Obtaining the Maximal Singularity-free Workspace of 6-UPS Parallel Mechanisms via Convex Optimization

Amirhossein Karimi, Mehdi Tale Masouleh and Philippe Cardou

Abstract This paper explores the maximal volume inscribed ellipsoid in the singularity free constant-orientation workspace of two classes of 6-UPS parallel mechanisms, namely, *quadratic* and *quasi-quadratic* Gough-Stewart platforms. It is of paramount importance to obtain the optimum singularity-free ellipsoid by taking into account the stroke of actuators. Convex optimization is applied as the fundamental optimization tool of this paper. For this purpose, a matrix modeling for the kinematic properties of Gough-Stewart platform is proposed. The main contribution of this paper consists in improving an existing method in a such a way that it leads to a global optimum rather than a suboptimal solution. The proposed algorithm could be regarded as one of the most reliable, in terms of obtaining the global extremum, and propitious approaches, in terms of computational time in comparison with other approaches proposed in the literature for obtaining the singularity-free workspace which make it suitable for real-time applications.

Key words: Parallel mechanism, Gough-Stewart platform, Singularity-free workspace, Convex optimization.

1 Introduction

Despite many controversial deterrents, parallel mechanisms (PMs) [1] are now widely used in different industrial contexts, such as parallel kinematic machines and pick-and-place applications. The workspace and singularities of PMs have been

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extensively studied in precious literature, due to their importance in the kinetostatic performance of the mechanism [2] , [3]. To the best knowledge of the authors, few of these studies focused on analyzing the singularity-free workspace of Gough-Stewart platform, which is a definite asset in practice for path planning and control [4]-[6].

This paper aims at obtaining the singularity-free ellipsoid of a class of 6-DOF PMs, known as Gough-Stewart platforms, for which the singularity loci expression, for a prescribed orientation, is a second degree algebraic polynomial. These mechanisms are referred to as *quadratic* Gough-Stewart PMs. By the same token, the *quasi-quadratic* Gough-Stewart PM is defined as a mechanism for which the singularity loci expression, for a prescribed orientation, is of degree two upon fixing one translational DOF.

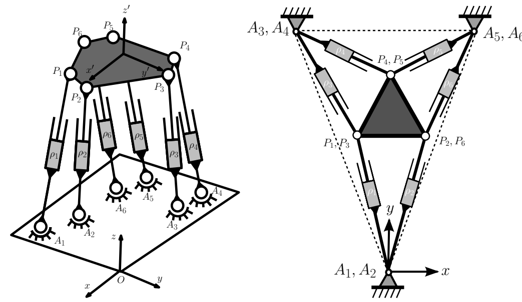
Convex optimization, the framework of this paper, can be regarded as a robust and reliable approach which is becoming the state-of-the-art in different disciplines due to its remarkable performance in terms of (1) computational time, (2) guaranteeing the optimality of the obtained solution and (3) providing analytical formulation of the problem.

The remainder of the paper is organized as follows. First the mechanism under study in this paper, the 6-U$\underline{\text{P}}$S PM, is broadly reviewed. We then touch briefly upon some preliminary definitions about convex optimization. Two classes of 6-U$\underline{\text{P}}$S PMs are introduced, namely, quadratic and quasi-quadratic PMs. For each case, a generalized algorithm is proposed in which the optimal ellipsoid is found within the singularity-free workspace. As case studies, the singularity-free workspaces of two given architectures of 6-U$\underline{\text{P}}$S PMs are obtained. Finally, the paper concludes by providing some remarks and describing related ongoing work.

2 Architecture Review and Kinematic Modeling

As depicted in Fig. 1, a Gough-Stewart platform is composed of six identical limbs of the U$\underline{\text{P}}$S type, which connect the base to the moving platform. Here and throughout this paper, R, P and S stand respectively for a revolute, a prismatic and a spherical joint while the underlined joint is the actuated. The platform generates 6-DOFs by adjusting the lengths ρ_i, $i = 1,\ldots,6$ of its prismatic joints. The pose (position and orientation) of the mobile platform is described by the two coordinate systems shown in Fig. 1(a), namely, O_{xyz} and $O_{x'y'z'}$ for the fixed and the mobile platforms, respectively. Point A_i and P_i, $i = 1,\ldots,6$, stand respectively for the coordinate of the U and S joints with respect to the fixed frame, O_{xyz}.

