Exact Workspace Synthesis for RCCR linkages

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Abstract A tool for the exact kinematic synthesis of a given workspace may be of interest when designing closed linkages. In these cases, finite-position synthesis cannot ensure smoothness of motion between task positions. In order to keep the simplicity of the finite-position synthesis approach, the workspace of relative displacements is described as a set of finite screws forming a screw surface. The screw surface is characterized by a number of screws which are used to generate the whole surface, and in turn to perform the dimensional synthesis. The methodology is here applied to the overconstrained RCCR closed linkage, for which the workspace of finite displacements yields a point-path synthesis problem.

1 Introduction

The dimensional synthesis of parallel robots has focused mainly on optimizing performance indices [7], [9] and reachable workspace sizing [1], [11], [3]; see also [12] for a comprehensive approach.

The use of a prescribed set of positions for the design of parallel robots by synthesizing all supporting legs has been applied in [20] for n-RRS parallel manipulators; also [10] and [16] perform partial kinematic synthesis of a 3-RPS parallel manipulator. In general, the finite-position method does not allow the control of the final trajectory of the parallel system and issues such as circuit defect may appear; in the most extreme cases, it may yield a system with negative mobility, that can be assembled at each task positions but cannot be driven from task position to task position.

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The kinematic mapping is used for the synthesis of planar and spherical linkages in order to state design equations and to provide a tool for visualizing the workspace and trajectories of the linkage. See [17] and more recent applications [4], [18] and [21]. For spatial motion, Study's kinematic mapping is used to obtain simplified equations for analysis and synthesis, see [8] and [2].

Considering the workspace of the linkage as a set of finite screws corresponding to finite displacements of the end-effector, and using Parkin's definition for pitch [13], the workspace takes a simple expression in some cases, see [5]. If the expression for the workspace is known, a finite set of positions can define the workspace and synthesize the corresponding linkage [14].

In this paper we apply the technique to the closed, movable RCCR linkage, studied by Waldron [19]. Its finite-screw workspace [15] has a constant orientation, hence the relative motion consists of translations only. The relative translations of the RC chain form a quadric surface, which is related to the finite set of translations used for the synthesis in order to perform exact workspace synthesis. The method yields a maximum of six different solutions; intersecting pairs of real solutions, several RCCR workspaces are generated.

2 The Workspace of Finite Screws

The workspace of relative displacements of an articulated chain can be expressed as a set of screws with a magnitude and a pitch; Parkin's definition of pitch [13] is used. Parkin's pitch appears naturally in the forward kinematics equations when using the Cifford algebra of dual quaternions [15]. When using this pitch, the screws corresponding to finite displacements of some linkages form screw systems; however, the relative workspaces of many linkages are nonlinear screw surfaces.

Given a finite-displacement screw $\mathsf{J} = (1+\mu\varepsilon)\mathsf{S}$, where μ is the pitch and $\mathsf{S} = (\mathbf{s}, \mathbf{c}\times\mathbf{s})$ are the Plucker coordinates of the screw axis, the exponential of the screw $\frac{\theta}{2}\mathsf{J}$ can be computed using the Clifford algebra product, to yield

$$e^{\frac{\theta}{2}\mathsf{J}} = (\cos\frac{\theta}{2} - \frac{d}{2}\sin\frac{\theta}{2}\varepsilon) + (\sin\frac{\theta}{2} + \frac{d}{2}\cos\frac{\theta}{2}\varepsilon)\mathsf{S} = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\mathsf{S}. \qquad (1)$$

The exponential of the screw defines a unit dual quaternion, corresponding to a relative displacement from an initial position to a final position in terms of a rotation around and a slide along axis S.

