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GENERATION OF UNDER-ACTUATED PARALLEL ROBOTS WITH NON-HOLONOMIC JOINTS AND KINETOSTATIC ANALYSIS OF A CASE STUDY

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ABSTRACT

It will be shown how to generate under-actuated manipulators by substituting non-holonomic spherical pairs (nS pairs) for (holonomic) spherical pairs (S pairs) in fully-parallel manipulators (FPMs). Through this pair substitution, an under-actuated manipulator, previously proposed by one of the authors, will be demonstrated to be generated from an inversion of the 6-3 FPM. Moreover, the kinetostatic analysis of this manipulator will be reconsidered to obtain a simple and compact formulation. This reformulated analysis can be used both in the design of the under-actuated manipulator, and in its control.

Keywords: *kinetostatics, non-holonomic constraint, under-actuated manipulator, parallel manipulator.*

NOMENCLATURE

P prismatic pair.
U universal joint.
S spherical pair.
nS non-holonomic spherical pair.
SPU spherical-prismatic-universal kinematic chain (limb).
nSPU (non-holonomic spherical)-prismatic-universal kinematic chain (limb).
FPM fully parallel manipulator.
DPA direct position analysis.
dof degrees-of-freedom.
RB rigid body.

1 INTRODUCTION

Non-holonomic constraints arise in many different areas of robotics such as motion planning and control of mobile robots, reorientation of free-flying space robots, rolling contacts of multi-fingered hands, etc. In all these cases, the non-holonomic constraints are inherent to the problem, but there are some cases in which the artificial introduction of this kind of constraints can provide important advantages.

In pick-and-place applications of manipulators, only the initial and the final poses (position and orientation) of the end effector are assigned by the task, whereas the end-effector path between them is free. The ideal manipulator for these applications should be able to make the end effector reach any pose in the six-dimensional operational space, and, by exploiting the free fly of the end effector, it should be able to satisfy additional design conditions that reduce its hardware complexity. Joints with non-holonomic constraints do not reduce the reachable relative poses of the links connected by the joint since non-holonomic constraints have the only effect of reducing the set of paths that can be covered for moving between two reachable relative poses. This reduction of practicable paths is accompanied by the rising of new reaction forces in the joint which can be usefully exploited to eliminate actuators. Thus, designing a manipula-

tor with fewer actuators than the degrees-of-freedom (dof)[1] of its configuration space —to reduce bulk, weight and expense— becomes feasible by introducing mechanical elements that lead to non-holonomic constraints.

The literature on the use of non-holonomic devices in the design of manipulators is limited to few examples. In [3], Stammers et al. presents a robot wrist that can attain any orientation with two motors only. This is achieved by means of a friction drive, using rollers on a spherical ball to which the end effector is fixed, and by fixing the two motors to the arm. In [4], Peshkin et al. present a passive spherical robot which can display programmable constraints. The device is based on a non-holonomic element involving a sphere and three reorientable rollers. In [5], Nakamura et al. describe an n-joint serial robot which can reach any pose in its n-dimensional configuration space with only two actuators. The joints of this manipulator are coupled by (n-1) non-holonomic devices, based on spheres and rollers, so that it reaches a desired pose by following a path whose computation is algorithmically equivalent to maneuvering a car with n-trailers. More recently, in [6], Ben-Horin and Thomas proposed a three-legged parallel robot where each leg is connected to the base through a sphere whose motion is constrained by a roller. This latter parallel architecture permits to attain any position and orientation for the platform using only three prismatic actuators.

Despite the difference of purpose, all mentioned examples include at least one sphere whose motion is constrained by a roller that can freely roll in contact with the sphere without slipping laterally. This no-slip constraint is a non-holonomic constraint, a constraint relating the velocities of the sphere and the roller. The kinematics of this sphere-roller assembly is equivalent to that of a unicycle on a sphere whose equations of motion can be represented by first-order differential equations [7].

Many research efforts have been made to clarify different aspects of non-holonomic mechanical systems including its controllability, stability, feedback stabilization, time-periodic control, chained form transformation, etc. but, in any case, achieving a formulation for the kinematics of the system, as compact and simple as possible, is essential to explore the applicability of all these results available in the literature.

Herein, the under-actuated parallel architecture presented in [6] is reconsidered from a different point of view which allows to see it a particular case of under-actuated manipulator obtained through the substitution of a spherical pair (S pair) by a non-holonomic pair (nS pair) in fully parallel manipulators (FPMs). Moreover, its kinetostatic analysis is reformulated so that a simple and compact formulation necessary for its design and control is obtained.

[1]The degrees-of-freedom (dof) of the configuration space, also called configuration (or finite) dof [1], are the minimum number of geometric parameters necessary to uniquely identify the configuration of the mechanical system [2]. They may be different from the instantaneous dof, also called velocity dof [1], of the same mechanical system.

This paper is structured as follows. Next section describes how to generate under-actuated parallel manipulators from FPMs. Section 3 is devoted to the studied under-actuated parallel architecture: a compact formulation for its instantaneous kinematics and statics is obtained, and some clues for the characterization of its singularities are provided. Eventually, section 4 offers the conclusions.

