i,i+1,\dots,j}$ is the dimension of $L_{i,i+1,\dots,j}$.

For a parallel 6R linkage L in (2), we make a special transformation as following:

$$h_1' := P_1 h_1 \overline{P_1},\; h_6' := P_1 h_6 \overline{P_1},\; h_3' := P_2 h_3 \overline{P_2},\; h_4' := P_2 h_4 \overline{P_2},$$

where $\overline{P_i}$ denote the conjugations of P_i for $i = 1,2$, and P_1 and P_2 are translations such that h_1', h_2', h_3' meet in a common point. This is equivalent to the statement that the dimension of coupling space L_{123}' is 4. Furthermore, we have $(t_1 - h_6)(t_1 - h_1) = (t_1 - h_6')(t_1 - h_1')$ and $(t_3 - h_3)(t_3 - h_4) = (t_3 - h_3')(t_3 - h_4')$, and we get the following.

Lemma 1. *Parallel 6R linkage L and its transformed linkage L' as above have the same quasi-angle-symmetric configuration space K_{qsym}.*

Three consecutive rotation axes through the same point can be replaced by a spherical joint. The next lemma follows from the classification of *S3R* linkages.

Lemma 2. *For the transformed parallel linkage L', we have $l_{654}' = 4$ or 6.*

If $l_{654}' = 4$, then the lines h_4', h_5', and h_6' also meet in a common point. There is an unique translation P that maps the common point of h_1', h_2, h_3' to the common point of h_4', h_5, h_6'. So, P maps h_1' to h_6', h_2 to h_5, and h_3' to h_4'. But then, P also maps h_1 to h_6 and h_3 to h_4.

Conversely, assume that for six lines $h_1, \dots, h_6$, there exists a translation taking h_1 to h_6, h_2 to h_5, and h_3 to h_6. Then the linkage $L = [h_1, \dots, h_6]$ is mobile.

If $l_{654}' = 6$, then two cases are possible: either L' is a composition of a spherical linkage $[h_1', h_2, h_3', h_7]$ and a Bennett linkage $[h_6', h_5, h_4', h_7]$, with a suitable line h_7, or L' is a composition of a spherical linkage $[h_1', h_2, h_3', h_7, h_8]$ and a Goldberg 5R linkage $[h_6', h_5, h_4', h_7, h_8]$, with suitable lines h_7, h_8 passing through the common point of h_1', h_2, h_3'. In both cases, we get $t_1 = t_3$, so the linkage L' – therefore also L – is angle-symmetric in the sense of [10]. The first case coincides with the "rank 3" case in [10], and the second case is subsumed by the "rank 4" case in [10].

We have sketched the proof of the following theorem.

Theorem 1. *If L is a parallel linkage, then it either has the translation property, or four of the rotation angles are equal.*

3 Constructions

All constructions in this section are given in algebraic terms, using dual quaternions. The examples have been produced by an implementation of the constructions in Maple$^{\text{TM}}$.

3.1 Translation property

Here is a construction of parallel 6R linkage with translation property.

Construction 1 *(Parallel 6R Linkage with Translation Property)*
I. Choose three rotation axes h_1, h_2, h_3, i.e. dual quaternions such that $h_i^2 = -1$.
II. Choose a translation $P = 1 + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, with a, b, c in the set of real numbers.
III. Set $h_4 = -Ph_3\overline{P}$, $h_5 = -Ph_2\overline{P}$ and $h_6 = -Ph_1\overline{P}$.
IV. Our parallel 6R Linkage with translation property is $L = [h_1, h_2, h_3, h_4, h_5, h_6]$.
□