For a serial chain with n joints, with joint parameters $\Delta\hat{\Theta} = (\Theta - \Theta_0 + (\mathbf{d} - \mathbf{d}_0)\varepsilon)$ around and along the axis S_i, $i = 1, \ldots, k$, the product of exponentials defines the relative workspace from a reference configuration,

$$\hat{D}(\Delta\hat{\Theta}) = \cos\frac{\hat{\Psi}}{2} + \sin\frac{\hat{\Psi}}{2}\mathsf{S} = e^{\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_k}{2}\mathsf{S}_k}. \qquad (2)$$

It is immediate to find the screw axis S, magnitude and pitch from this expression,

$$\sin\frac{\hat{\psi}}{2}\mathsf{S} = (\sin\frac{\psi}{2} + \varepsilon\frac{t}{2}\cos\frac{\psi}{2})\mathsf{S} = \sin\frac{\psi}{2}(1+\varepsilon\frac{\frac{t}{2}}{\tan\frac{\psi}{2}})\mathsf{S}, \tag{3}$$

so that the finite-screw relative workspace is a set of screw axes with magnitude $\sin\frac{\psi}{2}$ and Parkin's pitch $\frac{t/2}{\tan\frac{\psi}{2}}$. The value of the magnitude is unique and can be calculated using the scalar part of the forward kinematics, see [6].

3 The Closed RCCR Linkage

The closed RC-CR linkage is overconstrained and able to move with one degree of freedom [19] when the cylindrical (C) and revolute (R) joints of each pair are parallel, while both pairs are skew one to each other, see Figure 1.

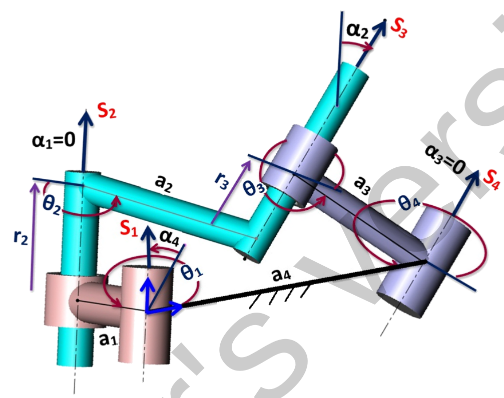

Fig. 1 RCCR linkage

For arbitrarily-positioned axes, the mobility of this spatial four-bar linkage is, using CKG formula, equal to zero. However it is possible to obtain a one-dof linkage for some special geometry.

The geometric features and the joint variable functions can be derived, for instance, by equating the forward kinematics of both RC serial chains at their end-effector [19]. According to the coordinate frame shown in Figure 1, and applying the needed condition of parallel axes, that is, $\alpha_1 = \alpha_3 = 0$, the forward kinematics of both RC chains 1-2 and 4-3 are,

$$[D_{RC1}] = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2)c\alpha_2 & s(\theta_1+\theta_2)s\alpha_2 & a_2c(\theta_1+\theta_2)+a_1c\theta_1 \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2)c\alpha_2 & -c(\theta_1+\theta_2)s\alpha_2 & a_2s(\theta_1+\theta_2)+a_1s\theta_1 \\ 0 & s\alpha_2 & c\alpha_2 & r_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

$$[D_{RC2}] = \begin{bmatrix} c(\theta_3+\theta_4) & s(\theta_3+\theta_4) & 0 & -a_3 c\theta_4 - a_4 \\ -s(\theta_3+\theta_4)c\alpha_4 & c(\theta_3+\theta_4)c\alpha_4 & s\alpha_4 & -r_3 s\alpha_4 + a_3 s\theta_4 c\alpha_4 \\ s(\theta_3+\theta_4)s\alpha_4 & -c(\theta_3+\theta_4)s\alpha_4 & c\alpha_4 & -r_3 c\alpha_4 - a_3 s\theta_4 s\alpha_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5)$$

where s and c stand for the *sin* and *cos* functions respectively.

