New Geometric Approaches to the Analysis and Design of Stewart-Gough Platforms

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***Abstract*— In general, rearranging the legs of a Stewart-Gough platform, *i.e.*, changing the locations of its leg attachments, modifies the platform singularity locus in a rather unexpected way. Nevertheless, some leg rearrangements have been recently found to leave singularities invariant. Identification of such rearrangements is useful not only for the kinematic analysis of the platforms, but also as a tool to redesign manipulators avoiding the implementation of multiple spherical joints, which are difficult to construct and have a small motion range.**

In this work, a summary of these singularity-invariant leg rearrangements is presented, and their practical implications are illustrated with several examples including well-known architectures.

I. Introduction

Parallel platforms have been widely studied during the last decades because of their advantages with respect to serial robots: improved stiffness-to-load ratio, lower inertia, enhanced dynamics and better accuracy. Among them, the Stewart-Gough platform [1], [2] has attracted the interest of many researchers and it is still the focus of several important research projects for many applications: micro-force sensors [3], positioning tools [4], [5], micro-precision interferometers [5], milling machines [6], flight simulators [7], radio-telescopes [8], [9], cable-driven robots [10] or supporting devices for rehabilitation and surgery interventions [11], [12].

The Stewart-Gough platform is defined as a 6-DoF parallel mechanism with six identical S$\underline{\text{P}}$S legs. Despite its geometric simplicity, its analysis translates into challenging mathematical problems [13], [14]. Forward kinematics usually involves solving high-order polynomial systems with no possible close-form solution, *i.e.* they must be approached with computationally costly numerical methods [15], [16]. The singularities of a Stewart-Gough platform are those poses for which the manipulator loses stiffness. Characterizing such unstable poses is essential for improving the performance capacities of the robot, but has revealed as a challenging problem, resulting in an extensive literature in the scientific kinematic world [17], [18], [19].

Closed-form forward kinematics and the characterization of singularities has only been completely solved for some particular architectures. Indeed, there are specific designs of Stewart-Gough platforms with nice symmetry properties [20], closed-form solution kinematics [21], [22], decoupled motions for position and orientation [23], [24] or a complete geometric understanding of its singularities [25], [26], [27]. Despite their nice properties, these designs are not usually chosen for practical applications, mainly due to a characteristic they all have in common: some spherical joints in the platform, the base, or both, share the same center of rotation forming *multiple spherical joints*. Such multiple spherical joints are difficult to construct and have a small range of motion.

Several solutions have been proposed in literature to solve this issue. The most usual approach for practical applications consists in substituting the multiple spherical joint by a group of single spherical joints with small offsets between them, thus simplifying the implementation of the platform but significantly increasing the complexity of the kinematic solution and the characterization of its singularities. If such offsets are neglected, then errors arise in the computations [28]. Other researchers have made efforts to design equivalent-motion mechanisms to substitute the multiple ball and socket joint. Such designs present several drawbacks, such as a complex design, expensive implementation, small range of motion or poor rigidity and accuracy [28], [29], [30], [31], [32], [33].

The present work proposes a new approach to solve this issue: finding leg rearrangements in a given Stewart-Gough platform that leave its kinematic solutions and singularity locus invariant. In other words, finding how to redesign the geometry of the platform so that the resulting architecture has its singularities located at the same positions of the workspace as the previous design. Even when there is no known solution to a given mathematical problem, it is always possible to try to find the set of transformations to the problem that leave its solution invariant. Although this does not solve the problem itself, it provides a lot of insight into its nature. This way of thinking is the one applied herein for the characterization of the singularity loci of Stewart-Gough platforms and it leads to a complete characterization of all the singularity invariant leg rearrangements.

It will be shown how such rearrangements provide a guide to substitute a multiple spherical joint by a group of single spherical joints separated by small offsets following a specific geometry, so that the kinematics and the singularities of the platform remain the same as those of the original architecture. In other words, for the previously mentioned designs, this work proposes a methodology to redesign them in a way that their nice kinematic and geometric properties are preserved, but avoiding the use of multiple spherical joints.

The presented *singularity-invariant leg rearrangements* are also useful for other reasons:

(a) If the singularity locus of the platform at hand has already been characterized, it could be interesting to modify the location of its legs to optimize some other platform

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The authors gratefully acknowledge funding from the Generalitat de Catalunya through the Robotics group (SRG0155).

characteristics without altering such locus.

