Technical Report

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Multi-body Singularity Equations IRI Technical Report

On how to obtain the equations for computation with CUIK

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Abstract

This technical report explains how to obtain a system of equations that encodes the singularities of a multi-body system with respect to some of its configuration variables. The system is obtained in a way that makes it appropiate for the CUIK software.

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1 Introduction

The allowable positions and orientations of all links in a multi-body system are usually encoded in an n_q-dimensional vector of generalized coordinates $\mathbf{q}$, subject to a system of $n_e \leq n_q$ equations of the form

$$\Phi(\mathbf{q}) = 0, \tag{1}$$

that express the kinematic constraints imposed by the joints. The solution of this system of equations is, in general, a manifold $\mathcal{C}$ called *configuration space*. In this document we suppose that equations (1) are already available but that they do not take into account articular limits. Firstly, we will complete this system of equations in order to introduce joint limits and then we will extend it in order to obtain a new set of equations whose solution contains the singularities of the multi-body system with respect to some of the variables in $\mathbf{q}$. All these new equations must maintain an appopiate formulation to use with the software CUIK.

The outline of this report is as follows. Section 2 describes how joint limit constraints in the form of inequalities can be converted into equations in order to complete the original system of equations. Section 3 explains how to obtain singular points by extending the previous system with the condition of Jacobian rank deficiency and, finally, some examples are given in Section 4.

2 Joint Limits equations

The n_q-dimensional vector of generalized coordinates $\mathbf{q}$ contains variables that describe the position of the links and, in particular, some of them correspond to the configuration of the joints. There are two types of variables in $\mathbf{q}$ that we may use when describing joint limits: variables refering to distance and variables refering to angular position. Typically they correspond to slider and revolute joints, respectively; but using only these two types of variables we can define articular limits for any lower-pair joint.

Articular limits for a generalized variable q_i are normally expressed by an inequality of the form

$$q_{min} \leq q_i \leq q_{max}. \tag{2}$$

We next show how to convert these inequalities into equations of the appopiate form using the *slack variable* technique of Optimization [1] to extend Eq. (1) in order to take into account joint limits.

2.1 Distance constraints

If l is the distance variable, we can transform an inequality of the form

$$l_{min} \leq l \leq l_{max} \tag{3}$$

into

$$l = m + h \sin t, \tag{4}$$

where $m = \frac{l_{max}+l_{min}}{2}$, $h = \frac{l_{max}-l_{min}}{2}$ and t is an auxiliary variable that can take any value but that we will restrain between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ to avoid obtaining multiple solutions of t for a given l. Figure 1 illustrates the situation.

As we cannot introduce trigonometric terms into CUIK, we perform another transformation to obtain

$$\begin{aligned} l &= m + hs, \\ c^2 + s^2 &= 1, \end{aligned} \tag{5}$$

with $s \in [-1, 1]$ and $c \in [0, 1]$, which is equivalent to $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

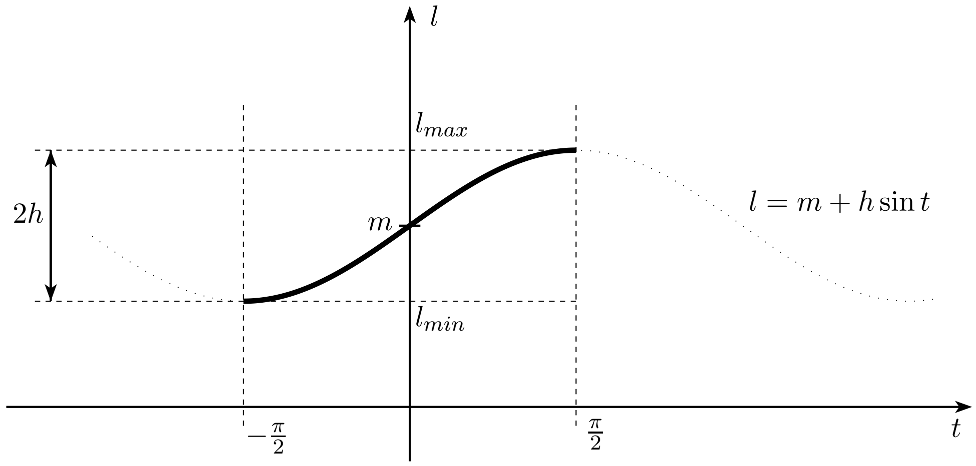

Figure 1: Distance constraint. The auxiliar variable t is constrained between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ to avoid multiple solutions.