2 GENERATION OF UNDER-ACTUATED MANIPULATORS

Two rigid bodies (RBs) connected by a spherical pair (S pair) can assume any relative orientation, and can move from one relative orientation to another by covering any spherical-motion path that joins the two relative orientations. Actually, the possibility of freely orientating two RBs with respect to one another is not related to the possibility of performing relative rotations around axes which pass through the center of spherical motion and have any direction. In fact, a suitable sequence (at least three) of finite rotations around coplanar axes that pass through the spherical-motion center can freely orientate one RB with respect to another. Thus, if the only free relative orientation of two RBs is required, the use of an S pair will be redundant. The use of a kinematic pair that allows only rotations around coplanar axes that pass through a fixed point would be sufficient.

Due to frictional forces, the rolling contact between a sphere and a roller forbids the sphere rotations around the axis through the sphere center, and perpendicular to the plane defined by the roller axis and the sphere center. By combining such a non-holonomic constraint with other constraints that forbid the relative translation between the sphere center and the roller axis, a non-holonomic joint will result. This joint constrains two RBs: one fixed to the sphere and the other fixed to the plane, defined by the roller axis and the sphere center. So that the resulting constrained motion permits only relative rotations around axes lying on the above-mentioned plane and passing through the sphere center. Hereafter, this type of joint will be called non-holonomic spherical pair (nS pair).

The constraint forces, which two RBs, joined by a nS pair, exert on one another through the joint, can be reduced to a resultant force applied on the sphere center and a torque perpendicular to the plane defined by the roller axis and the sphere center. The torque is the static effect of the non-holonomic constraint, whereas the resultant force on the sphere center is the same static effect that an S pair would have generated.

From a manufacturing point of view, it is worth noting that, in a nS pair, the presence of any number of roller-sphere contacts does not alter the kinetostatics of the nS pair, provided that all the roller axes lie on a same plane passing through the sphere center[2].

[2]In general, two rollers whose axes locate with the sphere center two different planes constrains the sphere to rotate around the intersection line between the

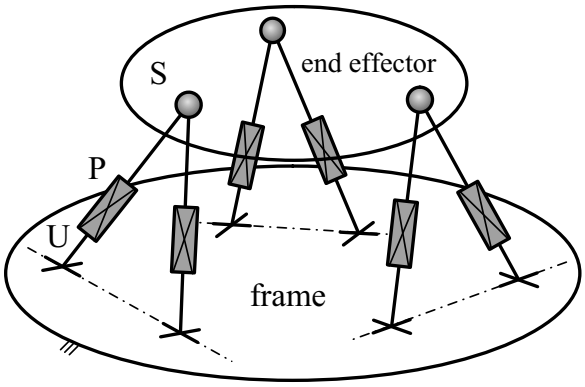

Figure 1. 6-3 FPM

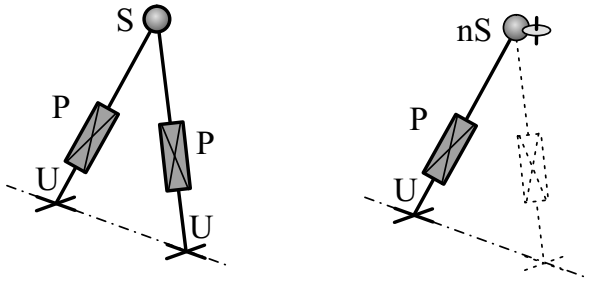

Figure 2. UPnS LIMB (**right**) GENERATED BY SUBSTITUTING AN nS PAIR FOR THE S PAIR IN THE TWO UPS LIMBS WITH COALESCED S PAIRS (**left**)

Moreover, the maximum torque transmitted through the nS pair, due to its frictional origin, can be fixed by suitably choosing the number of roller-sphere contacts together with the normal force transmitted through each contact.

The above discussion brings to the proposition: *i*) the substitution of a number of nS pairs for as many S pairs in a kinematic chain does not change the configuration space of that chain (i.e., neither the degrees-of-freedom (dof)[3] of the configuration space nor the reachable configurations change), it only reduces the practicable paths for moving that chain from one configuration to another.

Moreover, due to the torque that arises in a nS pair and to proposition *i*), the following proposition holds, too: *ii*) in a manipulator, the substitution of a number of nS pairs for as many S pairs does not change its workspace and allows the elimination of a number of actuators equal to the number of introduced nS pairs (i.e., generates an under-actuated manipulator).

Fully-parallel manipulators (FPMs) feature two platforms, one mobile (end effector) and the other fixed (frame), connected to each other by means of six universal(U)-prismatic(P)-spherical(S) kinematic chains (UPS limbs) where the prismatic pairs are the only actuated pairs [8]. In each limb, the centers of the universal joint and of the spherical pair (limb's attachment points) are points, fixed either to the end effector or to the frame, whose distance (limb length) is controlled by the actuated prismatic pair. Two or more attachment points, either in the end effector or in the frame, can coalesce into a unique point. According to the number of attachment points (no matter if they are multiple or not) in the end effector, say p, and in the frame, say q, different FPM architectures, named p-q FPM, are distinguished [8].