(b) If the singularity locus of the analyzed platform has not been characterized yet, it could be of interest to simplify the platform's geometry by changing the location of its legs, thus easing the task of obtaining this characterization.

In [34] it is shown that, for a leg rearrangement to be singularity-invariant, it is necessary and sufficient that the linear actuators' velocities, before and after the rearrangement, are linearly related. It is important to realize that, if this condition is satisfied, a one-to-one correspondence between the elements of the platform forward kinematics solution sets, before and after the rearrangement, exists. Actually, the invariance in the singularities and the assembly modes of a parallel platform are two faces of the same coin. These ideas are closely related to those that made possible the development of kinematic substitutions [35]. They are general in the sense that they can be applied to any kind of mechanism, not only parallel platforms. Indeed, there are several platforms with line-based singularities [36, Chapter 12, pp 272], or even three-legged platforms [37], [38], that are equivalent to 6-legged Stewart-Gough platforms.

This paper shows how the application of singularity-invariant leg rearrangements to well-studied platforms leads to interesting new results.

Section II introduces the notation used in the paper and Section III defines a singularity-invariant leg rearrangement in mathematical terms. Then, three case studies are presented (Sections IV, VI and V), with particular numerical examples showing interesting results, as well as the development and implementation of two prototypes based in them.

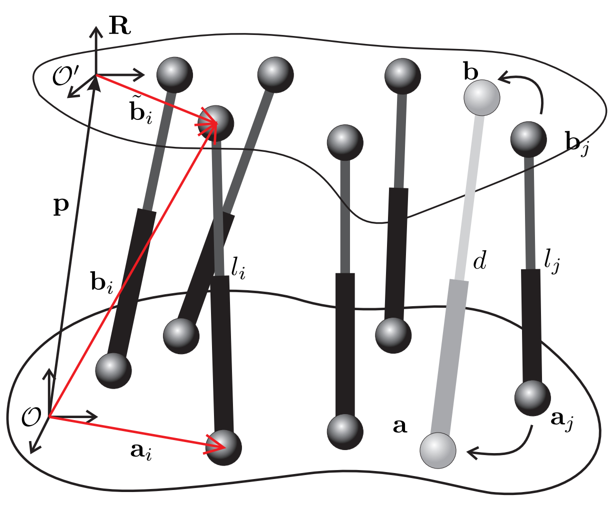

Fig. 1. A general Stewart-Gough platform with base attachments $\mathbf{a}_i$ and platform attachments at $\mathbf{b}_i$, $i = 1, \ldots, 6$. A single leg rearrangement consists in the substitution of one of the legs by a new one, in gray in the drawing.

II. Notation

A general Stewart-Gough platform is a 6-S$\underline{\text{P}}$S platform. In other words, it has six actuated prismatic legs with lengths l_i, $i = 1, \ldots, 6$, connecting two spherical passive joints centered at $\mathbf{a}_i = (x_i, y_i, z_i)^T$ and $\tilde{\mathbf{b}}_i = (r_i, s_i, t_i)^T$, given in base and platform reference frames, respectively (see Fig. 1). The pose of the platform is defined by a position vector $\mathbf{p} = (p_x, p_y, p_z)^T$ and a rotation matrix $\mathbf{R}$

$$\mathbf{R} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) = \begin{pmatrix} i_x & j_x & k_x \\ i_y & j_y & k_y \\ i_z & j_z & k_z \end{pmatrix},$$

so that the platform attachments can be written in the base reference frame as $\mathbf{b}_i = \mathbf{p} + \mathbf{R}\tilde{\mathbf{b}}_i$, for $i = 1, \ldots, 6$ (Fig. 1). To simplify the notation, the same name will be used to denote a point and its position vector.

There are 2 types of parameters that fully define a Stewart-Gough platform. The set of parameters that define the design of the manipulators:

Geometric parameters:

$$\mathcal{G} = (x_1, y_1, z_1, r_1, s_1, t_1, \ldots, x_6, y_6, z_6, r_6, s_6, t_6)$$

and two sets of parameters that can define the location of the manipulator within its workspace:

Pose parameters:

$$\mathbf{X} = (p_x, p_y, p_z, i_x, i_y, i_z, j_x, j_y, j_z, k_x, k_y, k_z)$$

Joint parameters:

$$\mathbf{\Theta} = (l_1, \ldots, l_6)$$

Finally, it will be useful to introduce a 6-dimensional space defined by the coordinates (x, y, z, r, s, t), called *the space of leg attachments*. Each point of this space defines a leg that goes from base attachment $\mathbf{a} = (x, y, z)^T$ to platform attachment $\tilde{\mathbf{b}} = (r, s, t)^T$.

