New Geometric Approaches to the Singularity Analysis of Parallel Platforms

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Abstract— **In general, rearranging the legs of a Stewart-Gough platform, *i.e.*, changing the locations of its leg attachments, modifies the platform singularity locus in a rather unexpected way. Nevertheless, some leg rearrangements have been recently found to leave singularities invariant. In this work, a summary of the some of such singularity-invariant leg rearrangements are presented, and their practical consequences are illustrated with several examples including well-known architectures.**

I. INTRODUCTION

The Stewart-Gough platform triggered the research on parallel manipulators, and it has remained one of the most widely studied because, despite its geometric simplicity, its analysis translates into challenging mathematical problems [19], [12]. One important part of this analysis corresponds to the characterization of its singularities.

The singularities of a Stewart-Gough platform are those poses for which the manipulator loses stiffness. Characterizing such unstable poses has revealed as a challenging problem during the last decades, resulting in an extensive literature in the scientific kinematic world [15], [23], [14], [3], [13].

The Stewart-Gough platform is defined as a 6-DoF parallel mechanism with six identical S̱PS legs. The geometric and topological characterization of its singularity locus in its six-dimensional configuration space is, in general, a huge task which has only been completely solved for some specializations —*i.e.,* designs in which some spherical joints in the platform, the base, or both, coalesce to form multiple spherical joints [2], [1].

The kinematics group at the Institut de Robòtica i Informàtica Industrial at Barcelona studies new approaches to the singularity analysis of parallel platforms. This work presents one of their indirect approaches: even when there is no known solution to a given mathematical problem, it is always possible to try to find the set of transformations to the problem that leave its solution invariant. Although this does not solve the problem itself, it provides a lot of insight into its nature. This way of thinking is the one applied herein for the characterization of the singularity loci of Stewart-Gough platforms. In this context, this approach means finding leg rearrangements in a given Stewart-Gough platform that leave its singularity locus invariant.

Such *singularity-invariant leg rearrangements* are useful for two main reasons: (a)

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1) If the singularity locus of the platform at hand has already been characterized, it could be interesting to modify the location of its legs to optimize some other platform characteristics without altering such locus.
2) If the singularity locus of the analyzed platform has not been characterized yet, it could be of interest to simplify the platform's geometry by changing the location of its legs, thus easing the task of obtaining this characterization.

In [5] it is shown how, for a leg rearrangement to be singularity-invariant, it is necessary and sufficient that the linear actuators' velocities, before and after the rearrangement, are linearly related. It is important to realize that, if this condition is satisfied, a one-to-one correspondence between the elements of the platform forward kinematics solution sets, before and after the rearrangement, exists. Actually, the invariance in the singularities and the assembly modes of a parallel platform are two faces of the same coin. These ideas are closely related to those that made possible the development of kinematic substitutions [18]. They are general in the sense that they can be applied to any kind of mechanism, not only parallel platforms.

This paper shows how the application of singularity-invariant leg rearrangements to well-studied platforms leads to interesting new results.

Section II introduces the notation used in the paper and Section III defines a singularity-invariant leg rearrangement in mathematical terms. Then, three case studies are presented (Sections IV, V and VI), with particular numerical examples showing interesting results and the development and implementation of two prototypes based in them.

II. NOTATION

A general Stewart-Gough platform is a 6-S̱PS platform. In other words, it has six actuated prismatic legs with lengths l_i, $i = 1, \ldots, 6$, connecting two spherical passive joints centered at $\mathbf{a}_i = (x_i, y_i, z_i)^T$ and $\tilde{\mathbf{b}}_i = (r_i, s_i, t_i)^T$, given in base and platform reference frames, respectively (see Fig. 1). The pose of the platform is defined by a position vector $\mathbf{p} = (p_x, p_y, p_z)^T$ and a rotation matrix $\mathbf{R}$

$$\mathbf{R} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) = \begin{pmatrix} i_x & j_x & k_x \\ i_y & j_y & k_y \\ i_z & j_z & k_z \end{pmatrix},$$

so that the platform attachments can be written in the base reference frame as $\mathbf{b}_i = \mathbf{p} + \mathbf{R}\tilde{\mathbf{b}}_i$, for $i = 1, \ldots, 6$ (Fig. 1). To simplify the notation, the same name will be used to denote a point and its position vector.

$$\mathbf{P} = \begin{pmatrix} -r_1 & -s_1 & -t_1 & x_1 & y_1 & z_1 & r_1x_1 & r_1y_1 & r_1z_1 & s_1x_1 & s_1y_1 & s_1z_1 & t_1x_1 & t_1y_1 & t_1z_1 & 1 \\ -r_2 & -s_2 & -t_2 & x_2 & y_2 & z_2 & r_2x_2 & r_2y_2 & r_2z_2 & s_2x_2 & s_2y_2 & s_2z_2 & t_2x_2 & t_2y_2 & t_2z_2 & 1 \\ -r_3 & -s_3 & -t_3 & x_3 & y_3 & z_3 & r_3x_3 & r_3y_3 & r_3z_3 & s_3x_3 & s_3y_3 & s_3z_3 & t_3x_3 & t_3y_3 & t_3z_3 & 1 \\ -r_4 & -s_4 & -t_4 & x_4 & y_4 & z_4 & r_4x_4 & r_4y_4 & r_4z_4 & s_4x_4 & s_4y_4 & s_4z_4 & t_4x_4 & t_4y_4 & t_4z_4 & 1 \\ -r_5 & -s_5 & -t_5 & x_5 & y_5 & z_5 & r_5x_5 & r_5y_5 & r_5z_5 & s_5x_5 & s_5y_5 & s_5z_5 & t_5x_5 & t_5y_5 & t_5z_5 & 1 \\ -r_6 & -s_6 & -t_6 & x_6 & y_6 & z_6 & r_6x_6 & r_6y_6 & r_6z_6 & s_6x_6 & s_6y_6 & s_6z_6 & t_6x_6 & t_6y_6 & t_6z_6 & 1 \\ -r & -s & -t & x & y & z & rx & ry & rz & sx & sy & sz & tx & ty & tz & 1 \end{pmatrix}. \tag{1}$$
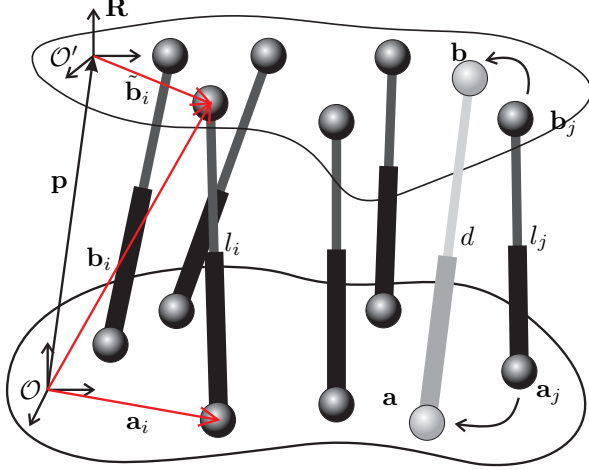