Some geometrical constraints and angular relations are obtained from equating these two transformations,

$$\begin{aligned} &\cos\alpha_2 = \cos\alpha_4 \Longrightarrow \alpha_4 = \pm\alpha_2, \\ &\cos(\theta_1+\theta_2) = \pm 1, \quad \sin(\theta_1+\theta_2) = 0 \Longrightarrow \theta_2 = n*\pi - \theta_1, \\ &\cos(\theta_3+\theta_4) = \pm 1, \quad \sin(\theta_3+\theta_4) = 0 \Longrightarrow \theta_3 = n*\pi - \theta_4, \end{aligned} \quad (6)$$

in which the directions of the fixed joints are parallel to the directions of the moving joints, with coupled rotation angles. We can also derive the following joint variable relations:

$$\begin{aligned} &\theta_4 = \pm\arccos\left(\frac{\pm a_2 - a_4 - a_1\cos\theta_1}{a_3}\right) \\ &r_3 = \frac{a_1\sin\theta_1 - a_3\cos\alpha_4\sin\theta_4}{\sin\alpha_4}, \quad r_2 = \frac{a_1\cos\alpha_4\sin\theta_1 - a_3\sin\theta_4}{\sin\alpha_4} \end{aligned} \quad (7)$$

4 The Workspace of Finite Displacements of the RC-CR Linkage

We denote the RC chain with parallel axes and angles $\theta_2 = -\theta_1$ a *parallel RC chain*. For solving the design problem, it is advantageous to compute the workspace of relative displacements with respect to a reference configuration. The reference configuration can be arbitrarily selected, with $\Delta r_2 = r_2 - r_{20}$ and $\Delta\theta_i = \theta_i - \theta_{i0}$. The workspace of relative displacements a parallel RC chain is

$$\begin{aligned} \hat{D} =& \hat{R}(\Delta\theta_1)\hat{C}(\Delta\theta_2, \Delta r_2) \\ =& 1 + \varepsilon\frac{1}{2}\big(\Delta r_2\mathbf{s}_1 + (\cos\Delta\theta_1 - 1)(\mathbf{c}_2 - \mathbf{c}_1) - \sin\Delta\theta_1(\mathbf{c}_2 - \mathbf{c}_1)\times\mathbf{s}_1\big), \end{aligned} \quad (8)$$

where $\hat{R}(\Delta\theta_1)$ is a rotation about an axis with Plucker coordinates S_1, and $\hat{C}(\Delta\theta_2, \Delta r_2)$ is a rotation and a translation about and along an axis with Plucker coordinates S_2. Both axes share the same direction $\mathbf{s}_1$ and their rotations are $\Delta\theta_1$ and $\Delta\theta_2 = -\Delta\theta_1$; the points $\mathbf{c}_1$ and $\mathbf{c}_2$ are any points on the axes along a common normal line. Notice that the relative displacements have no change in orientation, so that the chain has a constant-orientation workspace.

The workspace of relative translations for the RCCR linkage is given by the intersection of the workspaces of two parallel RC chains. Figure 2 shows the workspace of a parallel RC chain and the intersection workspace for two chains.

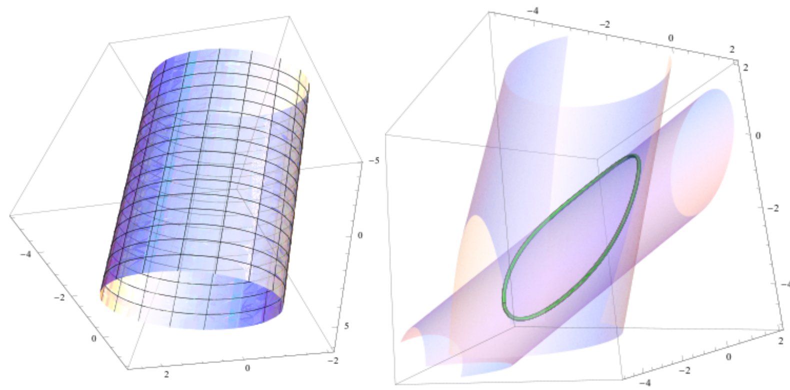

Fig. 2 Workspace of relative displacements for the parallel RC chain, left; for the RC-CR linkage, right.