III. Singularity-invariant leg rearrangements

A leg rearrangement consists in a relocation of the attachments of the manipulator, without modifying the pose of the platform, and thus, leading to new leg lengths $d_1, d_2, \ldots, d_6$ (Fig. 1). In general, such rearrangement completely modifies the kinematics of the manipulator and also the location of its singularities, because the solution of the forward kinematics of the rearranged platform changes, which leads to a different number of assembly modes and to a different set of singularities.

Despite this, recently, we have been able to identify leg rearrangements that do not modify the singularity locus of the platform, nor the solution of its forward kinematics. In other words, for the rearranged platform, the location of the singular poses within the workspace of the manipulator remain at the same position. This kind of rearrangements are called *singularity-invariant leg rearrangements*, and where characterized in detail in [34].

In Fig. 1 we show the rearrangement of the leg j, that is, the relocation of the attachments $\mathbf{a}_j$ and $\tilde{\mathbf{b}}_j$ to the new coordinates $\mathbf{a} = (x, y, z)^T$ and $\tilde{\mathbf{b}} = (r, s, t)^T$. In [34], it was shown that such rearrangement is singularity invariant if, and only if, the coordinates (x, y, z, r, s, t) make the matrix $\mathbf{P}$ in (1) to be rank defective. Details of how we obtained such matrix can be found in the appendix.

Note that the first 6 rows of $\mathbf{P}$ contain only geometric parameters of the manipulator, while the last row depends on the coordinates of the new attachments of the rearranged leg.

$$
\mathbf{P} = \begin{pmatrix}
-r_1 & -s_1 & -t_1 & x_1 & y_1 & z_1 & r_1x_1 & r_1y_1 & r_1z_1 & s_1x_1 & s_1y_1 & s_1z_1 & t_1x_1 & t_1y_1 & t_1z_1 & 1 \\
-r_2 & -s_2 & -t_2 & x_2 & y_2 & z_2 & r_2x_2 & r_2y_2 & r_2z_2 & s_2x_2 & s_2y_2 & s_2z_2 & t_2x_2 & t_2y_2 & t_2z_2 & 1 \\
-r_3 & -s_3 & -t_3 & x_3 & y_3 & z_3 & r_3x_3 & r_3y_3 & r_3z_3 & s_3x_3 & s_3y_3 & s_3z_3 & t_3x_3 & t_3y_3 & t_3z_3 & 1 \\
-r_4 & -s_4 & -t_4 & x_4 & y_4 & z_4 & r_4x_4 & r_4y_4 & r_4z_4 & s_4x_4 & s_4y_4 & s_4z_4 & t_4x_4 & t_4y_4 & t_4z_4 & 1 \\
-r_5 & -s_5 & -t_5 & x_5 & y_5 & z_5 & r_5x_5 & r_5y_5 & r_5z_5 & s_5x_5 & s_5y_5 & s_5z_5 & t_5x_5 & t_5y_5 & t_5z_5 & 1 \\
-r_6 & -s_6 & -t_6 & x_6 & y_6 & z_6 & r_6x_6 & r_6y_6 & r_6z_6 & s_6x_6 & s_6y_6 & s_6z_6 & t_6x_6 & t_6y_6 & t_6z_6 & 1 \\
-r & -s & -t & x & y & z & rx & ry & rz & sx & sy & sz & tx & ty & tz & 1
\end{pmatrix}. \quad (1)
$$

The 6 first rows of $\mathbf{P}$ where used in [39], [40] to characterize architectural singularities. With this additional row, we are able to characterize any singularity-invariant leg rearrangement by studying the rank of $\mathbf{P}$.

Gaussian Elimination uses elementary row operations to reduce a given matrix into a rank-equivalent one, with an upper triangular shape. After it is applied to a matrix, rank deficiency occurs when all the elements of the last row are zero. Matrix $\mathbf{P}$ is 7×16 and, if we apply Gaussian Elimination, the last row of the resulting matrix can be expressed as:

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{P}_1 & \dots & \mathbf{P}_{10} \end{pmatrix}, \quad (2)
$$

where $\mathbf{P}_i$, for $i = 1, \dots, 10$, are polynomials in the unknowns (x, y, z, r, s, t), and we can state that $\mathbf{P}$ is rank defective if, and only if, the 10 polynomials do simultaneously vanish.