The position vector of the operation point of the mobile platform with respect to the fixed frame is represented by $\mathbf{p} = [x,y,z]^T$. This operation point is chosen to be point P_1. The position vectors of point P_i in the fixed and mobile frames are denoted by $\mathbf{p}_i$ and $\mathbf{p'}_i$, respectively. The position vector of point A_i attached to the base is $\mathbf{a}_i$ with respect to the fixed frame. The rotation of the mobile platform is represented by the proper orthogonal matrix $\mathbf{Q}$ obtained from $[\phi,\theta,\psi]$, i.e., Roll-Pitch-Yaw angles respectively.



(a) A General Gough-Stewart PM (b) A MSSM (top view)

Fig. 1 6-DOF Gough-Stewart platform.

2.1 Workspace Analysis

In this paper, the constant-orientation workspace is considered and the procedure to obtain the workspace of the Gough-Stewart platform is according to that proposed in [2]. The constant-orientation workspace is the set of possible Cartesian positions of the end-effector operation point for a prescribed orientation [6]. The kinematic equations corresponding to the i^{th} limb can be expressed as:

$$\rho_i^2 = (\mathbf{p}_i - \mathbf{a}_i)^T (\mathbf{p}_i - \mathbf{a}_i) \tag{1}$$

where $\mathbf{p}_i = \mathbf{p} + \mathbf{Q}\mathbf{p}'_i$ is the position vector of each point P_i in the fixed frame. The stroke of the actuator is represent by the interval $[\rho_{\min}, \rho_{\max}]$. Substituting $\rho_{\min}$ and $\rho_{\max}$ in Eq. (1) leads to the vertex spaces of the corresponding leg, i.e., the workspace of a limb for a given orientation, with respect to the fixed frame [2]. By assuming

$$\mathbf{c}_i = \mathbf{a}_i - \mathbf{Q}\mathbf{p}'_i \tag{2}$$

as the coordinates of the center point of each sphere given in Eq. (1), one can represent the workspace, or more precisely the spheres, in a matrix formulation as follows:

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^T\mathbf{x} + c_i = 0, \quad \text{for } i = 1, \ldots, 12 \tag{3}$$
$$\mathbf{A}_i = \mathbf{I}_{3\times3}, \quad \mathbf{b}_i = -\mathbf{c}_i, \quad c_i = \mathbf{c}_i^T\mathbf{c}_i - \rho_i^2$$

where $\mathbf{I}_{3\times3}$ stands for the 3×3 identity matrix.

2.2 Singularity Analysis

Singularity usually refers to configurations in which the mechanism fails to preserve its innate rigidity and consequently the mobile platform gains or loses some DOFs.

The first-order kinematic equation of a general PM can be expressed as:

$$\mathbf{B}\dot{\boldsymbol{\theta}} = \mathbf{A}\mathbf{x} \tag{4}$$

where $\mathbf{x}$ and $\dot{\boldsymbol{\theta}}$ are the infinitesimal motion of the output and the input vector, respectively. Matrices $\mathbf{A}$ and $\mathbf{B}$ denote the so-called Jacobian matrices, the regularity of which is related to the singularity configurations of the mechanism. From the classification proposed in [3], each singularity configuration falls into one of three types. In this paper, we consider only Type II, which is known as the direct kinematic singularities, but which we refer to as *singularity* for the sake of brevity. Thus, the mechanism undergoes a singularity when $\mathbf{A}$ becomes rank deficient, i.e., when $\det(\mathbf{A}) = 0$.

The study carried out in [5] reveals that the quadratic Gough-Stewart platform corresponds to a design with similar base and platform. In this case, the moving platform and the base differ only by a scale factor while a quasi-quadratic Gough-Stewart platform is the one with planar base. In this architecture, all points A_i lie on a plane and the singularity equation in this case is a polynomial of degree three [5]. Furthermore, by inspection, a Gough-Stewart PM with a planar base is quadratic upon fixing z, the axis perpendicular to the fixed base. Hence we say that it is *quasi-quadratic*.

3 Review on Convex Optimization and General Mathematical Framework

A convex optimization problem can be expressed as the minimization of a convex objective function subject to inequality constraints which are all convex functions. The main feature of convex programming is that any locally optimal point is also globally optimal [7], [8]. In what follows, a preamble class of convex optimization problems is briefly introduced, which corresponds to the problem at hand in this paper.

