2009 IEEE International Conference on Robotics and Automation
Kobe International Conference Center
Kobe, Japan, May 12-17, 2009

Kinematics of Line-Plane Subassemblies in Stewart Platforms

Júlia Borràs and Federico Thomas

***Abstract*— When the attachments of five legs in a Stewart platform are collinear on one side and coplanar on the other, the platform is said to contain a line-plane subassembly. This paper is devoted to the kinematics analysis of this subassembly paying particular attention to the problem of moving the aforementioned attachments without altering the singularity locus of the platform. It is shown how this is always possible provided that some cross-ratios between lines —defined by points in the plane— are kept equal to other cross-ratios between points in the line. This result leads to two simple motion rules upon which complex changes in the location of the attachments can be performed. These rules have interesting practical consequences as they permit a designer to optimize aspects of a parallel robot containing the analyzed subassembly, such as its manipulability in a given region, without altering its singularity locus.**

***Index Terms*— Parallel manipulators, robot kinematics, architectural singularities, kinematics singularities, manipulator design.**

I. Introduction

The kinematic analysis of a Stewart platform gets greatly simplified when it contains rigid subassemblies. When this happens the platform forward kinematics can be solved in a modular fashion and the contribution of each subassembly to the singularity locus of the platform can be easily singled out. There are four basic rigid subassemblies in Stewart platforms involving linear geometric elements such as points, lines, and planes (Fig. 1). This paper is devoted to the kinematics analysis of the line-plane subassembly with the aim of (a) obtaining a simple characterization of its architectural singularities, and (b) giving simple and complete rules for modifying the location of the leg attachments in the subassembly without altering the singularity locus of the platform.

The number of possibly overlapping subassemblies in which a Stewart platform can be decomposed was analyzed by Kong and Gosselin [1]. Gao and col. extended this analysis to generalized Stewart platforms involving distances between points, lines and planes instead of only six pairwise distances between points [2]. Zhang and Song solved, for the first time, the forward kinematics of general Stewart platform containing a line-plane subassembly [3]. They showed how the line in the line-plane subassembly of such a platform can have up to eight configurations with respect to the plane and, as a consequence, the platform can have up to 16 assembly modes. The eight configurations of the line correspond to

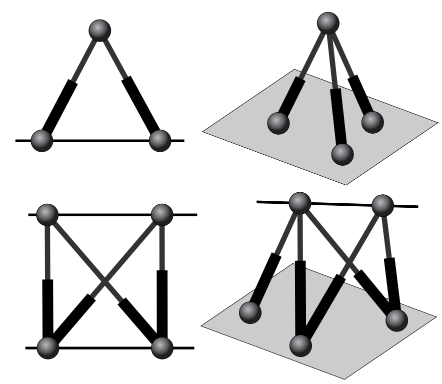

Fig. 1. Four rigid subassemblies involving linear geometric elements.

the roots of a bi-quartic polynomial. Therefore, the existence of an algebraic expression for these configurations in function of the five input distances was proved. Husty and Karger studied the conditions for this subassembly being architecturally singular and found two algebraic conditions that must be simultaneously satisfied [4]. To the best of our knowledge, no further insights into the analysis of the line-plane subassembly have been presented.

This work can be seen as a continuation of the one presented in [5], where we studied the line-line subassembly kinematics and where the role of cross-ratios between attachments in the characterization of architectural singularities, and in the singularity invariant modification of the attachments locations, was first acknowledged. We show herein how these ideas can be extended to the line-plane subassembly.

The paper is organized as follows. The next section introduces the notation used throughout this paper, and presents some preliminaries concerning the factorization of the Jacobian for a platform containing a line-plane subassembly. Section III deals with architectural singularities and presents a new simple geometric condition, in terms of cross-ratios, to decide if a given line-plane subassembly is architecturally singular. Section IV shows how to change the location of the attachments in a line-plane subassembly without changing the singularities of the platform. Section V presents a simple formulation to compute the lengths of the legs resulting from changing their attachments, using the transformations presented in the previous section, in terms of the lengths of the original legs. Section VI shows how the forward kinematics of a line-plane subassembly can be fully formulated in terms of distances and solved in terms of trilaterations. Finally, Section VII summarizes the contributions of this work.

The authors are with the Institut de Robòtica i Informàtica Industrial, CSIC-UPC. Llorens Artigas 4-6, 08028 Barcelona, Spain. E-mails: {jborras, fthomas}@iri.upc.edu. This work has been partially supported by the Spanish Ministry of Education and Science, under the I+D project DPI2007-60858.

