DETERMINATION OF SINGULARITY LOCI PROPERTIES IN FULLY-PARALLEL MANIPULATORS THROUGH LAPLACE EXPANSIONS

Raffaele Di Gregorio
rdigregorio@ing.unife.it
Department of Engineering
University of Ferrara
Via Saragat,1
44100 FERRARA, Italy, EU
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INTRODUCTION

**Fully-Parallel Manipulators** (FPMs) feature a mobile rigid body (**platform**) connected to the frame (**base**) through 6 kinematic chains (**legs**) of type **SPS** (or **UPS**)

Many FPM architectures can be conceived according to the number of spherical pairs which coalesce either in the platform or in the base.

The most general FPM architecture is the 6-6 FPM (also called “**General Gough-Stewart Platform**”)

The Theoretical Results obtained for the 6-6 FPM can be applied to **ALL the other FPM architectures** by suitably changing the geometric constant which appear in the formulas.
INTRODUCTION (contd)

REFERENCES ON FPMs’ SINGULARITIES

General Classifications

Determinant Decomposition

Grassman-Cayley algebra and/or Plucker coordinates
- Borràs J., Thomas F., Torras C., Straightening-free algorithm for the singularity analysis of Steward-Gough Platforms with collinear/coplanar attachments, Computational Kinematics 2009, 2009

Special Laplace Expansions
- Di Gregorio, R., Singularity locus of 6-4 fully-parallel manipulators, ARK 2010, in press
INTRODUCTION (contd)

THE PROBLEM

\[
(P_i - B_i) \cdot \dot{P} + [(P_i - P) \times (P_i - B_i)] \cdot \omega = d_i \dot{d}_i, \quad i = 1, \ldots, 6
\]

\[
J^T \begin{bmatrix} \dot{P} \\ \omega \end{bmatrix} = D \dot{d}
\]

\[
u_i = P_i - B_i; \quad d_i = |u_i|; \quad v_i = (P_i - P) \times u_i \quad i=1,\ldots,6
\]

\[
d = [d_1, d_2, d_3, d_4, d_5, d_6]^T; \quad D = \text{diag}(d)
\]

\[
U = [u_1, u_2, u_3, u_4, u_5, u_6]; \quad J = \begin{bmatrix} U \\ V \end{bmatrix}
\]

SINGULARITY CONDITION:

\[
\det(J) = 0
\]
**LAPLACE EXPANSION**

*Let $A$ be an $n \times n$ matrix*

Definitions. Let $r = (r_1, r_2, \ldots, r_k)$ be a list of $k$ row indices for $A$, where $1 \leq k < n$ and $0 \leq r_1 < r_2 < \cdots < r_k < n$. Let $c = (c_1, c_2, \ldots, c_k)$ be a list of $k$ column indices for $A$, where $1 \leq k < n$ and $0 \leq c_1 < c_2 < \cdots < c_k < n$. The submatrix obtained by *keeping* the entries in the intersection of any row and column that are in the lists is denoted

$$S(A; r, c)$$

(11)

The submatrix obtained by *removing* the entries in the rows and columns that are in the list is denoted

$$S'(A; r, c)$$

(12)

and is the *complementary submatrix* for $S(A; r, c)$.

**LAPLACE EXPANSION THEOREM.** Let $A$ be an $n \times n$ matrix. Let $r = (r_1, r_2, \ldots, r_k)$ be a list of $k$ row indices, where $1 \leq k < n$ and $0 \leq r_1 < r_2 < \cdots r_k < n$. The determinant of $A$ is

$$\det(A) = (-1)^{|r|} \sum_c (-1)^{|c|} \det S(A; r, c) \det S'(A; r, c)$$

(13)

where $|r| = r_1 + r_2 + \cdots + r_k$, $|c| = c_1 + c_2 + \cdots + c_k$, and the summation is over all $k$-tuples $c = (c_1, c_2, \ldots, c_k)$ for which $1 \leq c_1 < c_2 < \cdots < c_k < n$. 

R. DI GREGORIO,  *Determination of Singularity Loci Properties in Fully-Parallel Manipulators Through Laplace Expansions*
LAPLACE EXPANSION (contd)

Example

Figure 4.2 A visualization of the expansion by rows 0 and 1 of a $4 \times 4$ matrix in order to compute the determinant.

Number of terms appearing in the expansion: \[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]
LAPLACE EXPANSION (contd)

Our Case

\[
\det(J) = 0
\]

where \( J = \begin{bmatrix} U \\ V \end{bmatrix} \) with \( U = [u_1, u_2, u_3, u_4, u_5, u_6] \), and \( V = [v_1, v_2, v_3, v_4, v_5, v_6] \).

The Laplace expansion with \( k = 3 \) and \( r = (0,1,2) \) yields the following expression with 20 terms

\[
\det(J) = u_{123}v_{456} - u_{124}v_{356} + u_{125}v_{346} - u_{126}v_{345} + u_{134}v_{256} - u_{135}v_{246} + u_{136}v_{245} - u_{145}v_{236} - u_{146}v_{235} + u_{156}v_{234} - u_{234}v_{156} + u_{235}v_{146} - u_{236}v_{145} - u_{245}v_{136} + u_{246}v_{135} + u_{256}v_{134} + u_{345}v_{126} - u_{346}v_{125} + u_{356}v_{124} - u_{456}v_{123}
\]

where \((u_i = P_i - B_i; \; d_i = |u_i|; \; v_i = (P_i - P) \times u_i)\) \quad i=1,\ldots,6

\[
u_{ijk} = \det([u_i, u_j, u_k]) = u_i \cdot u_j \times u_k, \quad i,j,k = 1,\ldots,6\
v_{ijk} = \det([v_i, v_j, v_k]) = v_i \cdot v_j \times v_k, \quad i,j,k = 1,\ldots,6\]
LAPLACE EXPANSION (contd)

Since the choice of point $P$ is arbitrary, without losing generality, it can be chosen coincident with $P_1$.