3.1 Maximum Volume Ellipsoid Inscribed in the Intersection of Second Order Surfaces

In this case, the problem consists in obtaining $\mathscr{E}$

$$\mathscr{E} = \{\mathbf{x} \quad | \quad \mathbf{x^T P x} + 2\mathbf{q^T x} + r \leq 1\} \tag{5}$$

as the maximum volume ellipsoid which satisfies all the constraints $\mathscr{C}_i$ defined as

$$\mathscr{C}_i = \{\mathbf{x} \quad | \quad \mathbf{x}^T \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^T \mathbf{x} + c_i < 0\}, \quad i = 1, ..., m \tag{6}$$

where $\mathbf{A}_i$ is a $n \times n$ matrix, $\mathbf{P} \in \mathbf{S}^n_{++}$ ($\mathbf{S}^n_{++}$ represents the set of $n \times n$ symmetric positive definite matrices) and $\mathbf{q}$ is a n-dimensional vector. Since the algorithm proposed in [9] to solve such a problem leads to a suboptimal solution, we introduce here an extension to the latter approach. We propose a judicious iterative procedure in order to converge to the optimal solution. The results obtained in [9] reveal that this problem can be formulated as the following convex optimization problem:

$$\begin{aligned} \max \quad & 1 - \mathrm{tr}(\mathbf{SP}) - 2\mathbf{q}^{\mathbf{T}}\overline{\mathbf{x}} - r \\ \text{s.t.} \quad & \lambda_1, ..., \lambda_m \geq 0, \\ & \begin{bmatrix} \mathbf{P} - \lambda_i \mathbf{A}_i & \mathbf{q} - \lambda_i \mathbf{b}_i \\ (\mathbf{q} - \lambda_i \mathbf{b}_i)^T & r - 1 - \lambda_i c_i \end{bmatrix} \succeq 0, \quad i = 1, ..., m, \\ & \begin{bmatrix} \mathbf{P} & \mathbf{q} \\ \mathbf{q}^T & r \end{bmatrix} \succeq 0 \end{aligned} \tag{7}$$

which is convex in variables $\mathbf{P}$, $\mathbf{q}$, r, and $\lambda_1, ..., \lambda_m$. The objective function in Eq. (7) is the greatest lower bound on $\mathrm{Prob}(\mathbf{X} \in \bigcap_1^m \mathscr{C}_i)$, where $\mathbf{X}$ is a random variable on $\mathbf{R}^n$, and $\mathrm{prob}(\cdot)$ is the probability function. There is no information about the distribution of $\mathbf{X}$ except that the first and second moments $\overline{\mathbf{x}} = \mathrm{E}(\mathbf{X})$ and $\mathbf{S} = \mathrm{E}(\mathbf{X}\mathbf{X}^{\mathbf{T}})$. Therefore, Eq. (7) is referred to as a lower bound SDP (SemiDefinite Programming) [9]. The obtained ellipsoid is the locally maximum volume ellipsoid tangent to the boundaries of $\mathscr{C}_i$ at some points, which contains point $\overline{\mathbf{x}}$. Notice that it should be regarded as a suboptimal solution since it is generated by an initial guess.

In what follows, an improved approach referred to as improved lower bound SDP is proposed in order to circumvent the latter problem. The main part of this algorithm consists in defining the initial information of the probability distribution, i.e., vector $\overline{\mathbf{x}}_0$ and the matrix $\mathbf{S}_0$, the so-called initial guess. The challenge consists in defining the first guess in order to launch properly the improved approach to find the optimal ellipsoid, i.e., to find $\mathbf{P}_{\mathrm{opt}}$, $\mathbf{q}_{\mathrm{opt}}$ and r_{opt}. To do so, first, an arbitrary point satisfying all the constraints of the problem should be defined for $\overline{\mathbf{x}}_0$ and $\mathbf{S}_0$. To ease the selection of the first guess, the second moment matrix can be computed as follows from the first moment vector:

$$\mathbf{S}_0 = \overline{\mathbf{x}}_0\, \overline{\mathbf{x}}_0^T + w\mathbf{I}_{3\times 3} \tag{8}$$

For a small w, upon applying the above equation into Eq. (7), one can compute the optimal ellipsoid around the mean vector $\overline{\mathbf{x}}_0$, but this is not the final and optimal solution to the problem. In order to find the optimal ellipsoid, an improved algorithm should be considered in which for each iteration the center of the expanded ellipsoid is computed as mean vector $\overline{\mathbf{x}}_i$ and the second moment matrix is computed from Eq. (8) for the new mean vector $\overline{\mathbf{x}}_i$. This means that after the first guess $\overline{\mathbf{x}}_0$, for the rest of the algorithm, the results obtained for $\mathbf{P}$, $\mathbf{q}$ and r are considered as the initial guess for pursuing the iteration.