Due to the high number of S pairs appearing in FPMs, the substitutions of nS pairs for S pairs, accompanied by as many eliminations of actuators in the prismatic pairs, can be operated in many ways in all the FPM architectures. By exploiting all the possible substitutions, a lot of new under-actuated parallel manipulators can be generated. It is worth noting that a passive UPS limb only affects the workspace borders since it has connectivity six, and, if this effect is not necessary, the elimination of the actuator in a prismatic pair could be accompanied by the elimination of the whole resulting passive UPS limb.

3 CASE STUDY

In this section, an under-actuated parallel manipulator generated from the 6-3 FPM (Fig. 1) is studied.

The 6-3 FPM architecture features three couples of UPS limbs with coalesced S pairs in the end effector. This architecture was proposed first by Stewart [9], in the 1965, for a flight simulator. Successively, with the renewed interest for the parallel architectures, started at the end of the eighties, it was diffusely studied. In particular, regarding the direct position analysis of the 6-3 FPM, Innocenti and Parenti-Castelli [10] demonstrated that at most sixteen end-effector poses correspond to a given set of limb lengths. Then, Parenti-Castelli and Di Gregorio [11] demonstrated that the end-effector pose is uniquely determined when the value of one passive joint variable is measured besides the six limb lengths.

Starting from the 6-3 architecture, each couple of UPS limbs with coalesced S pairs (Fig. 2(left)) can be transformed into an UPnS limb, as shown in Fig 2(right), without affecting the workspace of the manipulator (see proposition *i*)). By operating this substitution in all the three couples of UPS limbs together with the inversion of the end effector with the frame, the under-actuated manipulator with topology 3-nSPU, shown in Fig. 3, is obtained. This under-actuated manipulator is able to move the

two planes, whereas three rollers whose axes locate with the sphere center three different planes lock the sphere

[3]The presence of non-holonomic constraints does not change the configuration dof [1, 2]. It only affects the instantaneous dof of the mechanism. Hereafter, the acronym dof used alone will mean configuration dof.

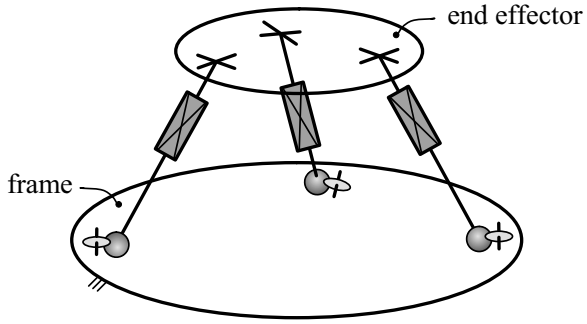

Figure 3. UNDER-ACTUATED MANIPULATOR WITH TOPOLOGY 3-nSPU

end effector in a six-dof workspace by changing only the three limb lengths.

Regarding the direct position analysis (DPA) of the 3-nSPU, since its configuration space has six dof, a number of closure equations equals to the number of unknowns can be written if, and only if, over the three limb lengths, three more passive joint variables are assigned (measured). By assigning (measuring) the three joint variables of the three revolute pairs[4] not adjacent to the end effector, the closure equation system coincides with the one of the 6-3 FPM for assigned limb lengths [11], and admits at most sixteen solution for the end-effector pose. Moreover, if the joint variable of a revolute pair adjacent to the end effector is measured (or coherently assigned) too, only one end-effector pose satisfies the closure equations [11].

3.1 Instantaneous Kinematics

Fig. 4 shows the i^{th} limb, $i = 1,2,3$, together with the notation that will be used. $\mathbf{w}_{1i}$ and $\mathbf{w}_{2i}$ are two any mutually orthogonal unit vectors fixed to the frame and lying on the plane defined by the roller axis and the center, A_i, of the sphere, in the roller-sphere contact. $\mathbf{w}_{3i}$ and $\mathbf{w}_{4i}$ are the two mutually orthogonal unit vectors of the axes of the two revolute pairs constituting the U joint. B_i is the center of the U joint. $\mathbf{a}_i$ and $\mathbf{b}_i$ are the two position vectors which locate the points A_i and B_i, respectively, in a generic Cartesian reference fixed to the frame, whereas $\mathbf{p}$ is the position vector of an end-effector point, P, in the same Cartesian reference. θ_{ji}, for $j = 1,\ldots,4$, is a joint variable denoting a rotation angle around the joint-axis defined by $\mathbf{w}_{ji}$, for $j = 1,\ldots,4$, and positive if counterclockwise with respect to $\mathbf{w}_{ji}$. The length of the i^{th} limb is equal to $\|\mathbf{b}_i - \mathbf{a}_i\|$, and it will be denoted l_i. Moreover, the limb-axis' unit vector, $\mathbf{g}_i$, and the unit vector, $\mathbf{h}_i$ ($\mathbf{r}_i$) normal to the plane located by the U-joint's revolute-pair axes (by the roller axis together with the sphere center in the nS joint) satisfy the following relationships:

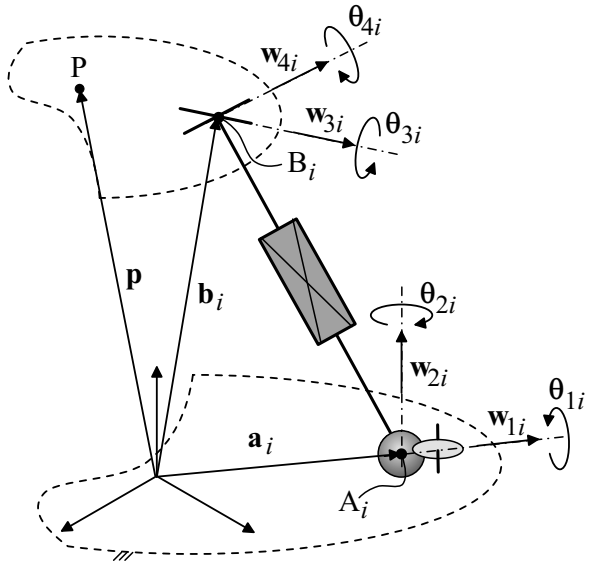

Figure 4. i^{th} LIMB OF TYPE nSPU: NOTATIONS

$$l_i \mathbf{g}_i = \mathbf{b}_i - \mathbf{a}_i, \quad \mathbf{h}_i = \mathbf{w}_{3i} \times \mathbf{w}_{4i}, \quad \text{and} \quad \mathbf{r}_i = \mathbf{w}_{1i} \times \mathbf{w}_{2i}. \tag{1}$$

The time differentiation of the first of the relationships (1) yields

$$\dot{l}_i\, \mathbf{g}_i + l_i\, \dot{\mathbf{g}}_i = \dot{\mathbf{b}}_i. \tag{2}$$

Since $\dot{\mathbf{g}}_i = (\dot{\theta}_{1i}\ \mathbf{w}_{1i} + \dot{\theta}_{2i}\ \mathbf{w}_{2i}) \times \mathbf{g}_i$, and $\dot{\mathbf{b}}_i = \dot{\mathbf{p}} + \boldsymbol{\omega} \times (\mathbf{b}_i - \mathbf{p})$, where $\boldsymbol{\omega}$ denotes the end-effector angular velocity, equation (2) can be rewritten as

$$\dot{l}_i\, \mathbf{g}_i + l_i\, \left[\dot{\theta}_{1i}\, (\mathbf{w}_{1i} \times \mathbf{g}_i) + \dot{\theta}_{2i}\, (\mathbf{w}_{2i} \times \mathbf{g}_i)\right] = \dot{\mathbf{p}} + \boldsymbol{\omega} \times (\mathbf{b}_i - \mathbf{p}). \tag{3}$$

The dot products of (3) by $\mathbf{w}_{1i}$ and $\mathbf{w}_{2i}$ yield the following two scalar equations:

$$\dot{l}_i\, (\mathbf{g}_i \cdot \mathbf{w}_{1i}) + l_i\, \dot{\theta}_{2i}\, (\mathbf{w}_{2i} \times \mathbf{g}_i \cdot \mathbf{w}_{1i}) = \dot{\mathbf{p}} \cdot \mathbf{w}_{1i} + \boldsymbol{\omega} \times (\mathbf{b}_i - \mathbf{p}) \cdot \mathbf{w}_{1i}, \tag{4}$$

$$\dot{l}_i\, (\mathbf{g}_i \cdot \mathbf{w}_{2i}) + l_i\, \dot{\theta}_{1i}\, (\mathbf{w}_{1i} \times \mathbf{g}_i \cdot \mathbf{w}_{2i}) = \dot{\mathbf{p}} \cdot \mathbf{w}_{2i} + \boldsymbol{\omega} \times (\mathbf{b}_i - \mathbf{p}) \cdot \mathbf{w}_{2i}. \tag{5}$$

On the other hand (see Fig. 4), the end-effector angular velocity is equal to $\sum_{j=1,4} \dot{\theta}_{ji}\ \mathbf{w}_{ji}$, whose dot product by $\mathbf{h}_i$ gives the following expression

$$\boldsymbol{\omega} \cdot \mathbf{h}_i = \dot{\theta}_{1i}\, (\mathbf{w}_{1i} \cdot \mathbf{h}_i) + \dot{\theta}_{2i}\, (\mathbf{w}_{2i} \cdot \mathbf{h}_i).$$

[4] Each U joint is constituted by two revolute pairs: one adjacent to, and the other not adjacent to the end effector.

Solving (4) and (5) for $\dot{\theta}_{2i}$ and $\dot{\theta}_{1i}$, respectively, and replacing the result in the above equation, yields:

$$\boldsymbol{\omega}\cdot\mathbf{h}_i = \left[\frac{\dot{\mathbf{p}}\cdot\mathbf{w}_{2i}+\boldsymbol{\omega}\times(\mathbf{b}_i-\mathbf{p})\cdot\mathbf{w}_{2i}-\dot{l}_i\,\mathbf{g}_i\cdot\mathbf{w}_{2i}}{l_i\,(\mathbf{w}_{1i}\times\mathbf{g}_i\cdot\mathbf{w}_{2i})}\right](\mathbf{w}_{1i}\cdot\mathbf{h}_i) + \left[\frac{\dot{\mathbf{p}}\cdot\mathbf{w}_{1i}+\boldsymbol{\omega}\times(\mathbf{b}_i-\mathbf{p})\cdot\mathbf{w}_{1i}-\dot{l}_i\,\mathbf{g}_i\cdot\mathbf{w}_{1i}}{l_i\,(\mathbf{w}_{2i}\times\mathbf{g}_i\cdot\mathbf{w}_{1i})}\right](\mathbf{w}_{2i}\cdot\mathbf{h}_i). \tag{6}$$