II. Notation and Preliminaries

Let us consider the line-plane subassembly contained in the Stewart platform appearing in Fig. 2. We assume that no four points in the plane are collinear. Otherwise, this subassembly would contain a line-line subassembly that could be studied separately [5]. We also assume that no two attachments, either in the plane or in the line, are coincident.

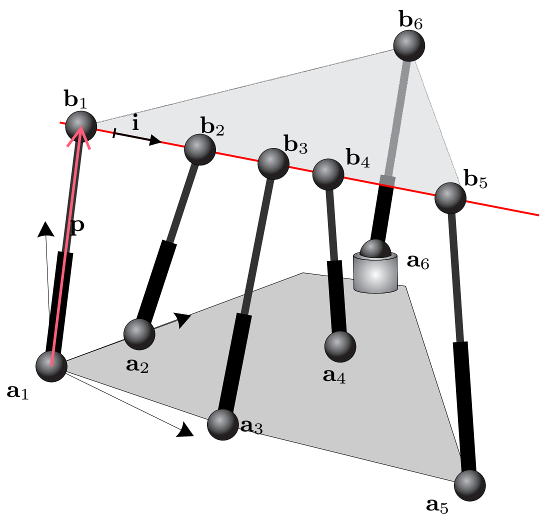

Fig. 2. Stewart platform containing a line-plane subassembly.

The attachments in the plane have coordinates $\mathbf{a}_i = (x_i, y_i, 0)$, for $i = 1, \ldots, 5$. The pose of the line with respect to the plane can be described by the position vector $\mathbf{p} = (x, y, z)$ and the unit vector $\mathbf{i} = (u, v, w)$ in the direction of the line. Thus, the coordinates of the attachments in the line, expressed in the base reference frame, can be written as $\mathbf{b}_i = \mathbf{p} + z_i\mathbf{i}$. Without loss of generality, we can set $x_1 = y_1 = y_2 = z_1 = 0$ by properly locating the base reference frame. In order to lighten the notation, an slight abuse of language is made by using the same symbol to denote a point and its position vector.

Then, the Plücker coordinates of the five leg lines in the line-plane subassembly can be written as:

$$\mathbf{l}_i = (\mathbf{b}_i - \mathbf{a}_i, (\mathbf{b}_i - \mathbf{a}_i) \times \mathbf{a}_i) = \begin{pmatrix} x + z_i u - x_i \\ y + z_i v - y_i \\ z + z_i w \\ -y_i(z + z_i w) \\ x_i(z + z_i w) \\ y_i(x + z_i u - x_i) - x_i(y + z_i v - y_i) \end{pmatrix}, \quad (1)$$

and that of the sixth leg, simply as $\mathbf{l}_6 = (\mathbf{v}, \mathbf{m})^T$.

The singularity locus of the platform is defined as the root locus of $\det(\mathbf{J})$ [6], where $\mathbf{J}$ is the matrix $\mathbf{J} = (\mathbf{l}_1, \ldots, \mathbf{l}_6)$. It can be checked that this determinant factors as follows:

$$\det(\mathbf{J}) = \det(\mathbf{T}) K(\mathbf{a}_6, \mathbf{b}_6), \quad (2)$$

where $K(\mathbf{a}_6, \mathbf{b}_6)$ is zero if, and only if, $\mathbf{a}_6$ lies on the platform plane, and $\mathbf{T}$ is:

$$\begin{pmatrix} wz & w(zu - xw) & w(zv - yw) & z(xw - zu) & z(yw - zv) \\ z_2 & x_2 & 0 & x_2 z_2 & 0 \\ z_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 \\ z_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 \\ z_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 \end{pmatrix}. \quad (3)$$

Therefore, the root locus of $\det(\mathbf{J})$ can be decomposed into the root locus due to the line-plane subassembly and that introduced by the sixth leg. The part of the singularity locus due to the line-plane subassembly corresponds to the root locus of the polynomial resulting from expanding $\det(\mathbf{T})$, i.e.,

$$C_1 wz + C_2 w(zu - xw) + C_3 w(zv - yw) + C_4 z(xw - zu) + C_5 z(yw - zv) = 0, \quad (4)$$

where C_i, for $i = 1, \ldots 5$, is the cofactor of the element i of the first row of $\mathbf{T}$.