This iterative procedure stops and returns $\mathbf{P}_{\mathrm{opt}}$, $\mathbf{q}_{\mathrm{opt}}$ and r_{opt} as optimal solutions when the distance between the centers of two consecutive expanded ellipsoids is smaller than a given value, ε, which means that there is not a significant change

Table 1 Geometric parameters of the PMs under study.

(a) Similar base and platform

i	1	2	3	4	5	6
x_{a_i}	0	1	2	1	-1	-1
y_{a_i}	0	0	1	2	2	1
z_{a_i}	0	0	0	2	1	0
x'_{p_i}	0	0.5	1	0.5	-0.5	-0.5
y'_{p_i}	0	0	0.5	1	1	0.5
z'_{p_i}	0	0	0	1	0.5	0
$(\rho_{\min})_i$	1	1	1	1	1	1
$(\rho_{\max})_i$	4	4	4	4	4	4

(b) MSSM

i	1	2	3	4	5	6
x_{a_i}	0	0	$\frac{1}{\sqrt[4]{3}}$	$\frac{1}{\sqrt[4]{3}}$	$-\frac{1}{\sqrt[4]{3}}$	$-\frac{1}{\sqrt[4]{3}}$
y_{a_i}	0	0	$\sqrt[4]{3}$	$\sqrt[4]{3}$	$\sqrt[4]{3}$	$\sqrt[4]{3}$
z_{a_i}	0	0	0	0	0	0
x'_{p_i}	$-\frac{\sqrt[4]{27}}{5}$	$\frac{\sqrt[4]{27}}{5}$	$\frac{\sqrt[4]{27}}{5}$	0	0	$-\frac{\sqrt[4]{27}}{5}$
y'_{p_i}	0	0	0	$\frac{3\sqrt[4]{3}}{5}$	$\frac{3\sqrt[4]{3}}{5}$	0
z'_{p_i}	0	0	0	0	0	0
$(\rho_{\min})_i$	1	1	1	1	1	1
$(\rho_{\max})_i$	4	4	4	4	4	4

in the center of the obtained ellipsoid by pursuing the procedure and the optimum ellipsoid is attained:

$$||\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_{i-1}||_2 \leq \varepsilon \tag{9}$$

4 Convex Modeling of Singularity-free Workspace

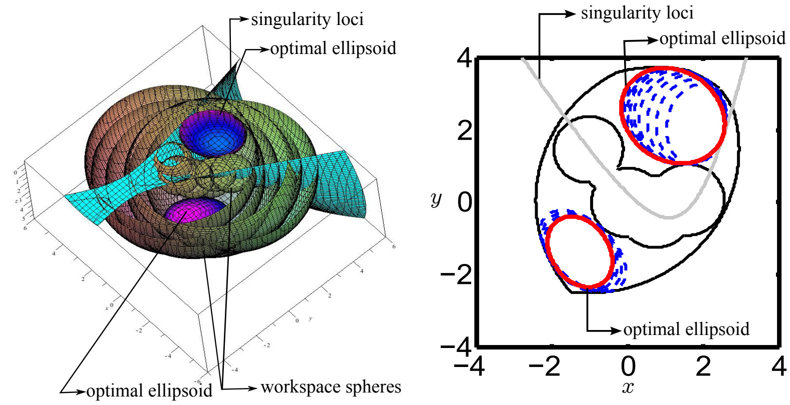
In this section, the aforementioned improved lower bound SDP algorithm is applied in order to investigate the singularity-free workspaces of two Gough-Stewart platforms. The optimization problem presented in Eq. (7) can be solved by resorting to the CVX package, a convex optimization package implemented in MATLAB by Grant and Boyd (2011) [10].

4.1 Case Study I: Similar Base and Platform

As the first case study, the Gough-Stewart PM with geometric parameters given in Table 1(a) is considered. Its base and moving platform being similar, thus this Gough-Stewart PM is quadratic.