Example 1. A random instance of the above construction is

$$h_1 = \left(\frac{7}{9} - \frac{80}{81}\varepsilon\right)\mathbf{i} - \left(\frac{4}{9} + \frac{34}{81}\varepsilon\right)\mathbf{j} + \left(\frac{4}{9} + \frac{106}{81}\varepsilon\right)\mathbf{k},$$
$$h_2 = \left(\frac{3}{5} + \frac{8}{25}\varepsilon\right)\mathbf{i} - \frac{8}{5}\varepsilon\mathbf{j} - \left(\frac{4}{5} - \frac{6}{25}\varepsilon\right)\mathbf{k},$$
$$h_3 = -\left(\frac{1}{3} - \frac{4}{9}\varepsilon\right)\mathbf{i} - \left(\frac{2}{3} + \frac{4}{9}\varepsilon\right)\mathbf{j} - \left(\frac{2}{3} - \frac{2}{9}\varepsilon\right)\mathbf{k},$$
$$P = 1 - \frac{16}{27}\varepsilon\mathbf{i} - \frac{20}{27}\varepsilon\mathbf{j} + \frac{8}{27}\varepsilon\mathbf{k},$$
$$h_4 = \left(\frac{1}{3} - \frac{148}{81}\varepsilon\right)\mathbf{i} + \left(\frac{2}{3} + \frac{116}{81}\varepsilon\right)\mathbf{j} + \left(\frac{2}{3} - \frac{14}{27}\varepsilon\right)\mathbf{k},$$

$$h_5 = -\left(\frac{3}{5} + \frac{1016}{675}\varepsilon\right)\mathbf{i} + \frac{296}{135}\varepsilon\mathbf{j} + \left(\frac{4}{5} - \frac{254}{225}\varepsilon\right)\mathbf{k},$$
$$h_6 = -\left(\frac{7}{9} - \frac{112}{81}\varepsilon\right)\mathbf{i} + \left(\frac{4}{9} - \frac{46}{81}\varepsilon\right)\mathbf{j} - \left(\frac{4}{9} + \frac{242}{81}\varepsilon\right)\mathbf{k}.$$

Its configuration curve is irreducible of genus 1. Its equations are:

$$-21t_1^2 + 9t_1^2t_2 + 25t_2^2t_1 + 6t_1t_2 - 9t_1 + 6 - 9t_2 - 15t_2^2 = 0,$$
$$-21 + 63t_1 + 5t_2 - 27t_1t_2 - 6t_3 + 72t_3t_2 = 0.$$

Here are the Denavit-Hartenberg parameters [3] of the above linkage. These are the orthogonal distance between two adjacent joint axes a_{ij}, the distance d_i between the two footpoints of the two neighboring axes on the $i-th$ axis, and the twist angle between two adjacent joint axes α_{ij}, for $i = 1,\ldots,6$ and $j = i+1$ (modulo 6). For any parallel linkage with translation property, the parameters fulfill the conditions

$$a_{12} = a_{56},\ a_{23} = a_{45},$$
$$d_1 = d_4 = 0,\ d_2 = d_5,\ d_3^2 + a_{34}^2 = d_6^2 + a_{61}^2,$$
$$\alpha_{34} = \alpha_{61} = 0,\ \alpha_{23} = \alpha_{45},\ \alpha_{56} = \alpha_{12}.$$

In the example, the values are

$$a_{12} = a_{56} = \frac{58\sqrt{5}}{225},\ a_{23} = a_{45} = \frac{2\sqrt{2}}{3},\ a_{34} = \frac{8\sqrt{305}}{81},\ a_{61} = \frac{8\sqrt{5}}{9},$$
$$\alpha_{34} = \alpha_{61} = 0,\ \alpha_{23} = \alpha_{45} = \arccos\left(\frac{1}{3}\right),\ \alpha_{56} = \alpha_{12} = \arccos\left(\frac{1}{9}\right),$$
$$d_1 = d_4 = 0,\ d_2 = d_5 = \frac{11}{25},\ d_3 = \frac{80}{81},\ d_6 = 0.$$

3.2 Parallel 6R linkage with angle-symmetric property

There are two constructions, corresponding to the two sub cases of angle-symmetric parallel linkages. The first appeared in [10] gives Parallel 6R Linkage with angle-symmetric property (type 1). Here is the second construction.