In conclusion, if any of the legs is relocated to the new attachments $\mathbf{a} = (x, y, z)^T$ and $\tilde{\mathbf{b}} = (r, s, t)^T$, the resulting leg rearrangement is singularity-invariant if, and only if, $\{\mathbf{P}_1 = 0, \dots, \mathbf{P}_{10} = 0\}$.

This is an overdetermined system that has no solution for a generic case. We need to impose at least 5 more scalar equations to obtain a 1-dimensional set of solutions. Next we will see several cases for which matrix $\mathbf{P}$ is simplified and solutions of dimension 1 and 2 are obtained.

IV. Case study I: Doubly-planar Stewart-Gough platforms

For any doubly planar Stewart-Gough platform, the coordinates of the base and platform attachments can be written, without loss of generality, as $\mathbf{a}_i = (x_i, y_i, 0)$ and $\tilde{\mathbf{b}}_i = (z_i, t_i, 0)$. In this case, a leg rearrangement with coordinates (x, y, z, t) stands for the substitution of any of the legs by another one going from the base attachment located at $\mathbf{a} = (x, y, 0)^T$ to the platform attachment at $\mathbf{b} = \mathbf{p} + \mathbf{R}(z, t, 0)^T$. In this case, matrix $\mathbf{P}$ can be simplified to

$$
\mathbf{P} = \begin{pmatrix}
-z_1 & -t_1 & x_1 & y_1 & x_1z_1 & y_1z_1 & x_1t_1 & y_1t_1 & 1 \\
-z_2 & -t_2 & x_2 & y_2 & x_2z_2 & y_2z_2 & x_2t_2 & y_2t_2 & 1 \\
-z_3 & -t_3 & x_3 & y_3 & x_3z_3 & y_3z_3 & x_3t_3 & y_3t_3 & 1 \\
-z_4 & -t_4 & x_4 & y_4 & x_4z_4 & y_4z_4 & x_4t_4 & y_4t_4 & 1 \\
-z_5 & -t_5 & x_5 & y_5 & x_5z_5 & y_5z_5 & x_5t_5 & y_5t_5 & 1 \\
-z_6 & -t_6 & x_6 & y_6 & x_6z_6 & y_6z_6 & x_6t_6 & y_6t_6 & 1 \\
-z & -t & x & y & xz & yz & xt & yt & 1
\end{pmatrix}. \quad (3)
$$

Consider the example with attachment local coordinates appearing in Table I.

To check rank deficiency, Gaussian Elimination is applied to $\mathbf{P}$ with the corresponding numerical values substituted. In this case, the last row of the resulting matrix has only 3 nonzero terms dependent on x, y, z and t. Different but equivalent equations arise depending on the order of the columns. For example, Gaussian Elimination on matrix $\mathbf{P}$ as it appears in equation (3) leads to a matrix whose last row is

$$
\frac{1}{P_{789}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & P_{89} & P_{79} & P_{78} \end{pmatrix},
$$

where P_{ij} is the determinant of the submatrix obtained from $\mathbf{P}$ after deleting columns i and j, and P_{ijk} the determinant of the submatrix formed by the first 6 rows of $\mathbf{P}$ after deleting columns i, j and k. With the corresponding numerical values, $P_{789} = -12180$ and the singularity-invariant leg rearrangements are defined by the condition defined by $\{P_{89} = P_{79} = P_{78} = 0\}$, which reads

$$
\left.\begin{aligned}
-\tfrac{338}{609}xz + xt + \tfrac{3706}{3045}yz + \tfrac{1096}{1015}x - \tfrac{22713}{1015}y & \\
-\tfrac{27743}{3045}z + \tfrac{19302}{1015}t &= 0 \\
-\tfrac{470}{609}xz + \tfrac{10519}{3045}yz + yt + \tfrac{13274}{1015}x - \tfrac{61662}{1015}y & \\
-\tfrac{87557}{3045}z + \tfrac{51343}{1015}t &= 0 \\
\tfrac{17}{609}xz - \tfrac{38}{609}yz - \tfrac{67}{203}x + \tfrac{194}{203}y & \\
+\tfrac{247}{609}z - \tfrac{192}{203}t + 1 &= 0
\end{aligned}\right\} \quad (4)
$$