Taking into account the vector identities

$$\mathbf{h}_i\times\mathbf{r}_i = \mathbf{h}_i\times(\mathbf{w}_{1i}\times\mathbf{w}_{2i}) = (\mathbf{w}_{2i}\cdot\mathbf{h}_i)\mathbf{w}_{1i}-(\mathbf{w}_{1i}\cdot\mathbf{h}_i)\mathbf{w}_{2i},$$

$$\mathbf{r}_i\cdot\mathbf{g}_i = \mathbf{w}_{1i}\times\mathbf{w}_{2i}\cdot\mathbf{g}_i = -\mathbf{w}_{1i}\times\mathbf{g}_i\cdot\mathbf{w}_{2i} = \mathbf{w}_{2i}\times\mathbf{g}_i\cdot\mathbf{w}_{1i},$$

relationship (6) can be rewritten as:

$$\dot{l}_i\,\mathbf{g}_i\cdot(\mathbf{h}_i\times\mathbf{r}_i) = \dot{\mathbf{p}}\cdot(\mathbf{h}_i\times\mathbf{r}_i)+\boldsymbol{\omega}\cdot[(\mathbf{b}_i-\mathbf{p})\times(\mathbf{h}_i\times\mathbf{r}_i)\ldots \ldots.-l_i\,(\mathbf{r}_i\cdot\mathbf{g}_i)\mathbf{h}_i]. \tag{7}$$

Since $\dot{l}_i$ can also be obtained as the projection of $\dot{\mathbf{b}}_i$ on $\mathbf{g}_i$ [see Eq. (2)], the following expression holds

$$\dot{l}_i = \dot{\mathbf{b}}_i\cdot\mathbf{g}_i = \dot{\mathbf{p}}\cdot\mathbf{g}_i+\boldsymbol{\omega}\cdot[(\mathbf{b}_i-\mathbf{p})\times\mathbf{g}_i]. \tag{8}$$

Replacing expression (8) for $\dot{l}_i$ in (7), gives

$$\dot{\mathbf{p}}\cdot\mathbf{s}_i+\boldsymbol{\omega}\cdot[(\mathbf{b}_i-\mathbf{p})\times\mathbf{s}_i-l_i(\mathbf{r}_i\cdot\mathbf{g}_i)\mathbf{h}_i]=0 \tag{9}$$

where

$$\mathbf{s}_i = \mathbf{h}_i\times\mathbf{r}_i-[\mathbf{g}_i\cdot(\mathbf{h}_i\times\mathbf{r}_i)]\mathbf{g}_i \tag{10}$$

is the component of $\mathbf{h}_i\times\mathbf{r}_i$ perpendicular to $\mathbf{g}_i$.

Eventually, rewriting equations (8) and (9), for $i=1,2,3$, in matrix form yields

$$\begin{pmatrix}\mathbf{1}_{3\times3}\\ \mathbf{0}_{3\times3}\end{pmatrix}\dot{\mathbf{l}} = \begin{pmatrix}\mathbf{G}_{3\times3} & \mathbf{K}_{3\times3}\\ \mathbf{S}_{3\times3} & \mathbf{J}_{3\times3}\end{pmatrix}\begin{pmatrix}\dot{\mathbf{p}}\\ \boldsymbol{\omega}\end{pmatrix} \tag{11}$$

where $\mathbf{1}_{3\times3}$ and $\mathbf{0}_{3\times3}$ are the 3×3 identity and zero matrix, respectively, $\dot{\mathbf{l}}=(\dot{l}_1,\dot{l}_2,\dot{l}_3)$ is the vector collecting the joint rates of the actuated joints, and

$$\mathbf{K}^T[i,:] = (\mathbf{b}_i-\mathbf{p})\times\mathbf{g}_i \tag{12}$$
$$\mathbf{G}^T[i,:] = \mathbf{g}_i \tag{13}$$
$$\mathbf{J}^T[i,:] = (\mathbf{b}_i-\mathbf{p})\times\mathbf{s}_i-l_i(\mathbf{r}_i\cdot\mathbf{g}_i)\mathbf{h}_i \tag{14}$$
$$\mathbf{S}^T[i,:] = \mathbf{s}_i \tag{15}$$