Such a choice makes the vector $v_1 \left[= (P_1 - P) \times u_1 \right]$ vanish, which makes the $v_{1jk}$ factor vanish, too.

Thus, the number of terms contained in the previous expression of $\det(J)$ reduces to 10.

In Short, $P \equiv P_1$ yields

$$\det(J) = u_{123}v_{456} - u_{124}v_{356} + u_{125}v_{346} - u_{126}v_{345} + u_{134}v_{256} -$$
$$u_{135}v_{246} + u_{136}v_{245} + u_{145}v_{236} - u_{146}v_{235} + u_{156}v_{234}$$

This reduced, but still general, expression contains only terms which depend on the geometry of the manipulator, and on the platform pose.
GEOMETRIC CONDITIONS

Singularities can be identified by finding the geometric conditions which make the deduced expressions of $\det(J)$ vanish.

The terms appearing in the deduced expressions are mixed products of vectors. Thus, they have clear geometric meanings.

Geometric conditions that make all the terms simultaneously vanish are singular geometries that can be easily identified by analyzing the deduced expressions. Such singular geometries are:

(a) All the leg axes are parallel to a single plane.
(b) One leg length vanishes.
(c) One leg axis is perpendicular to all the straight lines normal to each couple of leg axes chosen among the remaining five leg axes.
GEOMETRIC CONDITIONS (contd)

(d) All the leg axes pass through a point, say $P'$

(e) All the leg axes intersect a single straight line

(f) Four leg axes lie on a plane and the other two intersect each other in a point, $P_\perp$, lying on the same plane
**ALGEBRAIC EXPRESSION**

\[ P = P_1 \quad \Rightarrow \quad \det(J) = u_{123}v_{456} - u_{124}v_{356} + u_{125}v_{346} - u_{126}v_{345} + u_{134}v_{256} - u_{135}v_{246} + u_{136}v_{245} + u_{145}v_{236} - u_{146}v_{235} + u_{156}v_{234} \]

This expression of \( \det(J) \) is the sum of ten terms of type \( u_{1ij}v_{kmn} \), where the \( i, j, k, m \) and \( n \) indices are obtained by permuting the integer sequence 2-3-4-5-6.

Therefore, \textit{an explicit expression of} \( \det(J) \), containing all the mechanism geometric parameters and the platform pose parameters, \textit{can be obtained with the following procedure:}

1. \textit{(i)} the generic mixed product \( u_{1ij} \) is written in explicit form;
2. \textit{(ii)} the generic mixed product \( v_{kmn} \) is written in explicit form;
3. \textit{(iii)} the symbolic explicit expression of the generic term \( u_{1ij}v_{kmn} \) is obtained by multiplying the explicit expressions of \( u_{1ij} \) (step (i)) and \( v_{kmn} \) (step (ii));
4. \textit{(iv)} the explicit expression of \( \det(J) \) is obtained by calculating the expression of each term from the symbolic explicit expression of the generic term (step (iii)) and by adding the computed terms.
ALGEBRAIC EXPRESSION (contd)

This procedure can be implemented without systematically resorting to computer algebra, which makes it possible to find the type of explicit expression of det(J) after the first two steps have been implemented.

*It has been implemented in:


*and the result is that*

- $u_{1ij}$ and $v_{kmn}$ are linear and quadratic, respectively, in the coordinates of $P_1$ and the entries of the rotation matrix, $R_{bp}$; thus, the product $u_{1ij}v_{kmn}$ is cubic both in $P_1$’s coordinates and in $R_{bp}$’s entries, and it contains the products of the monomials cubic in $P_1$’s coordinates and the monomials cubic in $R_{bp}$’s entries.

- det(J) is at most a sixth-degree polynomial, which is cubic both in $P_1$’s coordinates and in $R_{bp}$’s entries;

- the singularity condition det(J)=0 can be put in the form of a ninth-degree polynomial equation which is cubic in the position parameters and a sixth-degree in the Rodrigues parameters.
SPECIAL CASES

- 6-4 FPM with one triple spherical pair

\[ P = P_1 = P_2 = P_3 \]

\[ \text{det}(J) = u_{123}v_{456} \]

**Singularity Condition:**

\[ u_{123}v_{456} = 0 \]
SPECIAL CASES (contd)

- 6-4 FPM with two double spherical pair

![Diagram of 6-4 FPM with two double spherical pair]

SINGULARITY CONDITION:

\[
\begin{align*}
\text{det}(\mathbf{J}) &= u_{123}v_{456} - u_{124}v_{356} + u_{125}v_{346} - u_{126}v_{345} \\
\end{align*}
\]

This condition has been studied in

- Di Gregorio, R., Singularity locus of 6-4 fully-parallel manipulators, ARK 2010, in press

where it has been transformed in the following form \((\mathbf{n}_{12} = \mathbf{u}_1 \times \mathbf{u}_2; \mathbf{n}_{34} = \mathbf{u}_3 \times \mathbf{u}_4)\):

\[
\text{det}(\mathbf{J}) = \mathbf{n}^T_{12} A \mathbf{n}_{34}
\]
CONCLUSIONS

- Laplace Expansion with submatrices of dimension 3x3 gives a quite simple expression of the general singularity condition of FPM.

- Such an expression involves only geometric quantities (mixed product of vectors) and can be written without introducing reference systems.

- It allows some geometric singular conditions to be immediately deduced.

- It allows the deduction of the algebraic form of the singularity condition.

- It is useful for studying special FPM architectures.

- It would be interesting to use it for deducing geometric transformations that do not affect the singularity locus.