Construction 2 *(Parallel 6R Linkage with angle-symmetric property, type 2)*

I. Choose two rotation axes h_1 and h_2, i.e. dual quaternions such that $h_1^2 = h_2^2 = -1$.

II. Choose another rotation axis h_6 parallel to h_1; the primal part of h_6 should be the primal part of h_1 times -1.

III. Compute two rotation axes m_1 and m_2 such that h_1, h_2, m_1, m_2 form a Bennett 4R linkage. One way to do this is to use the factorization algorithm for motion polynomials [8].

IV. Compute two rotation axes m_3 and h_5 such that h_6, m_2, m_3, h_5 form a Bennett 4R linkage, and such that the configuration curve is equal to the one in step III. Again, this can be done by factorizing a motion polynomial.

V. Choose a translation $P = 1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where b, c, d are real numbers.

V. Set $h_3 = -Pm_1\overline{P}$, $h_4 = -Pm_3\overline{P}$.

VI. Our parallel 6R Linkage is $L = [h_1, h_2, h_3, h_4, h_5, h_6]$. □

Example 2. A random instance of the above construction is

$$\begin{aligned}
h_1 &= \left(\frac{1}{3} - \frac{4}{9}\varepsilon\right)\mathbf{i} - \left(\frac{2}{3} - \frac{2}{9}\varepsilon\right)\mathbf{j} + \left(\frac{2}{3} + \frac{4}{9}\varepsilon\right)\mathbf{k},\\
h_2 &= -\left(\frac{1}{3} + \frac{8}{9}\varepsilon\right)\mathbf{i} - \left(\frac{2}{3} - \frac{8}{9}\varepsilon\right)\mathbf{j} + \left(\frac{2}{3} + \frac{4}{9}\varepsilon\right)\mathbf{k},\\
h_6 &= -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k},\\
a &= \frac{1}{2},\\
m_1 &= \left(\frac{119}{411} + \frac{124340}{168921}\varepsilon\right)\mathbf{i} + \left(\frac{226}{411} - \frac{172130}{168921}\varepsilon\right)\mathbf{j} - \left(\frac{322}{411} + \frac{74860}{168921}\varepsilon\right)\mathbf{k},\\
m_2 &= -\left(\frac{119}{411} - \frac{100888}{168921}\varepsilon\right)\mathbf{i} + \left(\frac{322}{411} - \frac{15560}{168921}\varepsilon\right)\mathbf{j} - \left(\frac{226}{411} + \frac{75292}{168921}\varepsilon\right)\mathbf{k},\\
m_3 &= \left(\frac{11601824}{8614971}\varepsilon - \frac{119}{411}\right)\mathbf{i} - \left(\frac{226}{411} - \frac{13771184}{8614971}\varepsilon\right)\mathbf{j} + \left(\frac{322}{411} + \frac{4651040}{2871657}\varepsilon\right)\mathbf{k},\\
h_5 &= \left(\frac{1}{3} - \frac{344}{459}\varepsilon\right)\mathbf{i} + \left(\frac{2}{3} - \frac{776}{459}\varepsilon\right)\mathbf{j} - \left(\frac{2}{3} + \frac{316}{153}\varepsilon\right)\mathbf{k},\\
P &= 1 - \frac{2}{3}\varepsilon\mathbf{i} - \frac{1}{2}\varepsilon\mathbf{j} + \varepsilon\mathbf{k},\\
h_3 &= \left(\frac{119}{411} + \frac{177770}{168921}\varepsilon\right)\mathbf{i} + \left(\frac{226}{411} - \frac{10388}{18769}\varepsilon\right)\mathbf{j} - \left(\frac{322}{411} - \frac{79}{168921}\varepsilon\right)\mathbf{k},\\
h_4 &= -\left(\frac{119}{411} - \frac{8876894}{8614971}\varepsilon\right)\mathbf{i} - \left(\frac{226}{411} - \frac{9760646}{8614971}\varepsilon\right)\mathbf{j} + \left(\frac{322}{411} + \frac{3377077}{2871657}\varepsilon\right)\mathbf{k}.
\end{aligned}$$