2.2 Angular constraints

Consider that θ is the angle we need to constrain between $-\alpha$ and $+\alpha$, i.e.

$$-\alpha \leq \theta \leq \alpha. \tag{6}$$

The angle variable θ will appear in the CUIK equations by means of two variables, say c and s, that represent its cosine and sine, respectively. Because of that, we will constrain $c = \cos\theta$ instead of θ itself. So, Eq.(6) becomes

$$c \geq c_\alpha \tag{7}$$

where $c_\alpha = \cos\alpha$. Actually, we should write $c_\alpha \leq c \leq 1$, but c represents the cosine of an angle and it will always be lower than or equal to 1. Inequality (7) can be written as an equality by

$$c = t^2 + c_\alpha, \tag{8}$$

where t is an auxiliary variable that can take any value but that we will restrain to be non-negative to avoid obtaining multiple solutions of t for a given c. Figure 2 illustrates the situation.

Relative vs. Absolute angles

Note that usually the angle we want to constrain is a relative angle between two links of the mechanism. So, if the vector $\mathbf{q}$ of generalized coordinates contains only absolute angles, we will need to add some equation relating the absolute and relative angles. This can be easily done by defining a unit vector $\mathbf{v}_i = [c_i, s_i]^\mathsf{T}$ for each link, directed along the line of the link, where c_i and s_i are the cosine and sine of the absolute angle of the link. Then, the cosine of the relative angle can be obtained from appropiate dot products of the $\mathbf{v}_i$ vectors. If we want to constrain the angle between links $i-1$ and i, we can write

$$c_i^r = \mathbf{v_{i-1}}^\mathsf{T}\mathbf{v_i}, \tag{9}$$

i.e.,

$$c_i^r = c_{i-1}c_i + s_{i-1}s_i, \tag{10}$$

where c_i^r is the cosine of the relative angle to be constrained.

Table 1 summarizes the transformations from articular limits to equations that can be introduced in CUIK.

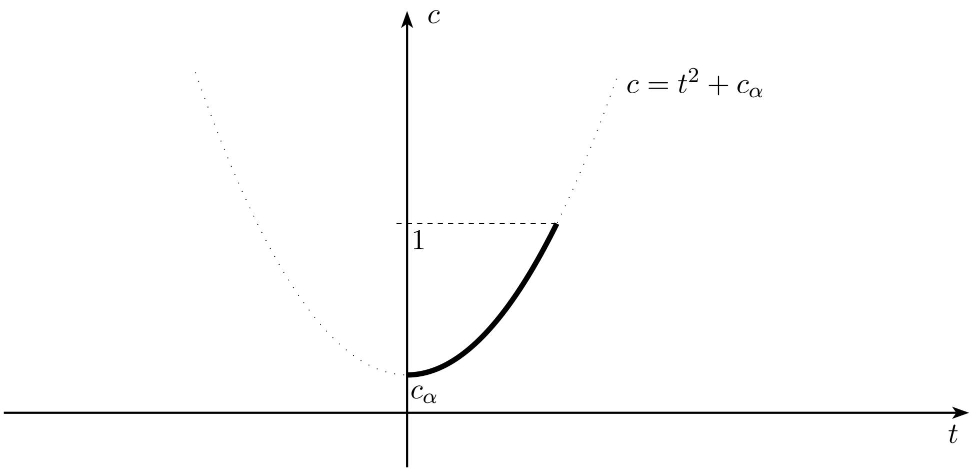

Figure 2: Angle constraint. The auxiliar variable t is constrained to be non-negative to avoid multiple solutions.