Note that any equation consisting of a submatrix determinant P_{ij} equated to zero will be bilinear in the unknowns, but with different monomials. As the system is linear, both in (x, y) and in (z, t), it can be rewritten in matrix form as

$$
\mathbf{S}_b \begin{pmatrix} z \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (5)
$$

where $\mathbf{S}_b$ is

$$
\begin{pmatrix}
\frac{-27743}{3045} + \frac{3706}{3045}y - \frac{338}{609}x & x + \frac{19302}{1015} & \frac{22713}{1015}y - \frac{1096}{1015}x \\
\frac{10519}{3045}y - \frac{87557}{3045} - \frac{470}{609}x & y + \frac{51343}{1015} & \frac{61662}{1015}y - \frac{13274}{1015}x \\
\frac{17}{609}x - \frac{38}{609}y + \frac{247}{609} & \frac{-192}{203} & -1 + \frac{67}{203}x - \frac{194}{203}y
\end{pmatrix}
$$

TABLE I

Attachment coordinates ($\mathbf{a}_i = (x_i, y_i, 0)^T$, $\tilde{\mathbf{b}}_i = (z_i, t_i, 0)^T$).

i	x_i	y_i	z_i	t_i
1	3	5	5	6
2	7	9	7	8
3	8	9	9	8
4	12	5	9	6
5	5	2	6	4
6	9	2	9	5

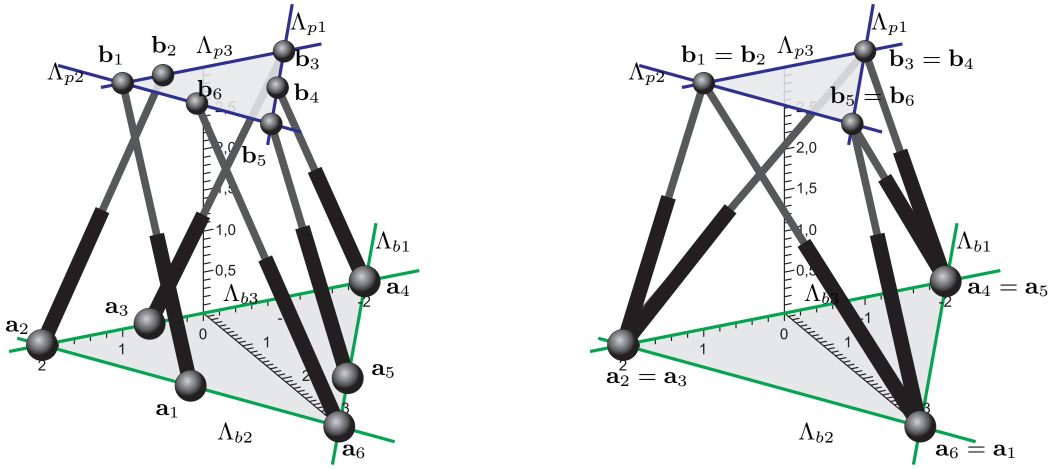

Fig. 4. Griffis-Duffy type I platform with the attachment coordinates given in Table II (left), and its equivalent octahedral manipulator after applying a leg rearrangement (right).

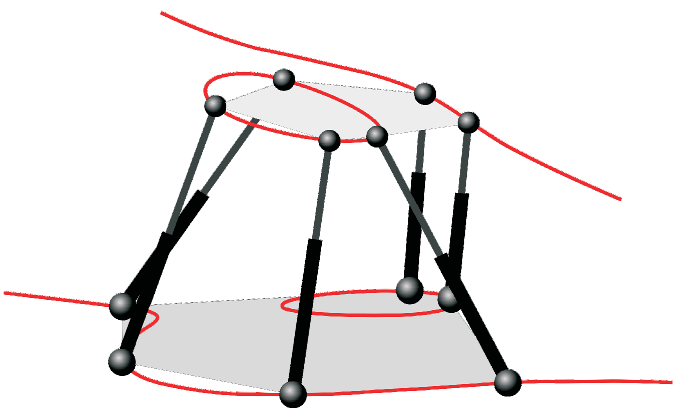

Fig. 2. A general singularity-invariant leg rearrangement for a doubly-planar Stewart-Gough platform.