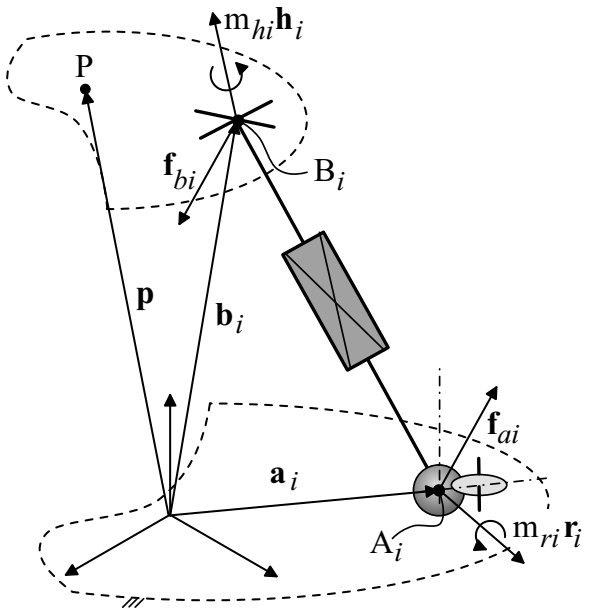

Figure 5. i^{th} LIMB OF TYPE nSPU: FREE-BODY DIAGRAM

with the notation $\mathbf{A}^T[i,:]$ to mean the i-th column of matrix $\mathbf{A}^T_{3\times3}$.

Matrix relationship (11) is the sought-after input-output instantaneous relationship necessary to implement the control algorithms of the 3-nSPU.

3.2 Static Analysis

The only input-output static relationship can be immediately deduced from (11) through the principle of virtual work. Nevertheless, in order to highlight how the loads act upon the limbs and are transmitted through the joints, the complete static analysis of the 3-nSPU will be developed here independently of (11).

Figure 5 shows the free-body diagram of the i^{th} limb. With reference to Fig. 5, the force $\mathbf{f}_{bi}$ ($\mathbf{f}_{ai}$), applied on B_i (A_i), together with the torque $m_{hi}\,\mathbf{h}_i$ ($m_{ri}\,\mathbf{r}_i$) are the resultants of constraint forces exerted by the end effector (frame) on the i^{th} limb through the U joint (the nS joint). Moreover, the force $-\mathbf{f}_{ext}$, applied on the end-effector point P, together with the torque $-\mathbf{m}_{ext}$ will denote the resultants of the interaction forces exerted on the end effector. The force $-\tau_i\,\mathbf{g}_i$ will denote the axial force exerted on the upper part of the i^{th} limb by the actuator in the prismatic pair. It is worth noting that the force equilibrium, along the limb axis, of the upper part of the i^{th} limb yields the following relationship $\tau_i=\mathbf{f}_{bi}\cdot\mathbf{g}_i$.

With these notations, the equilibrium of the forces applied on the i^{th} limb yields $\mathbf{f}_{bi}+\mathbf{f}_{ai}=0$; whereas, taking A_i as reference point, the equilibrium of the moments applied on the same

limb is:

$$m_{hi}\,\mathbf{h}_i + m_{ri}\,\mathbf{r}_i + l_i\,\mathbf{g}_i \times \mathbf{f}_{bi} = \mathbf{0}. \tag{16}$$

The dot product of Eq. (16) by $\mathbf{g}_i$, yields the relationship

$$m_{ri} = -m_{hi}\frac{\mathbf{h}_i \cdot \mathbf{g}_i}{\mathbf{r}_i \cdot \mathbf{g}_i}. \tag{17}$$

whose substitution for m_{ri} in (16) leads to

$$\begin{aligned}\frac{m_{hi}}{\mathbf{r}_i \cdot \mathbf{g}_i}\left[(\mathbf{r}_i \cdot \mathbf{g}_i)\mathbf{h}_i - (\mathbf{h}_i \cdot \mathbf{g}_i)\mathbf{r}_i\right] + l_i\,\mathbf{g}_i \times \mathbf{f}_{bi} = \ \ldots \\ \ldots.\ \frac{m_{hi}}{\mathbf{r}_i \cdot \mathbf{g}_i}\left[\mathbf{g}_i \times (\mathbf{h}_i \times \mathbf{r}_i)\right] + l_i\,\mathbf{g}_i \times \mathbf{f}_{bi} = \mathbf{0}.\end{aligned} \tag{18}$$

where the vector identity $\mathbf{g}_i \times (\mathbf{h}_i \times \mathbf{r}_i) = (\mathbf{r}_i \cdot \mathbf{g}_i)\mathbf{h}_i - (\mathbf{h}_i \cdot \mathbf{g}_i)\mathbf{r}_i$ has been used.

The dot product of Eq. (18) by $\mathbf{h}_i \times \mathbf{r}_i$ yields the relationship: $(\mathbf{g}_i \times \mathbf{f}_{bi}) \cdot (\mathbf{h}_i \times \mathbf{r}_i) = 0$. Such a relationship is satisfied if, and only if, $\mathbf{f}_{bi}$ is a linear combination of $\mathbf{g}_i$ and $\mathbf{h}_i \times \mathbf{r}_i$. Subtracting from $\mathbf{h}_i \times \mathbf{r}_i$ its component along $\mathbf{g}_i$, $\mathbf{s}_i$ is obtained. Since $\mathbf{g}_i$ and $\mathbf{s}_i$ are two orthogonal vectors that span the same subspace as $\mathbf{g}_i$ and $\mathbf{h}_i \times \mathbf{r}_i$, $\mathbf{f}_{bi}$ can be expressed as follows:

$$\mathbf{f}_{bi} = \tau_i\,\mathbf{g}_i + \tau_i^{\perp}\,\mathbf{s}_i. \tag{19}$$