Type	Constraint	Inequality	Equivalent CUIK equations
Distance	$l \in [a, b]$	$a \leq l \leq b$	$l = \frac{b+a}{2} + \frac{b-a}{2}s$ $c^2 + s^2 = 1$ $s \in [-1, 1]$ $c \in [0, 1]$
Relative angle	$\theta_i \in [-\alpha, \alpha]$ or $c_i^r \in [c_\alpha, 1]$	$c_i^r \geq c_\alpha$	$c_i^r = t^2 + c_\alpha$ $c_i^r = c_{i-1}c_i + s_{i-1}s_i$ $t \in [0, 1]$

Table 1: Transformation of articular limits to equivalent CUIK equations.

3 Jacobian rank deficiency equations

In this section we want to extend the system of equations (1) (which now takes into account articular limits if necessary) with some equations that represent the condition of Jacobian rank deficiency. The solution of the obtained system of equations will contain the singular points of $\mathcal{C}$ with respect to some of the variables in $\mathbf{q}$.

It is useful to view the vector $\mathbf{q}$ as partitioned into three subvectors

$$\mathbf{q} = [\mathbf{v}^\mathsf{T}, \mathbf{w}^\mathsf{T}, \mathbf{u}^\mathsf{T}]^\mathsf{T},$$

where $\mathbf{v}$ is a vector of n_v input variables corresponding to the actuated degrees of freedom of the multi-body system, $\mathbf{u}$ is a vector of n_u output variables encoding the pose of the end effector, and $\mathbf{w}$ is a n_w-dimensional vector encompassing the remaining intermediate variables. Here we will treat the case of finding end-effector singularities, i.e. singularities with respect to the $\mathbf{u}$ variables, but any other combination of variables should work. For instance, if searching actuator singularities, we should find singularities with respect to the $\mathbf{v}$ variables.

By defining $\mathbf{z} = [\mathbf{v}, \mathbf{w}]$, Eq. (1) can be written as

$$\Phi(\mathbf{z}, \mathbf{u}) = 0. \tag{11}$$

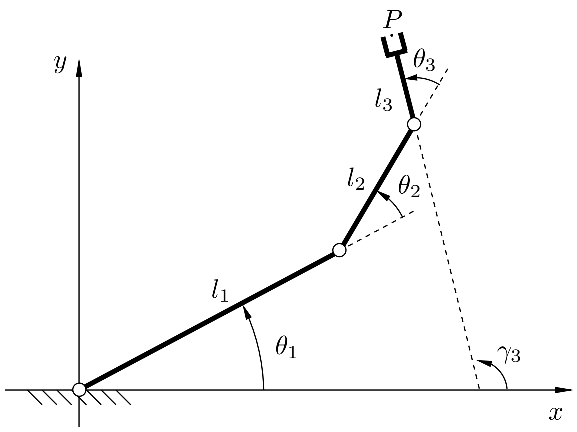

Figure 3: A 3R planar manipulator.

End-effector singularities are the points $\mathbf{q}$ that satisfy the system of equations

$$\left.\begin{array}{c}\Phi(\mathbf{z},\mathbf{u})=0\\ d\Phi_{\mathbf{z}}{}^{\mathsf{T}}\xi=0\\ \xi^{\mathsf{T}}\xi=1\end{array}\right\} \tag{12}$$

for some ξ, where ξ is an n_e-dimensional vector of unknowns. The first equation in (12) constrains the solutions to points on $\mathcal{C}$. The second and third equations impose the rank deficiency of $d\Phi_{\mathbf{z}}$ (the rows of this matrix are dependent whenever they yield a vanishing linear combination with non-null coefficients).

So, we only need to derive the equations (11) with respect to the $\mathbf{z}$ variables, transpose the obtained matrix and multiply it by a vector ξ of variables. This operation will give us n_z equations. If the system of equations (11) contains n_e equations and n_q variables, then we will have $n_e + n_z + 1$ equations and $n_q + n_e$ variables in (12).

This is best understood with the examples given in the next section.