Fig. 1. A general Stewart-Gough platform with base attachments $\mathbf{a}_i$ and platform attachments at $\mathbf{b}_i$, $i = 1, ..., 6$. A single leg rearrangement consist in the substitution of one of the legs by a new one, in gray in the drawing.

There are three types of parameters that fully define a Stewart-Gough platform

Pose parameters:

$\mathbf{X} = (p_x, p_y, p_z, i_x, j_x, k_x, i_y, j_y, k_y, i_z, j_z, k_z)$

Geometric parameters:

$\mathcal{G} = (x_1, y_1, z_1, r_1, s_1, t_1, \ldots, x_6, y_6, z_6, r_6, s_6, t_6)$

Joint parameters:

$\mathbf{\Theta} = (l_1, \ldots, l_6)$

Finally, it will be useful to introduce a 6-dimensional space defined by the coordinates (x, y, z, r, s, t), called *the space of leg attachments*. Each point of this space defines a leg that goes from base attachment $\mathbf{a} = (x, y, z)^T$ to platform attachment $\tilde{\mathbf{b}} = (r, s, t)^T$.

III. SINGULARITY-INVARIANT LEG REARRANGEMENTS

A leg rearrangement consists in a relocation of the attachments of the manipulator, without modifying the pose of the platform, and thus, leading to new leg lengths $d_1, d_2, \ldots, d_6$ (Fig. 1). In general, such rearrangement completely modifies the kinematics of the manipulator and also the location of its singularities, because the forward kinematics of the rearranged platform must be solved again, which leads to a different number of assembly modes and to a different set of singularities.

Despite of this, recently, we have been able to identify leg rearrangements that do not modify the singularity locus of the platform, nor the solution of its forward kinematics. In other words, for the rearranged platform, the location of the singularity poses within the workspace of the manipulator remain at the same position. This kind of rearrangement are called *singularity-invariant leg rearrangements*, and where characterized in detail in [5].

In Fig. 1 we show the rearrangement of the leg j. In other words, we relocate the attachments $\mathbf{a}_j$ and $\tilde{\mathbf{b}}_j$ to the new coordinates $\mathbf{a} = (x, y, z)^T$ and $\tilde{\mathbf{b}} = (r, s, t)^T$. In [5], it was shown how such leg rearrangement is singularity invariant if, and only if, the coordinates (x, y, z, r, s, t) make the matrix $\mathbf{P}$ in (1) to be rank defective.

Note that the first 6 rows of $\mathbf{P}$ contain only geometric parameters of the manipulator, while the last row depends on the coordinates of the new location of the rearranged leg. The 6 first rows of $\mathbf{P}$ where used in [11], [21] to characterize architectural singularities. With this additional row, we are able to characterize any singularity-invariant leg rearrangement by studying the rank of $\mathbf{P}$.

Gaussian Elimination uses elementary row operations to reduce a given matrix into a rank-equivalent one, with an upper triangular shape. After it is applied to a matrix, rank deficiency occurs when all the elements of the last row are zero. Matrix $\mathbf{P}$ is 7×16 and, if we apply Gaussian Elimination, the last row of the resulting matrix can be expressed as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{P}_1 & \ldots & \mathbf{P}_{10} \end{pmatrix}, \tag{2}$$

where $\mathbf{P}_i$, for $i = 1, \ldots, 10$, are polynomials in the unknowns (x, y, z, r, s, t), and we can state that $\mathbf{P}$ is rank defective if, and only if, the 10 polynomials are simultaneously vanished.

In conclusion, if any of the legs is relocated to the new attachments $\mathbf{a} = (x, y, z)^T$ and $\mathbf{b} = (r, s, t)$, the resulting leg rearrangement is singularity-invariant if, and only if, $\{\mathbf{P}_1 = 0, \ldots, \mathbf{P}_{10} = 0\}$.

This is an overdetermined system that has no solution for a generic case. We need to impose at least 5 more scalar equations to obtain a 1-dimensional set of solutions. Next we will see several cases for which matrix $\mathbf{P}$ is simplified and solutions of dimension 1 and 2 are obtained.