Replacing expression (19) for $\mathbf{f}_{bi}$ in (18), and taking into account that $\mathbf{g}_i \times (\mathbf{h}_i \times \mathbf{r}_i) = \mathbf{g}_i \times \mathbf{s}_i$ yields

$$\left(\frac{m_{hi}}{\mathbf{r}_i \cdot \mathbf{g}_i} + l_i\,\tau_i^{\perp}\right)(\mathbf{g}_i \times \mathbf{s}_i) = \mathbf{0}$$

which is satisfied if

$$m_{hi} = -l_i\,\tau_i^{\perp}\,(\mathbf{r}_i \cdot \mathbf{g}_i). \tag{20}$$

Using equation (17), Eq. (20) can be rewritten as

$$m_{ri} = l_i\,\tau_i^{\perp}(\mathbf{h}_i \cdot \mathbf{g}_i). \tag{21}$$

Regarding the end-effector equilibrium, the equilibrium of the forces is:

$$\mathbf{f}_{ext} = -\sum_{i=1}^{3}\mathbf{f}_{bi} = -\sum_{i=1}^{3}\tau_i\,\mathbf{g}_i - \sum_{i=1}^{3}\tau_i^{\perp}\,\mathbf{s}_i, \tag{22}$$

and, taking the end-effector point P as reference point, the equilibrium of the moments is:

$$\mathbf{m}_{ext} = -\sum_{i=1}^{3} m_{hi}\,\mathbf{h}_i - \sum_{i=1}^{3}(\mathbf{b}_i - \mathbf{p}) \times \mathbf{f}_{bi}. \tag{23}$$

The substitution of $\mathbf{f}_{bi}$, according to (19), and of m_{hi}, according to (20), into (23) yields

$$\mathbf{m}_{ext} = -\sum_{i=1}^{3}\tau_i(\mathbf{b}_i - \mathbf{p}) \times \mathbf{g}_i - \sum_{i=1}^{3}\tau_i^{\perp}\left[(\mathbf{b}_i - \mathbf{p}) \times \mathbf{s}_i - l_i\,(\mathbf{r}_i \cdot \mathbf{g}_i)\mathbf{h}_i\right] \tag{24}$$

Finally, equations (22) and (24) can be rewritten in matrix form as follows:

$$\begin{pmatrix}\mathbf{f}_{ext}\\ \mathbf{m}_{ext}\end{pmatrix} = -\begin{pmatrix}\mathbf{G}_{3\times3} & \mathbf{K}_{3\times3}\\ \mathbf{S}_{3\times3} & \mathbf{J}_{3\times3}\end{pmatrix}^{T}\begin{pmatrix}\boldsymbol{\tau}\\ \boldsymbol{\tau}^{\perp}\end{pmatrix} \tag{25}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is a vector collecting the signed magnitudes of the forces applied by the actuators in the prismatic pairs, and $\boldsymbol{\tau}^{\perp} = (\tau_1^{\perp}, \tau_2^{\perp}, \tau_3^{\perp})$.

Matrix relationship (25) is the input-output static relationship of the 3-nSPU. It is worth noting that, as expected, (11) and (25) satisfy the instantaneous power balance: $\mathbf{f}_{ext} \cdot \dot{\mathbf{p}} + \mathbf{m}_{ext} \cdot \boldsymbol{\omega} = -\boldsymbol{\tau} \cdot \dot{\mathbf{l}}$.

3.3 Singularity Analysis

Singularities are manipulator configurations where the relationship (input-output instantaneous relationship) between the rates of the actuated-joint variables and the end-effector twist, $(\dot{\mathbf{p}}, \boldsymbol{\omega})$, fails [12–14]. Three types of singularities can be distinguished [12]: (I) singularities of the inverse kinematic problem, (II) singularities of the direct kinematic problem, and (III) singularities both of the inverse and of the direct kinematic problems. Type-I singularities occur when the actuated joint rates cannot be uniquely computed for an assigned end-effector twist. Vice versa, type-II singularities occur when the end-effector twist cannot be uniquely determined for assigned actuated joint rates.

For the 3-nSPU, the input-output instantaneous relationship is (11) where the actuated joint rates are collected in the vector $\dot{\mathbf{l}}$. This relationship highlights that the 3-nSPU has only three instantaneous dof. Therefore, its singularity analysis can be addressed by using the scheme proposed in [15].

Regarding type-I singularities, provided that the assigned twist, $(\dot{\mathbf{p}}, \boldsymbol{\omega})$, satisfies the last three equation of system (11), it can be always solved[5].

[5] System (11) does not model the mobility limitations due to the physical constitution of the real joints, and to the real sizes of the links. Such limitations bound the workspace and, when correctly modeled, yield type-I singularities.

Regarding type-II singularities, the equation of the singularity locus is

$$\det\begin{pmatrix}\mathbf{G}_{3\times3} & \mathbf{K}_{3\times3}\\ \mathbf{S}_{3\times3} & \mathbf{J}_{3\times3}\end{pmatrix}=0. \tag{26}$$

The geometric interpretation of the above algebraic condition is not straightforward.