III. Architectural Singularities

In the particular case in which $\det(\mathbf{T})$ is identically zero, independently of the pose of the line with respect to the plane, the subassembly is said to be *architecturally singular* [7]. Next, we concentrate ourselves in the characterization of this kind of singularities.

Note that $\det(\mathbf{T})$ is zero —independently of the pose of the line— if, and only if, all the coefficients in equation (4) are zero. Since such coefficients are the cofactors of elements of the first row of $\mathbf{T}$, we can say that $\det(\mathbf{T})$ is identically zero if, and only if, the submatrix formed by the last four rows of $\mathbf{T}$, say $\hat{\mathbf{T}}$, is rank defective. This circumstance can be easily detected by applying Gaussian elimination on $\hat{\mathbf{T}}$.

After performing standard Gaussian elimination on $\hat{\mathbf{T}}$, the last row of the resulting matrix is:

$$\frac{1}{D}\begin{pmatrix} 0 & 0 & 0 & -C_4 & C_5 \end{pmatrix} \quad (5)$$

where C_i are the same cofactors appearing in (4), and

$$D = \begin{vmatrix} z_2 & x_2 & 0 \\ z_3 & x_3 & y_3 \\ z_4 & x_4 & y_4 \end{vmatrix}. \quad (6)$$

Thus, if $C_4 = C_5 = 0$, the line-plane subassembly will be architecturally singularity. It can be shown that, by permuting the indices of the attachment, we can always find a value for D different from zero.

It can be checked that the conditions $C_4 = 0$ and $C_5 = 0$ are one-to-one equivalent to the two algebraic conditions presented in [4, Theorem 1.6].

If we would perform Gaussian elimination on the vertically mirrored version of $\hat{\mathbf{T}}$, the last row in the resulting matrix would be:

$$\frac{1}{z_2 x_2 y_3 y_4 (z_3 - z_4)}\begin{pmatrix} 0 & 0 & 0 & -C_1 & C_2. \end{pmatrix} \quad (7)$$

Thus, we can alternatively say that a line-plane subassembly is architecturally singular if, and only if, $C_1 = C_2 = 0$. Actually, by permuting the columns of $\hat{\mathbf{T}}$, we can conclude that a line-plane subassembly is architecturally singular if, and only if, any two cofactors are zero. In any case, to avoid an undefined quotient, the denominator resulting from the applied Gaussian elimination must be different from zero, which is always possible by permuting the indices of the attachments.

It has been proved that a line-line subassembly is architecturally singular if, and only if, the cross-ratios between the attachments in both lines are equal [5]. It is interesting to see how the condition $C_1 = C_2 = 0$ can also be interpreted geometrically in terms of cross-ratios.

Given four collinear points with coordinates $\mathbf{p}_i = (n_i, 0, 0)$, for $i = 1, \ldots 4$, their cross-ratio is defined as:

$$CR(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{(n_3 - n_1)(n_4 - n_2)}{(n_4 - n_1)(n_3 - n_2)}. \quad (8)$$

Likewise, for a set of four coplanar and concurrent lines, l_1, l_2, l_3, l_4, their cross-ratio $CR_l(l_1, l_2, l_3, l_4)$ is defined as the cross-ratio of the four points resulting from intersecting these four lines with an arbitrary line, in general position, lying in the same plane [8, Section IV.3].

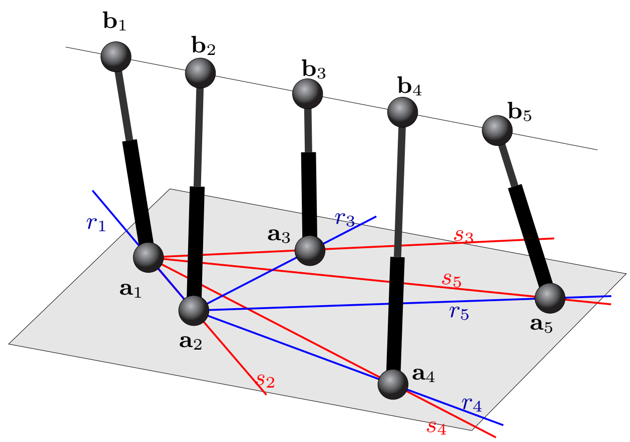

Fig. 3. A line-plane subassembly is architecturally singular if, and only if, $CR(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5) = CR_l(s_2, s_3, s_4, s_5)$ and $CR(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5) = CR_l(r_1, r_3, r_4, r_5)$. By permuting indices, up to ten equivalent sets of conditions can be derived.