4 Examples

4.1 3R manipulator

In this example we will obtain the *.cuik* file that, after computation, gives the set of end-effector singularities (including workspace boundaries) of the 3R manipulator of Fig. 3 for the case in which all θ_i angles are constrained to the $[-\frac{\pi}{3}, \frac{\pi}{3}]$ range. Here we only take into account singularities with respect to the position of the end-effector, not its orientation. With all these considerations Eq. (11) becomes:

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} x - l_1c_1 - l_2c_2 - l_3c_3 \\ y - l_1s_1 - l_2s_2 - l_3s_3 \\ c_1^2 + s_1^2 - 1 \\ c_2^2 + s_2^2 - 1 \\ c_3^2 + s_3^2 - 1 \\ cr_1 - c_1 \\ cr_2 - c_1c_2 - s_1s_2 \\ cr_3 - c_2c_3 - s_2s_3 \\ cr_1 - t_1^2 - \frac{1}{2} \\ cr_2 - t_2^2 - \frac{1}{2} \\ cr_3 - t_3^2 - \frac{1}{2} \end{bmatrix} = 0, \tag{13}$$

where $\mathbf{u} = [x, y]^\mathsf{T}$ and $\mathbf{z} = [c_1, s_1, c_2, s_2, c_3, s_3, cr_1, a_1, cr_2, a_2, cr_3, a_3]^\mathsf{T}$. Variables x, y respresent the position of point P, variables c_i and s_i represent, respectively, the cosine and sine of the absolute angles of the links with respect to the reference frame (for example, $c_3 = \cos\gamma_3$ and $s_3 = \sin\gamma_3$), variables c_i^r are the cosine of the relatives angles (i.e., $cr_2 = \cos\theta_2$) and variables t_i are the auxiliary variables used to impose articular limits.

The Jacobian of equations (13) with respect to the $\mathbf{z}$ variables is

$$d\Phi_z = \begin{bmatrix} -l_1 & 0 & -l_2 & 0 & -l_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -l_1 & 0 & -l_2 & 0 & -l_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2c_1 & 2s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2c_2 & 2s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3 & 2s_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & -s_2 & -c_1 & -s_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & -s_3 & -c_2 & -s_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2t_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2t_3 \end{bmatrix}. \tag{14}$$

Using equations (13) and (14) we can, now, build the *.cuik* file as follows:

```
[CONSTANTS]

l1:=4
l2:=2
l3:=1

[SYSTEM VARS]

x:[-l1-l2-l3 , l1+l2+l3]
y:[-l1-l2-l3 , l1+l2+l3]

c1:[-1,1]
s1:[-1,1]
c2:[-1,1]
s2:[-1,1]
c3:[-1,1]
s3:[-1,1]
```

```
cr1:[-1,1]
t1:[0,1]
cr2:[-1,1]
t2:[0,1]
cr3:[-1,1]
t3:[0,1]

xi1:[0,1]
xi2:[-1,1]
xi3:[-1,1]
xi4:[-1,1]
xi5:[-1,1]
xi6:[-1,1]
xi7:[-1,1]
xi8:[-1,1]
xi9:[-1,1]
xi10:[-1,1]
xi11:[-1,1]

[SYSTEM EQS]

l1*c1 + l2*c2 + l3*c3 - x = 0;
l1*s1 + l2*s2 + l3*s3 - y = 0;
c1^2 + s1^2 - 1 = 0;
c2^2 + s2^2 - 1 = 0;
c3^2 + s3^2 - 1 = 0;
cr1 - c1 = 0;
cr2 - c1*c2 - s1*s2 = 0;
cr3 - c2*c3 - s2*s3 = 0;
cr1 - t1^2 - 0.5 = 0;
cr2 - t2^2 - 0.5 = 0;
cr3 - t3^2 - 0.5 = 0;

- l1*xi1 + 2*c1*xi3 - xi6 - c2*xi7 = 0;
- l1*xi2 + 2*s1*xi3 - s2*xi7 = 0;
- l2*xi1 + 2*c2*xi4 - c1*xi7 - c3*xi8 = 0;
- l2*xi2 + 2*s2*xi4 - s1*xi7 - s3*xi8 = 0;
- l3*xi1 + 2*c3*xi5 - c2*xi8 = 0;
- l3*xi2 + 2*s3*xi5 - s2*xi8 = 0;
xi6 + xi9 = 0;
- 2*t1*xi9 = 0;
xi7 + xi10 = 0;
- 2*t2*xi10 = 0;
xi8 + xi11 = 0;
- 2*t3*xi11 = 0;

xi1^2 + xi2^2 + xi3^2 + xi4^2 + xi5^2 + xi6^2 + xi7^2 + xi8^2 + xi9^2 + xi10^2 +
    xi11^2 = 1;
```

The execution of this *.cuik* file using a $\sigma = 0.01$ gives as outupt the end-effector singularities shown in Fig. 4.