The problem can be formulated as follows. The matrix formulation of the 12 spheres, $\mathscr{C}_i$, six inner and six outer spheres, is obtained from Eq. (3). The singularity equation can be reformulated readily as Eq. (6). The optimization problem given in Eq. (7) is then solved to obtain $\mathbf{P}_{\text{opt}}$, $\mathbf{q}_{\text{opt}}$ and r_{opt}. Finally, we implement the improved lower bound SDP method introduced in Section 3, in order to obtain the maximal singularity-free ellipsoid.

It should be noted that the optimal solution obtained from the improved lower bound SDP can be changed according to the choice of the initial guess $\bar{\mathbf{x}}_\mathbf{0}$. Thus, selection of different initial guesses in each feasible subregion of the singularity-free workspace results in the same optimal solution. However, the optimal solution



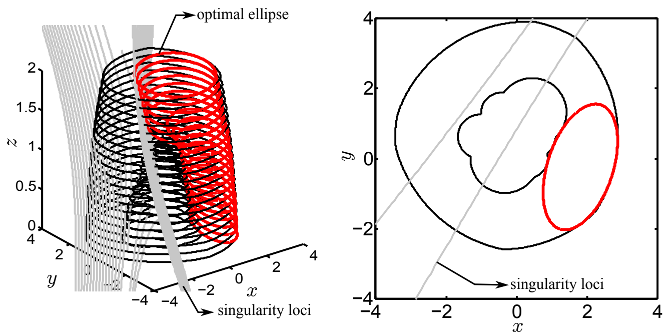
(a) The 3D view (b) The cross-section for $z = 0.3$

Fig. 2 The maximum-volume singularity-free ellipsoid in the workspace of 6-UPS for $\phi = \theta = \psi = \frac{\pi}{4}$.

varies based on the selection of $\bar{\mathbf{x}}_\mathbf{0}$ within other feasible subregions of singularity-free workspace. Figure 2 depicts the results obtained for $\phi = \theta = \psi = \frac{\pi}{4}$. Two initial guesses are considered, $\bar{\mathbf{x}}_0 = [2,2,0]^T$ and $\omega = 0.1$, and $\bar{\mathbf{x}}_0 = [-1,-1,1]^T$ and $\omega = 0.1$, which lead to two distinct optimal solutions as depicted in Fig. 2. Figure 2(b) represents the result for $z = 0.3$ including the iterative procedure, i.e., the dashed ellipses, to find the optimal solution, the solid ellipse. For this case, the computational time obtained by a PC equipped with an Intel(R) Core(TM) i5-2430M CPU @ 2.40GHz, and 4GB RAM is 0.6 s.

4.2 Case Study II: MSSM

As represented in Fig. 1(b), the Minimal Simplified Symmetric Manipulator (MSSM) is an architecture for which the base and the moving platform are isosceles triangles. In this case, a MSSM with an equilateral triangle base of unit area is considered. The moving platform is also an equilateral triangle with area of $\frac{9}{25}$. Table 1(b) represents the geometric parameters of this mechanism. Since in this architecture, the base of the mechanism is coplanar, the singularity equation is of quasi-quadratic type, i.e., the singularity equation becomes quadratic upon fixing z. Following the same reasoning explained for the previous case, the maximum area ellipse for each section, i.e., $z \in [0,2]$ in this case, can be obtained via the improved lower bound SDP approach. By starting from $\bar{\mathbf{x}}_0 = [1,0]^T$ and $\omega = 0.1$, after a computational time of 7.1 s, the optimal ellipse found as depicted in Fig. 3 for $\phi = \frac{\pi}{6}, \theta = \frac{\pi}{4}$, and $\psi = 0$. Figure 3(a) represents the general result with $\Delta z = 0.1$ as the increment value for each cross section while Fig. 3(b) depicts the result for a given cross section, $z = 0.5$.



(a) The isometric view for $0 \leq z \leq 2$ (b) The cross-section for $z = 0.5$

Fig. 3 The maximum-area singularity-free ellipses in the workspace of MSSM for $\phi = \frac{\pi}{6}, \theta = \frac{\pi}{4}, \psi = 0$.

5 Conclusion

This paper investigates the singularity-free workspace of two classes of 6-DOF parallel mechanisms referred to as quadratic and quasi-quadratic Gough-Stewart platforms. An extension to an existing approach was presented in order to converge to the optimal solution from an initial guess. In terms of computational time, the proposed algorithms provides some outstanding results with respect to others reported in the literature. Ongoing work include obtaining the appropriate design parameters for a prescribed singularity-free ellipsoid, namely, performing the dimensional synthesis of Gough-Stewart platforms.

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