A line-plane subassembly is architecturally singular if, and only if, $CR(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5) = CR_l(s_2, s_3, s_4, s_5)$ and $CR(\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5) = CR_l(r_1, r_3, r_4, r_5)$ (Fig.3). By permuting indices, up to ten equivalent sets of conditions can be derived. It can be proved that all of them are equivalent to the corresponding cofactor conditions $C_i = C_j = 0$ for $i, j \in \{1, ..., 5\}$, $i \neq j$ (in Fig.3, $C_1 = C_2 = 0$).

IV. Changing Attachments without Changing Singularities

Let us consider the multilinear equation

$$ax + by + cz + dxz + eyz + f = 0, \quad (9)$$

which implicitly defines a hypersurface in the space defined by $(x, y, z) \in \mathbb{R}^3$. The attachments of leg i of our line-plane subassembly define a point, (x_i, y_i, z_i), in this hypersurface. Since we have five legs (i.e., five points in this hypersurface), the coefficients a, b, c, e, and f are uniquely determined. Actually, (9) can be expressed in terms of these five points as:

$$\begin{vmatrix} z & x & y & xz & yz & 1 \\ z_1 & x_1 & y_1 & x_1 z_1 & y_1 z_1 & 1 \\ z_2 & x_2 & y_2 & x_2 z_2 & y_2 z_2 & 1 \\ z_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 & 1 \\ z_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 & 1 \\ z_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 & 1 \end{vmatrix} = 0.$$

Observe that, if we substitute one of the chosen five points by any other point in the hypersurface, the resulting equation will have the same coefficients up to a scalar multiple.

Since in our case $x_1 = y_1 = z_1 = y_2 = 0$, the above equation yields

$$\begin{vmatrix} z & x & y & xz & yz \\ z_2 & x_2 & 0 & x_2 z_2 & 0 \\ z_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 \\ z_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 \\ z_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 \end{vmatrix} = 0. \quad (10)$$

In other words,

$$C_1 z + C_2 x + C_3 y + C_4 zx + C_5 zy = 0 \quad (11)$$

where C_i are the cofactors referred in the previous section, i.e., the same coefficients appearing in (4).

Now, if we change the attachments of one leg so that the coordinates of the new attachments satisfy (10), the coefficients of the singularity polynomial in (4) remain the same up to a constant multiple and, as a consequence, its root locus remains invariant. This simple observation gives us the clue to change the attachments in a line-plane subassembly without changing the platform singularity locus.

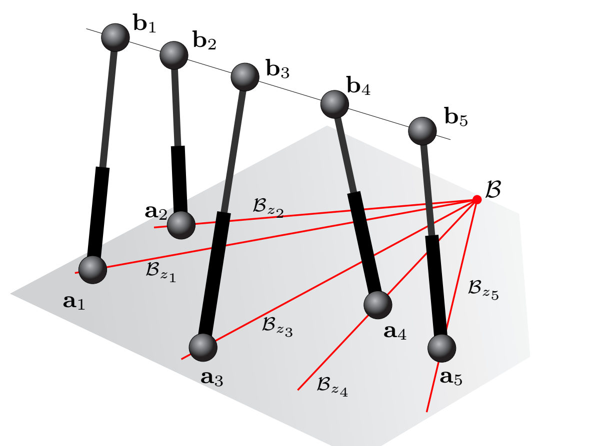

Fig. 4. The one-to-one correspondence between the attachments in the line and the lines of the pencil centered at $\mathcal{B}$. Each value of z_i defines a point in the line, $\mathbf{b}_i = \mathbf{p} + z_i \mathbf{i}$, and a line in the plane $\mathcal{B}_{z_i}$.