Here we found that the configuration curve is reducible. It has one non-degenerate component in K_{qsym}, with rational parametrization:

$$(t_1, t_2, t_3) = (t, t+1, t).$$

In Figure 1, we present twelve configuration positions of this linkage produced by Maple. □

Here are the numeric values of the Denavit-Hartenberg parameters.

$$a_{61}=\frac{2}{3},\ a_{12}=\frac{\sqrt{2}}{3},\ a_{23}=\frac{4151\sqrt{34}}{41922},\ a_{34}=\frac{274\sqrt{17}}{459},\ a_{45}=\frac{6617\sqrt{34}}{41992},\ a_{56}=\frac{86\sqrt{2}}{153},$$

$$\alpha_{34}=\alpha_{61}=0,\ \alpha_{23}=\alpha_{45}=\arccos\left(\frac{135}{137}\right),\ \alpha_{56}=\alpha_{12}=\arccos\left(\frac{7}{9}\right),$$

$$d_1=d_4=0,\ d_2=d_5=\frac{923}{1224},\ d_3=\frac{4795}{1836},\ d_6=\frac{225}{68}.$$

We do not know the general conditions of the Denavit-Hartenberg parameters of a linkage obtained by the construction.

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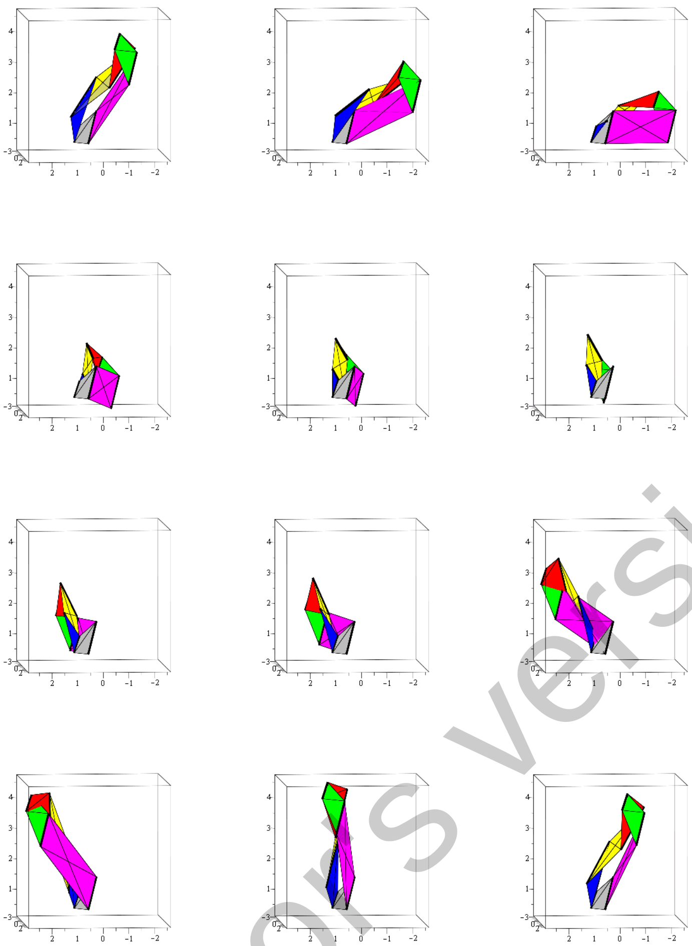

Fig. 1: A parallel angle-symmetric linkage of type 2 (described in Example 2). The four colored tetrahedra and the two colored parallelograms represent the six links, and the joints are the common edges of connected tetrahedra/parallelograms. Possible collisions of the links are just shown as overlapping links.

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