IV. CASE STUDY I: DOUBLY-PLANAR STEWART-GOUGH PLATFORMS

For any doubly planar Stewart-Gough platform, the coordinates of the base and platform attachments can be written,

without loss of generality, as $\mathbf{a}_i = (x_i, y_i, 0)$ and $\tilde{\mathbf{b}}_i = (z_i, t_i, 0)$. In this case, a leg rearrangement with coordinates (x, y, z, t) stands for the substitution of any of the legs by another one going from the base attachment located at $\mathbf{a} = (x, y, 0)^T$ to the platform attachment at $\mathbf{b} = \mathbf{p} + \mathbf{R}(z, t, 0)^T$. In this case, matrix $\mathbf{P}$ can be simplified to

$$
\mathbf{P} = \begin{pmatrix}
-z_1 & -t_1 & x_1 & y_1 & x_1 z_1 & y_1 z_1 & x_1 t_1 & y_1 t_1 & 1 \\
-z_2 & -t_2 & x_2 & y_2 & x_2 z_2 & y_2 z_2 & x_2 t_2 & y_2 t_2 & 1 \\
-z_3 & -t_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 & x_3 t_3 & y_3 t_3 & 1 \\
-z_4 & -t_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 & x_4 t_4 & y_4 t_4 & 1 \\
-z_5 & -t_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 & x_5 t_5 & y_5 t_5 & 1 \\
-z_6 & -t_6 & x_6 & y_6 & x_6 z_6 & y_6 z_6 & x_6 t_6 & y_6 t_6 & 1 \\
-z & -t & x & y & xz & yz & xt & yt & 1
\end{pmatrix}.
$$

$$(3)$$

Consider the example with attachment local coordinates appearing in Table I.

ATTACHMENT COORDINATES ($\mathbf{a}_i = (x_i, y_i, 0)^T$, $\tilde{\mathbf{b}}_i = (z_i, t_i, 0)^T$).

i	x_i	y_i	z_i	t_i
1	3	5	5	6
2	7	9	7	8
3	8	9	9	8
4	12	5	9	6
5	5	2	6	4
6	9	2	9	5

To check rank deficiency, Gaussian Elimination is applied on $\mathbf{P}$ with the corresponding numerical values substituted. In this case, the last row of the resulting matrix has only 3 nonzero terms dependent on x, y, z and t. Different but equivalent equations arise depending on the order of the columns. For example, Gaussian Elimination on matrix $\mathbf{P}$ as it appears in equation (3) leads to a matrix whose last row is

$$
\frac{1}{P_{789}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & P_{89} & P_{79} & P_{78} \end{pmatrix},
$$

where P_{ij} is the determinant of the submatrix obtained from $\mathbf{P}$ after deleting columns i and j, and P_{ijk} the determinant of the submatrix formed by the first 6 rows of $\mathbf{P}$ after deleting columns i, j and k. With the corresponding numerical values, $P_{789} = -12180$ and the singularity-invariant leg rearrangements are defined by the condition defined by $\{P_{89} = P_{79} = P_{78} = 0\}$, which reads as

$$
\left.
\begin{aligned}
-\frac{338}{609}xz + xt + \frac{3706}{3045}yz + \frac{1096}{1015}x - \frac{22713}{1015}y \\
- \frac{27743}{3045}z + \frac{19302}{1015}t &= 0 \\[6pt]
-\frac{470}{609}xz + \frac{10519}{3045}yz + yt + \frac{13274}{1015}x - \frac{61662}{1015}y \\
- \frac{87557}{3045}z + \frac{51343}{1015}t &= 0 \\[6pt]
\frac{17}{609}xz - \frac{38}{609}yz - \frac{67}{203}x + \frac{194}{203}y \\
+ \frac{247}{609}z - \frac{192}{203}t + 1 &= 0
\end{aligned}
\right\}
$$

$$(4)$$

Note that any equation consisting of a submatrix determinant P_{ij} equated to zero will be bilinear in the unknowns,
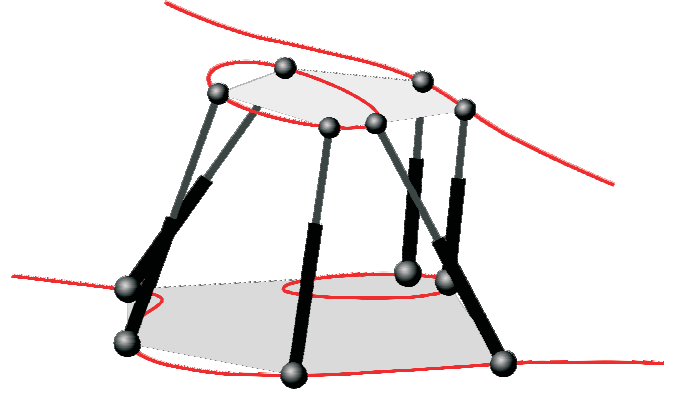

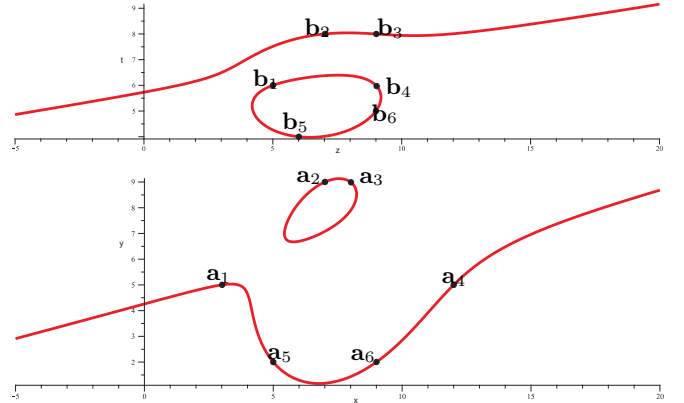Fig. 2. A general singularity-invariant leg rearrangement for a doubly-planar Stewart-Gough platform

Fig. 3. The base and the platform curves of the doubly-planar Stewart-Gough platform depicted in Fig. 2.