Equation (10) implicitly defines a one-to-one correspondence between points in the line and lines in the plane. Indeed, given an attachment in the plane with coordinates $(x, y, 0)$, we conclude from equation (10) that there is a unique corresponding attachment in the line with coordinate z. On the way round, given an attachment in the line, a line

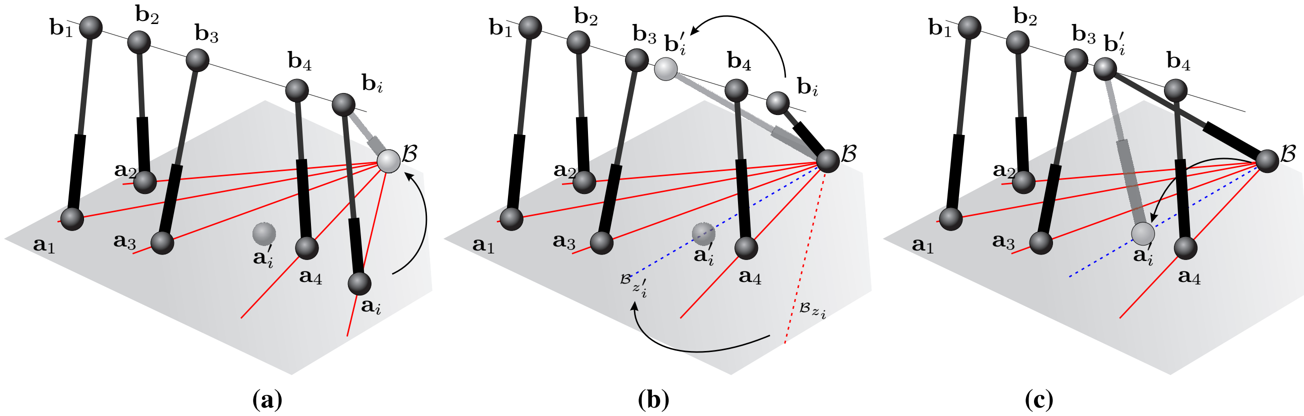

Fig. 5. It is possible to move an attachment in the plane to any arbitrary location by following three steps (see text for details).

is defined in the plane through equation (10). It is important to realize that, as z varies, a pencil of lines is generated in the plane. In other words, the generated lines intersect at a single point whose coordinates are:

$$\left(\frac{C_3C_1}{C_2C_5 - C_4C_3}, \frac{-C_2C_1}{C_2C_5 - C_4C_3}, 0\right). \tag{12}$$

In what follows, the point with the above coordinates will be called the *center of the correspondence*. It will be denoted by $\mathcal{B}$ and any line in the plane passing through it will be called a $\mathcal{B}$-line (Fig. 4).

Finally, two simple rules to move the attachments without altering the singularity locus naturally arise:

- all attachments in the plane can be freely moved along their $\mathcal{B}$-lines, and
- an attachment in the line can be freely moved if, and only if, the other attachment of the corresponding leg is located at $\mathcal{B}$.

Following these two rules, it is possible to move any attachment in the plane, say $\mathbf{a}_i$, to any arbitrary location, say $\mathbf{a}'_i$, in three steps (Fig. 5):

- move $\mathbf{a}_i$ along the corresponding $\mathcal{B}$-line until it meets $\mathcal{B}$,
- move $\mathbf{b}_i$ till its coordinate in the line determines a $\mathcal{B}$-line that contains $\mathbf{a}'_i$, and
- move $\mathbf{a}_i$ along the $\mathcal{B}$-line that contains $\mathbf{a}'_i$.

Thus, it is possible to carry out many complex transformations, but special attention must be paid to avoid that, at each step,

- no three attachments in the plane are located in the same $\mathcal{B}$-line because three leg lengths would become dependent, and
- no four attachments in the plane are collinear as, in this case, the line-plane subassembly would contain an architecturally singular line-line subassembly. This rather surprising result will become evident at the end of this section where the proposed transformations are interpreted in terms of cross-ratios.

Given a general line-plane subassembly, what is the maximum number of attachments that, following the above motion rules, can be made coincident? Two attachments in the plane can be readily made coincide by taking them along their $\mathcal{B}$-lines till they meet at $\mathcal{B}$ [Fig. 6(a)]. Afterwards, their corresponding attachments in the line can be moved to coincide with two other attachments [Fig. 6(b)]. Note that no further coincidences are possible without incurring in one of the two previous exceptions. It is obvious that other valid line-plane subassemblies including more coincident