In order to characterize the workspace of the parallel RC chain, we perform implicitization in Eq.(8) to eliminate the joint variables θ_1 and r_2. The elimination yields a quadratic surface of expression

$$\begin{aligned}\mathcal{Q}(x,y,z): \quad & (s_{1y}^2+s_{1z}^2)x^2+(s_{1x}^2+s_{1z}^2)y^2+(s_{1x}^2+s_{1y}^2)z^2 \\ & -2s_{1x}s_{1y}xy-2s_{1x}s_{1z}xz-2s_{1y}s_{1z}yz+c_{21x}x+c_{21y}y+c_{21z}z=0, \quad (9)\end{aligned}$$

where (x,y,z) is a point of the $\mathbb{R}^3$ space of relative translations, $\mathbf{s}_1=(s_{1x},s_{1y},s_{1z})$ is the direction for both joints, and $\mathbf{c}_{21}=\mathbf{c}_2-\mathbf{c}_1=(c_{21x},c_{21y},c_{21z})$ is the vector along the common normal between both joints.

This surface is classified as a circular cylinder, with radius $R=\sqrt{\mathbf{c}_{21}\cdot\mathbf{c}_{21}}$ and passing through the origin, which corresponds to the zero relative displacement. The intersection of two such circular cylinders yields a quartic curve which is the workspace of the RCCR linkage.

5 Dimensional Synthesis for the RCCR Linkage

The workspace of the RCCR linkage is a constant-orientation curve, and hence the synthesis problem can be reduced to a point-path synthesis problem. The point-path synthesis problem is stated as follows: given an initial point $\mathbf{P}_1$ (which we will use as reference configuration), relative displacements of the RC-CR chain will move this point to the rest of task points $\mathbf{P}_2, \mathbf{P}_3, \ldots, \mathbf{P}_n$.

The action of the chain on this point can be calculated using one of the conjugations in the Clifford algebra. If the forward kinematics of relative displacements of Eq.(8) is denoted by $\hat{D}=1+\varepsilon\mathbf{d}$, then

$$\mathbf{P}_i=\hat{D}\hat{P}_1\hat{D}^*, \quad i=2,\ldots,n, \qquad (10)$$

where $\hat{P}_1 = (1+\varepsilon\mathbf{P}_1)$ is the dual quaternion expression of the point $\mathbf{P}_1$, and the conjugation yields

$$(1+\varepsilon\frac{1}{2}\mathbf{d})(1+\varepsilon\mathbf{P}_1)(1+\varepsilon\frac{1}{2}\mathbf{d}) = 1+\varepsilon(\mathbf{P}_1+\mathbf{d}) \tag{11}$$

Notice that this is equivalent to equating the relative translations, $\mathbf{d} = \mathbf{P}_i - \mathbf{P}_1$ for $i = 2,\dots,n$.

Let us consider the case of the parallel RC chain, in which the values of θ_1 and r_2 are independent. This results in $3(n-1)$ design equations, with the structural variables $\mathbf{s}_1$ and $\mathbf{c}_{21} = \mathbf{c}_2 - \mathbf{c}_1$ and the joint variables r_2 and θ_1 for each point, for a total of $4+2(n-1)$ unknowns. Up to $n = 5$ point-positions can be defined in order to do exact point-path synthesis.

The standard finite-position synthesis technique equates the parameterized expression of the translation workspace to the task relative translations. However in this case, the implicit equation for the workspace has a simpler expression as a function of the chain structural parameters.

The four relative translations $\mathbf{P}_i$, $i = 2,\dots,5$, are used to shape the circular cylinder, and define the parallel RC chain that creates the motion. The system of design equations consists of six quadratic equations in six unknowns,

$$\begin{aligned} &\mathscr{Q}(\mathbf{P}_i) = 0, \quad i = 2,3,4,5; \\ &\mathbf{s}_1\cdot\mathbf{s}_1 = 1, \mathbf{s}_1\cdot\mathbf{c}_{21} = 0, \end{aligned} \tag{12}$$

which are easy enough to be solved using algebraic techniques. There are at most 6 different solutions.