Nevertheless, the last three equations of system (11) allows the elimination of $\dot{\mathbf{p}}$ provided that $\det(\mathbf{S}_{3\times3})=\mathbf{s}_1\cdot\mathbf{s}_2\times\mathbf{s}_3$ is different from zero. In this case, system (11) becomes

$$\dot{\mathbf{l}}=\mathbf{Q}\boldsymbol{\omega} \tag{27}$$

where $\mathbf{Q}$ is the 3×3 matrix $(\mathbf{K}_{3\times3}-\mathbf{G}_{3\times3}\mathbf{S}_{3\times3}^{-1}\mathbf{J}_{3\times3})$. Thus, the analytic expression of the singularity locus becomes

$$\det(\mathbf{Q})=\mathbf{q}_1\cdot\mathbf{q}_2\times\mathbf{q}_3=0 \tag{28}$$

where the vectors $\mathbf{q}_i$, for $i=1,2,3$, are the column vectors of matrix $\mathbf{Q}$. In conclusion, if the mixed product $\mathbf{s}_1\cdot\mathbf{s}_2\times\mathbf{s}_3$ is different from zero (i.e., the three vectors $\mathbf{s}_i$, for $i=1,2,3$, are neither coplanar nor null vectors), the type-II singularities are geometrically identified by either the coplanarity of the three vectors $\mathbf{q}_i$, for $i=1,2,3$, or by the fact that at least one of the $\mathbf{q}_i$ vectors is a null vector.

If the mixed product $\mathbf{s}_1\cdot\mathbf{s}_2\times\mathbf{s}_3$ is zero, the determinant of the whole 6×6 matrix appearing in (26) must be considered, and geometric interpretations of (26) are much more difficult to provide.

The zeroing of $\mathbf{s}_1\cdot\mathbf{s}_2\times\mathbf{s}_3$ can be geometrically identified since it occurs when either (a) at least one of the $\mathbf{s}_i$ vectors is a null vector, or (b) the three $\mathbf{s}_i$ vectors are coplanar. Vector $\mathbf{s}_i$ (see definition (10)) is related to the configuration of the i^{th} limb, and it is the component of $\mathbf{h}_i\times\mathbf{r}_i$ perpendicular to $\mathbf{g}_i$ (i.e., to the limb axis).

As a consequence, condition (a) occurs when, in at least one limb, either $(a.1)$ the two unit vectors $\mathbf{h}_i$ and $\mathbf{r}_i$ are parallel (i.e., when, in a limb, the revolute pair axes in the U joint are both parallel to the plane defined by the roller axis and the sphere center in the nS pair), or $(a.2)$ the limb axis is the intersection line between the plane, defined by the roller axis and the sphere center in the nS pair, and the plane, defined by the revolute pair axes of the U joint. Condition $(a.2)$ is forbidden in practice by the actual sizes of joints and links. Regarding condition $(a.1)$, a very special case occurs when $\mathbf{h}_i$ and $\mathbf{r}_i$ are parallel in all the limbs. This occurrence makes the matrix $\mathbf{S}_{3\times3}$ a null matrix and allows the determinant at the left-hand side of (26) to be factorized as $\det(\mathbf{G}_{3\times3})\det(\mathbf{J}_{3\times3})$ where $\det(\mathbf{G}_{3\times3})$ is equal to $\mathbf{g}_1\cdot\mathbf{g}_2\times\mathbf{g}_3$, whereas, in this case, $\det(\mathbf{J}_{3\times3})$ is equal to $-l_1l_2l_3(\mathbf{r}_1\cdot\mathbf{g}_1)(\mathbf{r}_2\cdot\mathbf{g}_2)(\mathbf{r}_3\cdot\mathbf{g}_3)\mathbf{h}_1\cdot\mathbf{h}_2\times\mathbf{h}_3$. Thus, in this case, a type-II singularity occurs when either the limb axes are all parallel to a unique plane, or the $\mathbf{h}_i$ vectors are coplanar, or, finally, in at least one limb, the limb axis lies on the plane defined by the roller axis and the sphere center of the nS pair. Moreover, it is worth noting that, in this case, the end effector performs an instantaneous translation, if neither $\det(\mathbf{G}_{3\times3})$ nor $\det(\mathbf{J}_{3\times3})$ are equal to zero (i.e., out of singularity).

Regarding condition (b) (i.e., the coplanarity of the $\mathbf{s}_i$ vectors), it occurs when the limb axes are all parallel, and in other configurations more difficult to visualize.

4 CONCLUSIONS

If a number of non-holonomic spherical pairs replaces as many spherical pairs in a manipulator, the same number of actuators can be eliminated. The resulting manipulator will keep the same workspace of the generating manipulator, but it will be under-actuated.

This technique for generating under-actuated manipulators can be applied to fully-parallel manipulators, where many spherical pairs are present, and the elimination of an actuator in an UPS limb can be accompanied with the elimination of the whole UPS limb.

Through this pair substitution, an under-actuated manipulator, previously proposed by one of the authors, has been generated from an inversion of the 6-3 FPM. The kinetostatic analysis of this manipulator has been reconsidered to obtain a simple and compact formulation. This reformulated analysis can be used both in the design of the under-actuated manipulator, and in its control. Further works on this manipulators will present an exhaustive singularity analysis, and will provide design criteria for increasing its useful workspace.

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