but with different monomials. As the system is linear, both in (x, y) and in (z, t), it can be rewritten in matrix form as

$$
\mathbf{S}_b \begin{pmatrix} z \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5}
$$

where $\mathbf{S}_b$ is

$$
\begin{pmatrix}
\frac{-27743}{3045} + \frac{3706}{3045}y - \frac{338}{609}x & x + \frac{19302}{1015} & \frac{22713}{1015}y - \frac{1096}{1015}x \\[4pt]
\frac{10519}{3045}y - \frac{87557}{3045} - \frac{470}{609}x & y + \frac{51343}{1015} & \frac{61662}{1015}y - \frac{13274}{1015}x \\[4pt]
\frac{17}{609}x - \frac{38}{609}y + \frac{247}{609} & \frac{-192}{203} & -1 + \frac{67}{203}x - \frac{194}{203}y
\end{pmatrix}
$$

which only depends on x and y (b refers to *base*, as x and y are the coordinates of the base attachments). The other way round, the system can also be written as

$$
\mathbf{S}_p \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6}
$$

where $\mathbf{S}_p$ is

$$
\begin{pmatrix}
t + \frac{1096}{1015} - \frac{338}{609}z & \frac{-22713}{1015} + \frac{3706}{3045}z & \frac{27743}{3045}z - \frac{19302}{1015}t \\[4pt]
\frac{13274}{1015} - \frac{470}{609}z & \frac{-61662}{1015} + \frac{10519}{3045}z + t & \frac{87557}{3045}z - \frac{51343}{1015}t \\[4pt]
\frac{-67}{203} + \frac{17}{609}z & \frac{-38}{609}z + \frac{194}{203} & \frac{-247}{609}z + \frac{192}{203}t - 1
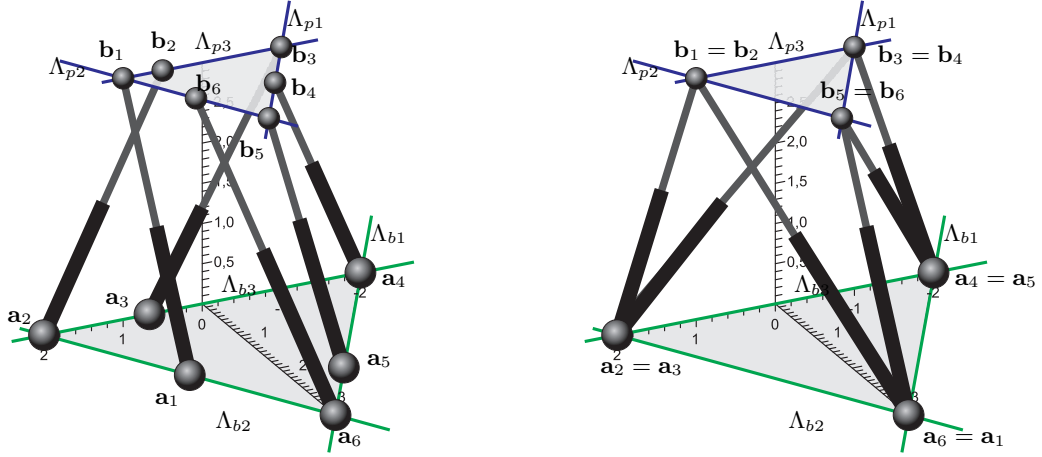\end{pmatrix}
$$

Fig. 4. Griffis-Duffy type I platform with the attachment coordinates given in in Table II (left), and its equivalent octahedral manipulator after applying a leg rearrangement (right).

that only depends on z and t (p refers to *platform*, as z and t are the coordinates of the platform attachments).

From equation (5) it is clear that the system has a solution for (z, t) only for those (x, y) that satisfy $\det(\mathbf{S}_b) = 0$, and this solution is unique (assuming that $\mathbf{S}_b$ has rank 2). In the same way, there exists a solution for (x, y) only for those (z, t) that make $\det(\mathbf{S}_p) = 0$. Both determinants define cubic curves on the base and platform planes, respectively. In other words, system (4) defines a one-to-one correspondence between generic points on two cubic curves. However, the correspondence may be not one-to-one for special points on the cubics for non-generic examples (see details in [10]).

For this particular example, the equations of the cubic on the base is

$$\frac{16}{145}x^3 - \frac{293}{609}x^2y + \frac{253}{1015}xy^2 - \frac{142}{609}y^3 + \frac{1061}{3045}x^2 +$$
$$\frac{4343}{1015}xy + \frac{2313}{1015}y^2 - \frac{17888}{1015}x - \frac{26032}{1015}y + \frac{261691}{3045} = 0$$

and on the platform

$$\frac{9}{145}z^3 - \frac{396}{1015}z^2t + \frac{293}{1015}zt^2 - \frac{192}{203}t^3 + \frac{282}{203}z^2 + \frac{1877}{1015}zt$$
$$+ \frac{2229}{145}t^2 - \frac{17799}{1015}z - \frac{98097}{1015}t + \frac{32922}{145} = 0$$

which have been plotted in Fig. 3. The curves attached to the manipulator base and platform are shown in Fig. 2.

Depending on the placement of the attachments, these curves can be generic curves of degree 3, or a line and a conic, or even 3 lines crossing 2 by 2. In the next example, one of these degenerate cases is analyzed.

A. An octahedral manipulator implementation

In 1993, Griffis and Duffy patented a manipulators named thereafter Griffis-Duffy platform [17]. The platform have his attachments distributed on triangles, three attachments on the vertexes and three on the midpoints of the edges, and platform is formed by joining the attachments on the midpoints on the base to the vertexes on the platform, as the example with attachment coordinates given in Table II. A

representation of these manipulators can be found in Fig. 4-(left).