6 Example

The task points used in this example are presented in Table 1. The system of equations (12) yields four real solutions, presented in Table 1, which can be assembled in pairs in order to create RCCR linkages. The number of different workspaces obtained is six.

Table 1 Goal points and solution RC chains

Point	Coordinates
P_1	$(2.31, 3.84, -1.08)$
P_2	$(0.34, -2.81, 0.89)$
P_3	$(2.21, -3.47, 0.63)$
P_4	$(2.18, 3.77, -2.66)$
P_5	$(-1.22, -1.42, -2.22)$

Solution	$\mathbf{s}_1$	$\mathbf{c}_2 - \mathbf{c}_1$
1	$(-0.09, -0.04, -0.99)$	$(0.03, 7.38, -0.30)$
2	$(-0.54, 0.42, -0.73)$	$(3.86, 5.19, 0.15)$
3	$(-0.10, 0.99, -0.08)$	$(4.11, 0.51, 1.22)$
4	$(0.54, 0.84, -0.03)$	$(-4.32, 2.85, 1.82)$

The solution workspaces for the parallel RC chains can also be intersected pairwise in order to create workspaces for the RCCR chains. The workspace equations can be used to visually assess the trajectory of the linkage and also to check for circuit defect. Figure 3 shows three of the six possible combinations for this example, and Figure 4 shows one of the possible RCCR chains.

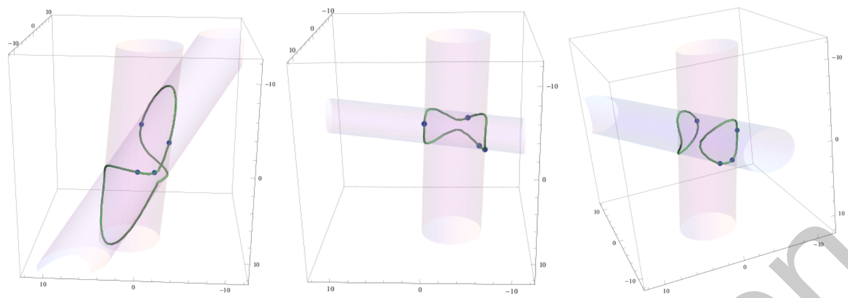

Fig. 3 Three of the six RCCR workspaces with the task points

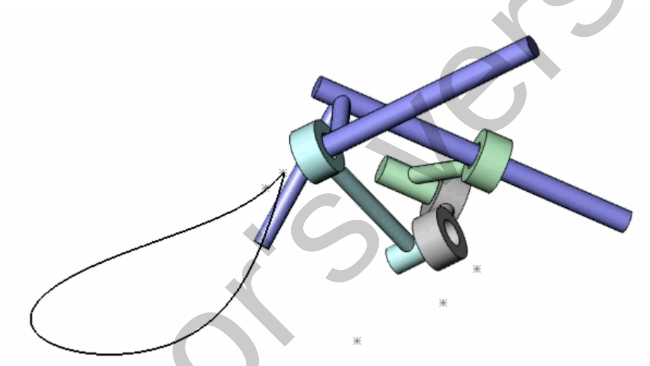

Fig. 4 RCCR linkage created with chains 2 and 3, passing through points on one of its two circuits.

7 Conclusions

This paper presents an exact-workspace synthesis method for the RCCR linkage, an overconstrained mechanism with mobility one. The implicitization of the algebraic equations of the workspace of relative displacements yields a circular cylinder that can be shaped using a set of finite positions. This simple case, in which the workspace has a constant orientation, is a building block towards a more general methodology for the exact workspace synthesis of spatial linkages.

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