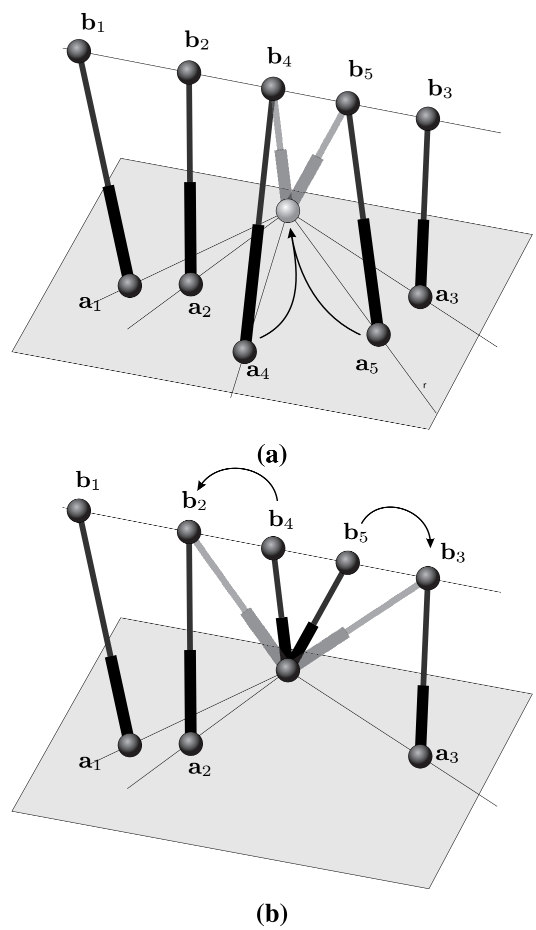

Fig. 6. Given a general line-plane subassembly, what is the maximum number of attachments that, following the proposed transformations, can be made coincident?

attachments exist, but we can ensure that their singularity loci will be essentially different from that of the general line-plane subassembly.

Finally, it is worth noting that a one-to-one correspondence between lines and/or points exists through a multilinear expression if, and only if, the cross-ratio of any set of four lines in a plane or four points in a line is equal to the cross-ratio of their correspondents. This fact derives from [9, Theorem II.2]. Since (10) defines a one-to-one correspondence through a multilinear function between lines and points, we can conclude that the cross-ratio of any four $\mathcal{B}$-lines must be equal to the cross-ratio of the corresponding attachments in the line. This provides and alternative way for computing $\mathcal{B}$ and, what is much more important, an alternative way of defining all valid changes in the location of the attachments as those that keep invariant the cross-ratios between the $\mathcal{B}$-lines in the plane and their corresponding attachments in the line.

V. Computing leg lengths

This section shows how to compute the length of the legs resulting from changing the location of their attachments, using the motion rules presented in the previous section, in terms of the lengths of the original legs.

The leg lengths of our line-plane subassembly can be expressed as $l_i^2 = \|\mathbf{b}_i - \mathbf{a}_i\|$, for $i = 1, \ldots, 5$. Then, if we subtract from the expression for l_i, $i = 2, \ldots, 5$, the equations $u^2+v^2+w^2 = 1$ and $l_1^2 = x^2+y^2+z^2$, quadratic terms cancel yielding

$$\begin{aligned} z_i t - x_i x - y_i y - x_i z_i u - y_i z_i v \\ + 1/2(x_i^2 + y_i^2 + z_i^2 + l_1^2 - l_i^2) = 0, \end{aligned} \tag{13}$$

for $i = 2, \ldots, 5$, where $t = -\mathbf{p} \cdot \mathbf{i}$, x, y, u and v are unknowns.

Now, suppose we want to compute the distance $d^2 = \|\mathbf{b} - \mathbf{a}\|$ between the point in the plane $\mathbf{a} = (x, y, 0)$ and the point in the line $\mathbf{b} = \mathbf{p} + z\mathbf{i}$, where $\{x, y, z\}$ satisfies equation (10). If we subtract from the expression for d^2 the equations $u^2 + v^2 + w^2 = 1$ and $l_1^2 = x^2 + y^2 + z^2$, quadratic terms cancel, as above. The resulting expression for d^2, together with the equations in (13), lead to a system of five equations in six unknowns. If we take t as parameter, the resulting linear system can be written as:

$$\begin{pmatrix} x_2 & 0 & x_2 z_2 & 0 & 0 \\ x_3 & y_3 & x_3 z_3 & y_3 z_3 & 0 \\ x_4 & y_4 & x_4 z_4 & y_4 z_4 & 0 \\ x_5 & y_5 & x_5 z_5 & y_5 z_5 & 0 \\ x & y & xz & yz & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \\ d^2 \end{pmatrix} = \begin{pmatrix} z_2 t + N_2 \\ z_3 t + N_3 \\ z_4 t + N_4 \\ z_5 t + N_5 \\ zt + N \end{pmatrix}, \tag{14}$$