TABLE II
COORDINATES OF THE ATTACHMENTS $\mathbf{a}_i = (x_i, y_i, 0)$ AND
$\mathbf{b}_i = \mathbf{p} + \mathbf{R}(z_i, t_i, 0)^T$ FOR THE ANALYZED ROBOTS

i	x_i	y_i	z_i	t_i
1	1	$\sqrt{3}$	1	0
2	2	0	$1/2$	0
3	$2/3$	0	-1	0
4	-2	0	$-1/2$	$\sqrt{3}/2$
5	$-2/3$	$(4/3)\sqrt{3}$	0	$\sqrt{3}$
6	0	$2\sqrt{3}$	$1/2$	$\sqrt{3}/2$

In this case, the system obtained by applying Gaussian elimination on the corresponding matrix $\mathbf{P}$ results in :

$$\left.\begin{array}{r} 2t - y + yz + xt = 0 \\ (\sqrt{3}z + t - \sqrt{3})y = 0 \\ -2\sqrt{3}z + 4t + \sqrt{3}x - y + \sqrt{3}xz + 3yz - 2\sqrt{3} = 0 \end{array}\right\} \tag{7}$$

The resolution of this system gives correspondences between base and platform attachments that leave the singularities invariant. The base and platform cubic curves, in this case, factorize into the 3 lines:

$$(\sqrt{3}z - t + \sqrt{3})(\sqrt{3}z + t - \sqrt{3})t = 0,$$

and

$$(-3x + \sqrt{3}y - 6)(3x + \sqrt{3}y - 6)y = 0,$$

respectively.

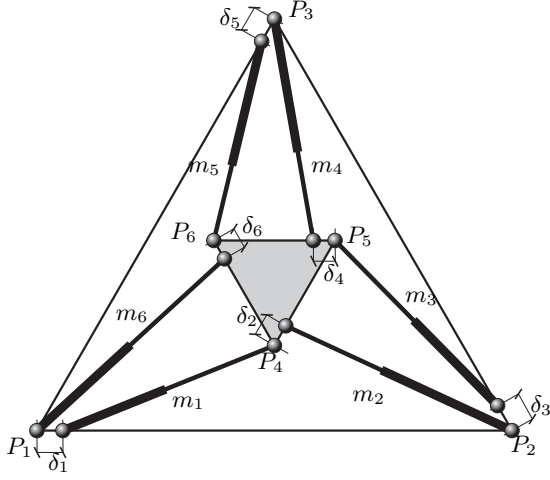Actually, it can be checked that system (7) has 6 sets of

Fig. 5. Contrary to what happens to the Stoughton-Arai approximation, the proposed modification leads to a 6-6 platform kinematically equivalent to the octahedral manipulator.

solutions

$$\Delta_{b1} = \{(x, y, z, t) \mid$$
$$x = \lambda, y = (\lambda_1 + 2)\sqrt{3}, z = 0, t = \sqrt{3}; \lambda_1 \in \mathbb{R}\},$$
$$\Delta_{b2} = \{(x, y, z, t) \mid$$
$$x = \lambda_2, y = (2 - \lambda_2)\sqrt{3}, z = 1, t = 0; \lambda_2 \in \mathbb{R}\},$$
$$\Delta_{b3} = \{(x, y, z, t) \mid$$
$$x = \lambda_3, y = 0, z = -1, t = 0; \lambda_3 \in \mathbb{R}\},$$
$$\Delta_{p1} = \{(x, y, z, t) \mid$$
$$x = -2, y = 0, z = \lambda_4, t = \sqrt{3}(\lambda_4 + 1); \lambda_4 \in \mathbb{R}\},$$
$$\Delta_{p2} = \{(x, y, z, t) \mid$$
$$x = 0, y = 2\sqrt{3}, z = \lambda_5, t = \sqrt{3}(1 - \lambda_5); \lambda_5 \in \mathbb{R}\},$$
$$\Delta_{p3} = \{(x, y, z, t) \mid$$
$$x = 2, y = 0, z = \lambda_6, t = 0; \lambda_6 \in \mathbb{R}\}.$$

In other words, these are 6 point-line correspondences, that is, to each vertex of the base (platform) triangle corresponds a line on the platform (base) triangle. This means that, for the Griffis-Duffy type manipulator, we can fix the attachments at the vertexes of the platform (base), and then rearrange the opposite attachments along a line in the base (platform) without modifying the kinematics of the platform.

As a result, by moving the six midpoint attachments along their supporting lines, the manipulator can be rearranged into the manipulator depicted in Fig. 4-(right), which is the widely known octahedral manipulator. This is an interesting result, because we can avoid the use of multiple spherical joints (that is, spherical joints sharing the same center) without loosing the properties of the celebrated octahedral architecture [14]. A manipulator has been constructed following the design in Fig. 5 in the Laboratory of Parallel Robots, at the Institut de Robòtica i Informàtica Industrial [22] (Fig. 6). Its advantage is that it is a 6-6 manipulator with the same
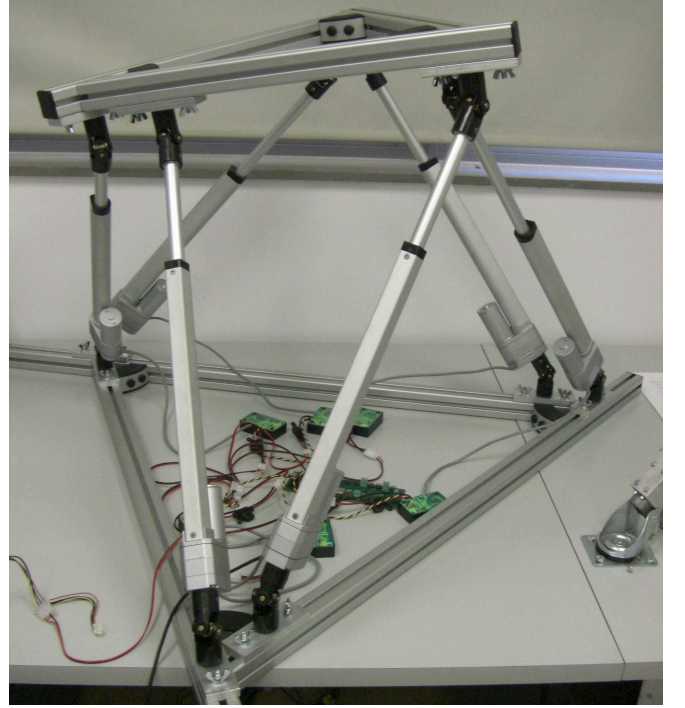