4.2 Planar Stewart Platform

In this example we will obtain the *.cuik* file that, after computation, gives the set of end-effector singularities of the Planar Stewart Platform of Fig. 5 for the case in which all leg lengths are constrained within prescribed ranges $[l_i^{min}, l_i^{max}]$, where $l_1^{min} = l_2^{min} = \sqrt{2}$, $l_3^{min} = 1$, $l_1^{max} = l_2^{max} = 2$, and $l_3^{max} = 3$.

For this, being $[x, y]^\mathsf{T}$ the coordinates of P, the slider variables l_i can be written as

$$\begin{aligned} l_1^2 &= y^2 - 2ys + s^2 + x^2 + 2x - 2xc - 2c + c^2 + 1,\\ l_2^2 &= y^2 - 2ys + s^2 + x^2 - 2x - 2xc + 2c + c^2 + 1,\\ l_3^2 &= y^2 + 2ys + s^2 + x^2 - 4x + 2xc - 4c + c^2 + 4, \end{aligned} \tag{15}$$

where c and s refer to the sine and cosine of ϕ, respectively, and thus must satisfy

$$c^2 + s^2 = 1. \tag{16}$$

In order to obtain Eq. (11), equations (15) and (16) need to be completed with the equations constraining the articular limits. By defining $m_i = \frac{l_i^{max}+l_i^{min}}{2}$ and $h_i = \frac{l_i^{max}-l_i^{min}}{2}$, these constraints are written as

$$\begin{aligned} l_i &= m_i + h_i s_i,\\ c_i^2 + s_i^2 &= 1, \end{aligned} \tag{17}$$