where $N_i = \frac{1}{2}(x_i^2 + y_i^2 + z_i^2 + l_1^2 - l_i^2)$ and $N = \frac{1}{2}(x^2 + y^2 + z^2 + l_1^2)$. The determinant of this linear system is $\frac{1}{2}C_1$, that is, the cofactor of the first element of $\hat{\mathbf{T}}$. If this cofactor is zero, we can always choose as parameter either x, y, u, or v, to reformulate the above linear system. Since for a non-architecturally singular line-plane subassembly no two cofactors are zero (as we have proved in Section III), we can always find a non-singular linear system by choosing the right parameter.

Solving the above system for d^2 using the Cramer's rule, we get

$$d^2 = \frac{\begin{vmatrix} x_2 & 0 & x_2 z_2 & 0 & z_2 t + N_2 \\ x_3 & y_3 & x_3 z_3 & y_3 z_3 & z_3 t + N_3 \\ x_4 & y_4 & x_4 z_4 & y_4 z_4 & z_4 t + N_4 \\ x_5 & y_5 & x_5 z_5 & y_5 z_5 & z_5 t + N_5 \\ x & y & xz & yz & zt + N \end{vmatrix}}{\frac{1}{2}C_1}. \tag{15}$$

In other words,

$$d^2 = \frac{rt + s}{\frac{1}{2}C_1}, \tag{16}$$

where

$$r = \begin{vmatrix} x_2 & 0 & x_2 z_2 & 0 & z_2 \\ x_3 & y_3 & x_3 z_3 & y_3 z_3 & z_3 \\ x_4 & y_4 & x_4 z_4 & y_4 z_4 & z_4 \\ x_5 & y_5 & x_5 z_5 & y_5 z_5 & z_5 \\ x & y & xz & yz & z \end{vmatrix} \tag{17}$$

and

$$s = \begin{vmatrix} x_2 & 0 & x_2 z_2 & 0 & N_2 \\ x_3 & y_3 & x_3 z_3 & y_3 z_3 & N_3 \\ x_4 & y_4 & x_4 z_4 & y_4 z_4 & N_4 \\ x_5 & y_5 & x_5 z_5 & y_5 z_5 & N_5 \\ x & y & xz & yz & N \end{vmatrix}. \tag{18}$$

Finally, notice that, for any set of values for x, y, and z satisfying (10), $r = 0$. Therefore, the length of any leg, resulting from the changes in its attachments following the rules proposed in the previous section, can be readily computed as the quotient of two determinants involving the attachment coordinates and the leg lengths prior to the changes.

VI. Forward kinematics

Given the general line-plane subassembly in Fig.7, the lengths of the dotted segments in blue, l_1, l_3, and l_4, can be obtained using the formula presented in the previous section. Then, the lengths of the dotted segments in red u_1, u_2, u_3 and u_4, can be obtained using standard techniques from Distance Geometry, as described below. Once these distances are known, the forward kinematics of our line-plane subassembly can trivially solved by a sequence of trilaterations [10].

The Cayley-Menger determinant, $D(\mathbf{p}_1, ..., \mathbf{p}_n)$ of the set of points $\mathbf{p}_1, ..., \mathbf{p}_n$ is defined as the determinant of the $(n+1) \times (n+1)$ matrix with the last row and column entries set to one and its ij entry, the square distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ [10]. This determinant is proportional to squared volume of the $(n-1)$-dimensional simplex defined by the n points. Thus, in three dimensions, the Cayley-Menger determinant of five or more points is necessarily zero. Then, we can establish the following quadratic relation between u_1^2 and u_2^2.

$$D(\mathbf{a}_2, \mathbf{b}_2, \mathcal{B}, \mathbf{b}_5, \mathbf{a}_5) = \sum_{i=0}^{2} \sum_{j=0}^{2} p_{ij} (u_1^2)^i (u_2^2)^j = 0 \tag{19}$$