Fig. 6. This platform consists of six extensible legs connecting a moving platform to a fixed base. We avoid the use of multiple spherical joints (that is, spherical joints sharing the same center) without loosing the properties of the celebrated octahedral architecture.

kinematics and singularities to the widely studied octahedral manipulator (see more details in [20], [6]).

V. CASE STUDY II: A DECOUPLED STEWART-GOUGH PLATFORM

Consider the manipulator in Fig. 7. It contains a tripod and 3 more legs, with all the base attachments coplanar. Thus, without loss of generality, we can write the coordinates of the attachments as $\mathbf{a}_i = (x_i, y_i, 0)^T$ and $\tilde{\mathbf{b}}_i = (r_i, s_i, t_i)^T$. This manipulator is said to be decoupled because the three legs forming the tripod give the position of the platform, while the three remaining ones orient it. When the tripod is rigid, *i. e.*, fixed at a position, this manipulator is also known as spherical [4], [16].

Consider the example with numeric coordinates appearing in Table III. After performing Gaussian Elimination on

TABLE III
ATTACHMENT COORDINATES $\mathbf{a}_i = (x_i, y_i, 0)$ AND
$\mathbf{b}_i = \mathbf{p} + \mathbf{R}(r_i, s_i, t_i)^T$

i	x_i	y_i	r_i	s_i	t_i
1	2	-1	2	2	0
2	5	4	2	2	0
3	-1	4	2	2	0
4	7	-2	5	0	1
5	2	7	2	5	1
6	-3	-2	-1	0	1

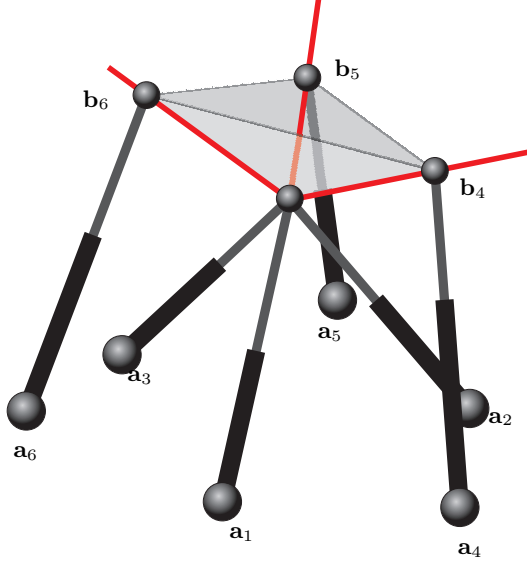the corresponding matrix $\mathbf{P}$, only six non-zero elements

Fig. 7. A decoupled manipulator with non-planar platform. In blue, its singularity-invariant leg rearrangement lines.

remain at the last row. That is, a leg rearrangement will be singularity-invariant if it fulfills the following 6 conditions

$$-2xr + yr + 4x - 2y + 6r - 6s + 18t = 0,$$
$$-4xr/3 + xs + 2x/3 + 6r - 6s + 12t = 0,$$
$$1/5(17xr + ys - 34x - 10y - 34r + 34s - 207t) = 0,$$
$$5xr/3 + xt - 10x/3 - 5r + 5s - 17t = 0,$$
$$9xr/5 + yt - 18x/5 - 18r/5 + 18s/5 - 89t/5 = 0,$$
$$-1xr/2 + x + r - 3s/2 + 9t/2 + 1 = 0$$

This system of equations has 4 sets of solutions:

$$\mathcal{T} = \{(x, y), (r, s, t) \mid$$
$$x = \lambda, y = \mu; r = 2, s = 2, t = 0, \lambda, \mu \in \mathbb{R}\},$$
$$\Delta_1 = \{(x, y), (r, s, t) \mid$$
$$x = 2, y = 7; r = 2, s = 2 + 3\lambda, t = \lambda, \lambda \in \mathbb{R}\},$$
$$\Delta_2 = \{(x, y), (r, s, t) \mid x = 7, y = -2;$$
$$r = 5 - 3\lambda/2, s = \lambda, t = 1 - \lambda/2, \lambda \in \mathbb{R}\},$$
$$\Delta_3 = \{(x, y), (r, s, t) \mid x = -3, y = -2;$$
$$r = 2 - 3\lambda, s = 2 - 2\lambda, t = \lambda, \lambda \in \mathbb{R}\}.$$

The first one corresponds to the tripod component and it means that base attachments can be rearranged to any point of the base plane as long as its corresponding platform attachment is the vertex of the tripod. The other 3 sets correspond to point-line correspondences as before, depicted as red lines in Fig. 7. This means that $\mathbf{b}_4$, $\mathbf{b}_5$ and $\mathbf{b}_6$ can be relocated to any other point of the red lines (as long as their corresponding base attachment remains the same).