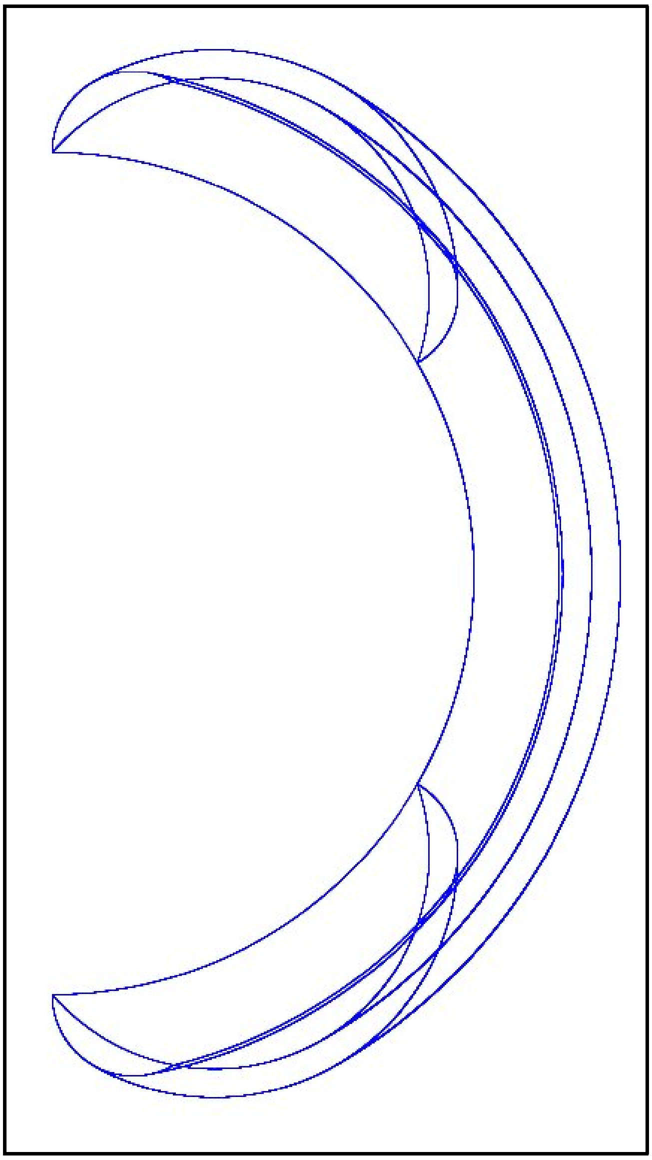

Figure 4: End-effector singularities containing workspace boundaries of the 3R manipulator.

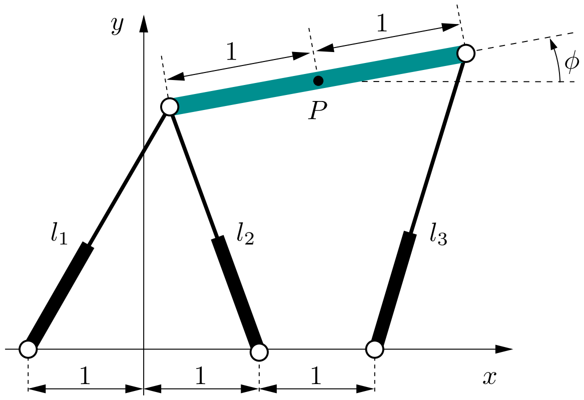

Figure 5: A planar Stewart platform.

for $i = 1, 2, 3$, which allows to integrate them readily into the previous equations. Thus, Eq. (11) is, in this case,

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} y^2 - 2ys + s^2 + x^2 + 2x - 2xc - 2c + c^2 + 1 - m_1^2 - 2m_1h_1s_1 - h_1^2s_1^2 \\ y^2 - 2ys + s^2 - 2x + x^2 + 2c - 2xc + c^2 + 1 - m_2^2 - 2m_2h_2s_2 - h_2^2s_2^2 \\ y^2 + 2ys + s^2 + x^2 - 4x + 2xc - 4c + c^2 + 4 - m_3^2 - 2m_3h_3s_3 - h_3^2s_3^2 \\ c^2 + s^2 - 1 \\ c_1^2 + s_1^2 - 1 \\ c_2^2 + s_2^2 - 1 \\ c_3^2 + s_3^2 - 1 \end{bmatrix} = 0, \quad (18)$$

where the l_i variables have been substitued using Eq. (17) in order to obtain a system with less variables and equations. Setting $\mathbf{u} = [x, y]^\mathsf{T}$ and $\mathbf{z} = [c_1, c_2, c_3, c, s, s_1, s_2, s_3]^\mathsf{T}$, the Jacobian with respect to the $\mathbf{z}$ variables is

$$d\Phi_z = \begin{bmatrix} 0 & 0 & 0 & -2x - 2 + 2c & -2y + 2s & -2m_1h_1 - 2h_1^2s_1 & 0 & 0 \\ 0 & 0 & 0 & 2 - 2x + 2c & -2y + 2s & 0 & -2m_2h_2 - 2h_2^2s_2 & 0 \\ 0 & 0 & 0 & 2x - 4 + 2c & 2y + 2s & 0 & 0 & -2m_3h_3 - 2h_3^2s_3 \\ 0 & 0 & 0 & 2c & 2s & 0 & 0 & 0 \\ 2c_1 & 0 & 0 & 0 & 0 & 2s_1 & 0 & 0 \\ 0 & 2c_2 & 0 & 0 & 0 & 0 & 2s_2 & 0 \\ 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 2s_3 \end{bmatrix}. \quad (19)$$