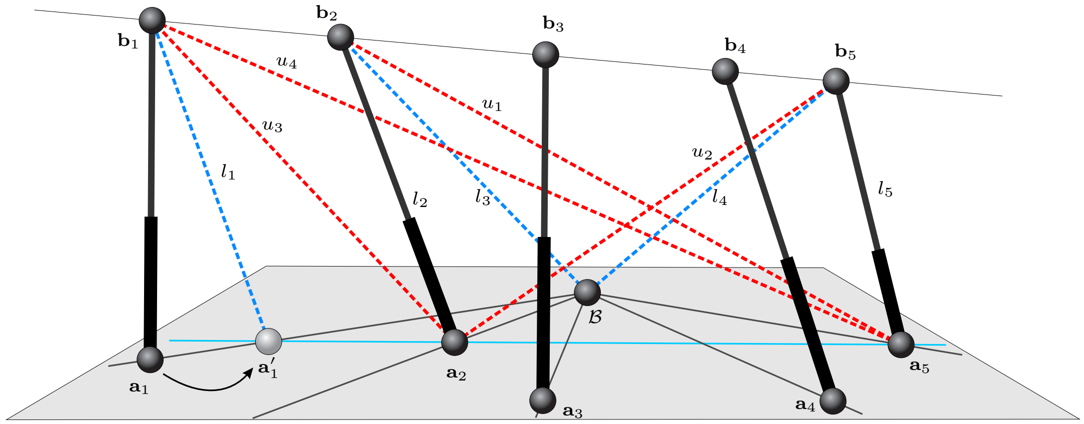

Fig. 7. The forward kinematics of a line-plane subassembly can be fully formulated in terms of distances. l_1, l_3 and l_4 can be computed using the distance formula presented in Section V. Then, u_i, for $i = 1, \ldots, 4$ can be obtained using standard Distance Geometry techniques.

Since the Cayley-Menger determinant of four coplanar points must also be zero, because the volume of the simplex defined by the four points is degenerate, the following linear equations in u_i^2, $i = 1, \ldots, 4$ can also be readily obtained:

$$D(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5, \mathbf{a}_2) = s_{12}u_2^2 + s_{25}u_3^2 + (s_{12} + s_{25})l_2^2 + s_{12}s_{25}(s_{12} + s_{25}) = 0 \tag{20}$$

$$D(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5, \mathbf{a}_5) = s_{12}l_5^2 + s_{25}u_4^2 + (s_{12} + s_{25})u_1^2 + s_{12}s_{25}(s_{12} + s_{25}) = 0 \tag{21}$$

$$D(\mathbf{a}'_1, \mathbf{a}_2, \mathbf{a}_5, \mathbf{b}_1) = r_{12}u_4^2 + r_{25}l_1^2 + (r_{12} + r_{25})u_3^2 + r_{12}r_{25}(r_{12} + r_{25}) = 0, \tag{22}$$

where r_{ij} are distances between base points $\mathbf{a}_i$ and $\mathbf{a}_j$ and s_{ij} distances between line points $\mathbf{b}_i$ and $\mathbf{b}_j$.

Using (20) and (21), we obtain values for u_3^2 and u_4^2 that can be substituted in (22), thus obtaining a linear equation in squared distances of the form:

$$p_1u_1^2 + p_2u_2^2 + p_3 = 0. \tag{23}$$

Finally, using (19) and (23), we obtain a resultant polynomial of degree four in u_1^2. Then, a closed-form solution for u_1^2 exists. For each possible value for u_1^2, four possible poses for the line with respect to the plane can be found by trilateration. Using the corresponding values for u_2^2, u_3^2 and u_4^2 to discriminate solutions, only two possible poses for each value of u_1^2 are possible. Actually, they are mirror solutions with respect to the plane. Thus, a line-plane subassembly can attain up to eight assembly modes, a result consistent with that presented by Zhang and Song in [3].

VII. Conclusions

A kinematics analysis of the line-plane subassembly with especial emphasis on singularity-invariant transformations in the locations of its leg attachments has been presented. It has been shown how these transformations can be interpreted as those that keep invariant some cross-ratios between points in the line and lines in the plane, and how similar cross-ratios also permit to decide if the given line-plane subassembly is architecturally singular. This remarkable characterization of architectural singularities and singularity-invariant transformations in terms of cross-ratios has given us much insight into both problems, mainly because of its straightforward geometric and visual interpretation. We actually conjecture that this characterization can be extended to general Stewart platforms.

Finally, it is worth noting that locating a line, with respect to a plane, from a set of pairwise distances between points in the line and in the plane is a basic operation that arises in constraint-based geometric modeling. As a consequence, it is worth investigating the repercussions the presented results to this problem.

Acknowledgement

The authors would like to thank Carme Torras and Maria Alberich-Carramiñana for their technical comments and helpful advice during the development of this work.

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