In Fig. 8 we show two possible singularity-invariant leg rearrangements of the manipulator at hand. For all of them, the decoupling properties remain the same as they are all equivalent manipulators.
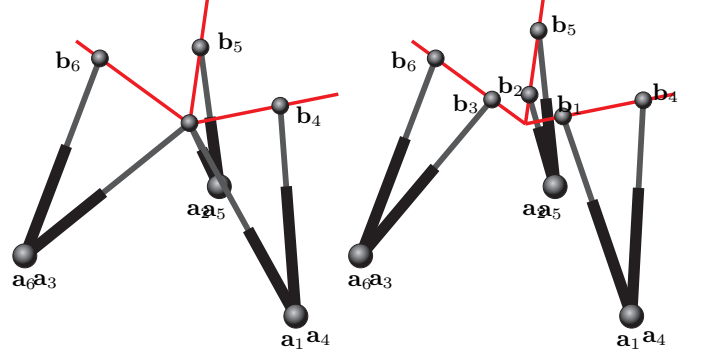

Fig. 8. Singularity-invariant leg rearrangements from the example in Fig. 7

VI. CASE STUDY III: PENTAPODS

A pentapod is usually defined as a 5-degree-of-freedom fully-parallel manipulator with an axial spindle as moving platform. This kind of manipulators have revealed as an interesting alternative to serial robots handling axisymmetric tools. The moving platform can freely rotate around the axis defined by the five aligned revolute joints, but if this rotation axis is made coincident with the symmetry axis of the tool, the uncontrolled motion becomes irrelevant in most cases. Their particular geometry permits that, in one tool axis, large inclination angles are possible thus overcoming the orientation limits of the classical Stewart-Gough platform.

A pentapod involves only 5 of the 6 legs of the Stewart-Gough platform, with the platform attachments collinear. This 5 legs form a rigid component by itself that can be studied separately. In addition to the platform attachments collinearity, if we consider all the base attachments coplanar, then we can write the coordinates of the attachments as $\mathbf{a}_i = (x_i, y_i, 0)^T$ and $\tilde{\mathbf{b}}_i = (z_i, 0, 0)^T$ for $i = 1..5$ and the corresponding matrix $\mathbf{P}$ after some simplifications reads as

$$\mathbf{P} = \begin{pmatrix} z_1 & x_1 & y_1 & x_1 z_1 & y_1 z_1 & 1 \\ z_2 & x_2 & y_2 & x_2 z_2 & y_2 z_2 & 1 \\ z_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 & 1 \\ z_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 & 1 \\ z_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 & 1 \\ z & x & y & xz & yz & 1 \end{pmatrix}. \tag{8}$$

In this case, $\mathbf{P}$ is a square matrix, so its rank deficiency is characterized only by the equation $\det(\mathbf{P}) = 0$. In [9] it was shown that such condition defines a one-to-one correspondence between the platform attachments and the lines of a pencil attached at the base. The center of this pencil, called $\mathcal{B}$-point in [9], [7], plays an important role in the geometric characterization of the manipulator singularities.

Consider the example with numerical coordinates appearing in table VI.

After substituting the numerical values in $\mathbf{P}$, we get that the condition for the singularity invariance is

$$\det(\mathbf{P}) = x - z = 0. \tag{9}$$

This means that any leg can be rearranged to a leg going from the base attachment $\mathbf{a} = (\lambda, y, 0)^T$ to $\tilde{\mathbf{b}} = (\lambda, 0, 0)^T$
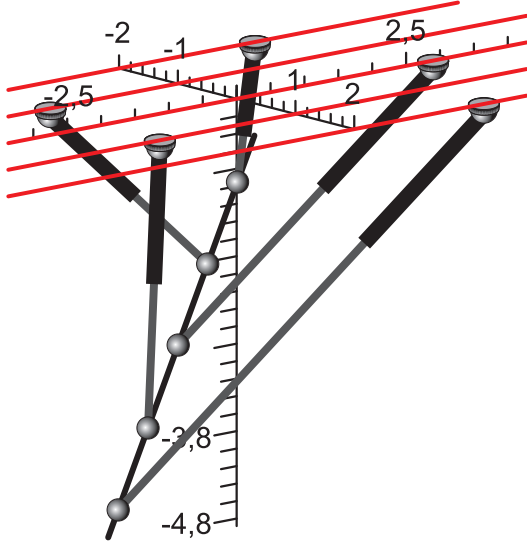
Fig. 9. Pentapod analyzed in Section VI. Note that it is in an upside-down configuration, so that the platform is located under the base.

TABLE IV
ATTACHMENTS $\mathbf{a}_i = (x_i, y_i, 0)$ AND $\tilde{\mathbf{b}}_i = (z_i, 0, 0)$

i	x_i	y_i	z_i
1	-2	2	-2
2	-1	-2	-1
3	0	3	0
4	1	-2	1
5	2	2	2

without modifying the singularity locus (where for a fixed λ, the y coordinate can take any value). This corresponds to the rearrangements plotted in Fig. 9, that is, a one-to-one correspondence between the attachments at the platform and a pencil of parallel lines attached at the base. In this case, the center of the pencil lies at infinity.

This particular architecture was proved to be quadratically solvable in [8], [9], that is, its forward kinematics can be solved by solving only 2 quadratic polynomials. If we fix the attachments of the platform, the corresponding base attachments can be relocated to any point of the red lines plotted in Fig.9. Taking advantage of that idea, at Laboratory of Parallel Robots at IRI we have developed a reconfigurable manipulator prototype based on this structure. Its base attachments can be reconfigured along actuated guides, without modifying the nature of its forward kinematics nor the singularities of the manipulator, and thus increasing the versatility of the manipulator, as for each task, the legs can be reconfigured to equally distribute the forces among its legs (Fig. 10).