Using equations (18) and (19) we can, now, build the *.cuik* file as follows:

```
[CONSTANTS]

m1:=1.70710678118
m2:=1.70710678118
m3:=2
h1:=0.292893218815
h2:=0.292893218815
h3:=1

[SYSTEM VARS]

x:[-50,50]
```

```
y:[-50,50]

c1:[0,1]
c2:[0,1]
c3:[0,1]
c:[-1,1]
s:[-1,1]
s1:[-1,1]
s2:[-1,1]
s3:[-1,1]

xi1:[0,1]
xi2:[-1,1]
xi3:[-1,1]
xi4:[-1,1]
xi5:[-1,1]
xi6:[-1,1]
xi7:[-1,1]

[SYSTEM EQS]

y^2-2*y*s+s^2+x^2+2*x+1-2*x*c-2*c+c^2-m1^2-2*m1*h1*s1-h1*h1*s1^2=0;
y^2-2*y*s+s^2+1-2*x+x^2+2*c-2*x*c+c^2-m2^2-2*m2*h2*s2-h2*h2*s2^2=0;
y^2+2*y*s+s^2+x^2-4*x+4+2*x*c-4*c+c^2-m3^2-2*m3*h3*s3-h3*h3*s3^2=0;

c^2+s^2-1=0;
c1^2+s1^2-1=0;
c2^2+s2^2-1=0;
c3^2+s3^2-1=0;

2*c1*xi5=0;
2*c2*xi6=0;
2*c3*xi7=0;
-2*x*xi1-2*xi1+2*c*xi1+2*xi2-2*x*xi2+2*c*xi2+2*x*xi3-4*xi3+2*c*xi3+2*c*xi4=0;
-2*y*xi1+2*s*xi1-2*y*xi2+2*s*xi2+2*y*xi3+2*s*xi3+2*s*xi4=0;
-2*m1*h1*xi1-2*h1*h1*s1*xi1+2*s1*xi5=0;
-2*m2*h2*xi2-2*h2*h2*s2*xi2+2*s2*xi6=0;
-2*m3*h3*xi3-2*h3*h3*s3*xi3+2*s3*xi7=0;

xi1^2+xi2^2+xi3^2+xi4^2+xi5^2+xi6^2+xi7^2-1=0;
```

The execution of this *.cuik* file using a $\sigma = 0.01$ gives as outupt the end-effector singularities shown in Fig. 6.

References

[1] D. G. Luenberger. *Introduction to linear and nonlinear programming.* Addison-Wesley Reading, MA, 1973.

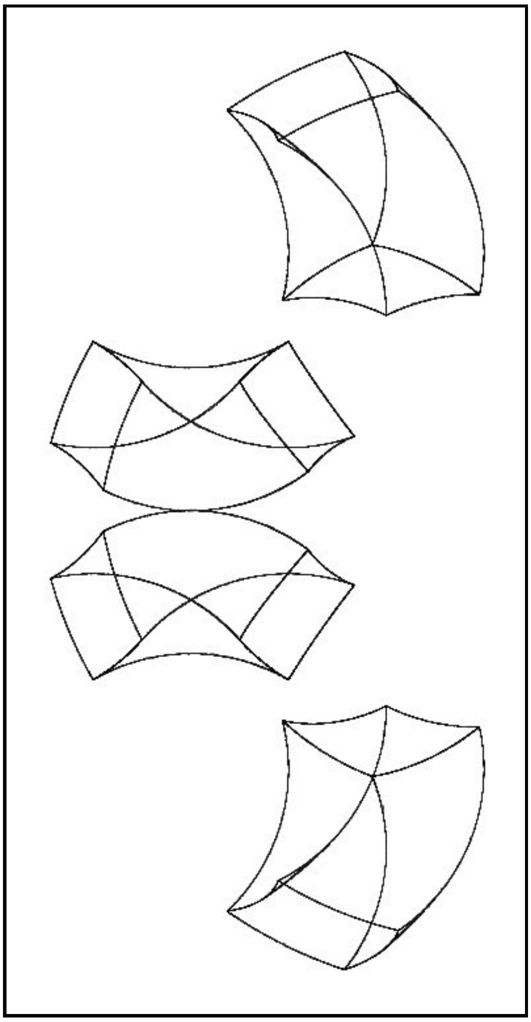

Figure 6: End-effector singularities containing workspace boundaries of the Planar Stewart platform.

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