VII. CONCLUSIONS

The present work shows how the application of singularity-invariant leg rearrangements provide a new geometric approach to the study of Stewart-Gough platform singularities. Indeed, we have presented three case studies that illustrate several new results.

Fig. 10. Prototype of the reconfigurable quadratically-solvable pentapod and its joint implementations.

We have presented a tool to detect equivalences between manipulators, which means that we can use previous known geometric interpretations of singularities to new architectures. That is the case of the Griffis-Duffy platform at Section IV. The 6-6 Stewart-Gough platform prototype shown in Fig. 6 has the same kinematic properties than the octahedral manipulator, that is, the same geometric interpretation for its singularities applies, as well as all other kinematic properties studied in the extensive literature about the octahedral manipulator.

We have also shown how decoupled manipulators can be rearranged to equivalent and apparently non-decoupled manipulators, with different configurations of their spherical joints that might be easy to construct.

Also, the hidden geometric structure reveled by these curves of singularity-invariant leg rearrangements can help in the simplification of the forward kinematics resolution. For example, in the case study III, we show a manipulator that is quadratically solvable.

Finally, new geometric interpretation of singularities have been found thanks to singularity-invariant leg rearrangements. For example, for pentapods with planar bases, the identified pencil of lines at the base of the manipulator reveals to be crucial for the geometric interpretation of its singularities. Similar interpretations represent a challenge for the future work.

In conclusion, this indirect approach to the analysis of Stewart-Gough platform singularities has succeed in finding new results in a topic with an extensive previous literature.

VIII. ACKNOWLEDGMENTS

The authors gratefully acknowledge

REFERENCES

[1] M. Alberich-Carramiñana, M. Garolera, F. Thomas, and C. Torras, "Partially-flagged parallel manipulators: Singularity charting and avoidance," *IEEE Transactions on Robotics*, vol. 25, no. 4, pp. 771–784, 2009.

[2] M. Alberich-Carramiñana, F. Thomas, and C. Torras, "Flagged parallel manipulators," *IEEE Transactions on Robotics*, vol. 23, no. 5, pp. 1013–1023, 2007.

[3] P. Ben-Horin and M. Shoham, "Singularity condition of six degree-of-freedom three-legged parallel robots based on Grassmann-Cayley algebra," *IEEE Transactions on Robotics*, vol. 22, no. 4, pp. 577–590, 2006.

[4] I. A. Bonev and C. M. Gosselin, "Analytical determination of the workspace of symmetrical spherical parallel mechanisms," *IEEE Transactions on Robotics*, vol. 22, no. 5, pp. 1011–1017, 2006.

[5] J. Borràs, "Singularity-invariant leg rearrangements in Stewart-Gough platforms," Ph.D. dissertation, Spanish National Research Council (CSIC)- Technical university of Catalonia (UPC), 2011.

[6] J. Borràs, F. Thomas, and C. Torras, "On Δ-transforms," *IEEE Transactions on Robotics*, vol. 25, no. 6, pp. 1225–1236, 2009.

[7] ——, "Architectural singularities of a class of pentapods," *Mechanism and Machine Theory*, 2010, to appear.

[8] ——, "A family of quadratically-solvable 5-UPS parallel robots," in *IEEE International Conference on Robotics and Automation*, 2010, pp. 4703–4708.

[9] ——, "Singularity-invariant families of 5-SPU platforms," *IEEE Transactions on Robotics*, 2010, to appear.

[10] ——, "Singularity invariant leg rearrangements in doubly-planar Stewart-Gough platforms," in *Robotics: Science and Systems VI*. The MIT Press, 2010.

[11] R. Bricard, "Mémoire sur les déplacements à trajectoires sphériques," *Journal de l'École Polytechnique*, vol. 11, no. 2, pp. 1–93, 1906.

[12] B. Dasguptaa and T. Mruthyunjayab, "The Stewart platform manipulator: a review," *Mechanism and Machine Theory*, vol. 35, pp. 15–40, 2000.

[13] P. Donelan, "Singularities in robot kinematics - a publications database," 2007, [Online; accessed 1-December-2008]. [Online]. Available: http://www.mcs.vuw.ac.nz/˜donelan/cgi-bin/rs/main

[14] D. Downing, A. Samuel, and K. Hunt, "Identification of the special configurations of the octahedral manipulator using the pure condition," *International Journal of Robotics Research*, vol. 21, no. 2, pp. 147–159, 2002.

[15] C. Gosselin and J. Angeles, "Singularity analysis of closed-loop kinematic chains," *IEEE Transactions on Robotics*, vol. 6, no. 3, pp. 281–290, 1990.

[16] C. M. Gosselin and E. St.-Pierre, "Development and experimentation of a fast 3-DOF camera-orienting device," *International Journal of Robotics Research*, vol. 16, no. 5, pp. 619–630, 7997.

[17] M. Griffis and J. Duffy, "Method and apparatus for controlling geometrically simple parallel mechanisms with distinctive connections," *US Patent 5,179,525*, 1993.

[18] K. Hunt, *Kinematic Geometry of Mechanisms*. Oxford University Press, 1978, p. 323.

[19] J.-P. Merlet, *Parallel Robots*. Springer, 2000.

[20] N. Rojas, J. Borràs, and F. Thomas, "A distance-based formulation of the octahedral manipulator kinematics," in *IFToMM Symposium on Mechanism Design for Robotics*, 2010.

[21] O. Röschel and S. Mick, "Characterisation of architecturally shaky platforms," in *International Symposium on Advances in Robot Kinematics*, 1998, pp. 465–474.

[22] F. Thomas, "Laboratory of parallel robots," 2011. [Online]. Available: http://www.iri.upc.edu/people/thomas/ID-Robots.htm

[23] D. Zlatanov, "Generalized singularity analysis of mechanisms." Ph.D. dissertation, University of Toronto, 1998.