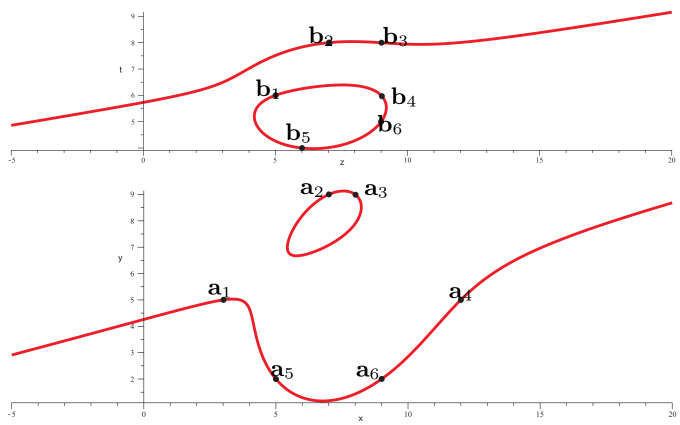

Fig. 3. The base and the platform curves of the doubly-planar Stewart-Gough platform depicted in Fig. 2.

which only depends on x and y (b refers to *base*, as x and y are the coordinates of the base attachments). The other way round, the system can also be written as

$$\mathbf{S}_p \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

where $\mathbf{S}_p$ is

$$\begin{pmatrix} t + \frac{1096}{1015} - \frac{338}{609} z & \frac{-22713}{1015} + \frac{3706}{3045} z & \frac{27743}{3045} z - \frac{19302}{1015} t \\ \frac{13274}{1015} - \frac{470}{609} z & \frac{-61662}{1015} + \frac{10519}{3045} z + t & \frac{87557}{3045} z - \frac{51343}{1015} t \\ \frac{-67}{203} + \frac{17}{609} z & \frac{-38}{609} z + \frac{194}{203} & \frac{-247}{609} z + \frac{192}{203} t - 1 \end{pmatrix}$$

that only depends on z and t (p refers to *platform*, as z and t are the coordinates of the platform attachments).

From equation (5) it is clear that the system has a solution for (z, t) only for those (x, y) that satisfy $\det(\mathbf{S}_b) = 0$, and this solution is unique (assuming that $\mathbf{S}_b$ has rank 2). In the same way, there exists a solution for (x, y) only for those (z, t) that make $\det(\mathbf{S}_p) = 0$. Both determinants define cubic curves on the base and platform planes, respectively. In other words, system (4) defines a one-to-one correspondence between generic points on two cubic curves. However, the correspondence may not be one-to-one for special points on the cubics for non-generic examples (see details in [41]).

For this particular example, the equation of the cubic on the base is

$$\frac{16}{145}x^3 - \frac{293}{609}x^2y + \frac{253}{1015}xy^2 - \frac{142}{609}y^3 + \frac{1061}{3045}x^2 + \frac{4343}{1015}xy + \frac{2313}{1015}y^2 - \frac{17888}{1015}x - \frac{26032}{1015}y + \frac{261691}{3045} = 0,$$

and that on the platform is

$$\frac{9}{145}z^3 - \frac{396}{1015}z^2t + \frac{293}{1015}zt^2 - \frac{192}{203}t^3 + \frac{282}{203}z^2 + \frac{1877}{1015}zt + \frac{2229}{145}t^2 - \frac{17799}{1015}z - \frac{98097}{1015}t + \frac{32922}{145} = 0,$$

which have been plotted in Fig. 3. The curves attached to the manipulator base and platform are shown in Fig. 2.

Depending on the placement of the attachments, these curves can be generic curves of degree 3, or a line and a conic, or even 3 lines crossing 2 by 2. In the next example, one of these degenerate cases is analyzed.

A. An octahedral manipulator implementation

In 1993, Griffis and Duffy patented a manipulator named thereafter Griffis-Duffy platform [42]. The platform has its attachments distributed on triangles, three attachments on the vertexes and three on the midpoints of the edges, and the platform is formed by joining the attachments on the midpoints on the base to the vertexes on the platform, as in the example with attachment coordinates given in Table II. A representation of this manipulator can be found in Fig. 4-(left).

TABLE II
COORDINATES OF THE ATTACHMENTS $\mathbf{a}_i = (x_i, y_i, 0)$ AND $\mathbf{b}_i = \mathbf{p} + \mathbf{R}(z_i, t_i, 0)^T$ FOR THE ANALYZED ROBOT

i	x_i	y_i	z_i	t_i
1	1	$\sqrt{3}$	1	0
2	2	0	$1/2$	0
3	$2/3$	0	-1	0
4	-2	0	$-1/2$	$\sqrt{3}/2$
5	$-2/3$	$(4/3)\sqrt{3}$	0	$\sqrt{3}$
6	0	$2\sqrt{3}$	$1/2$	$\sqrt{3}/2$

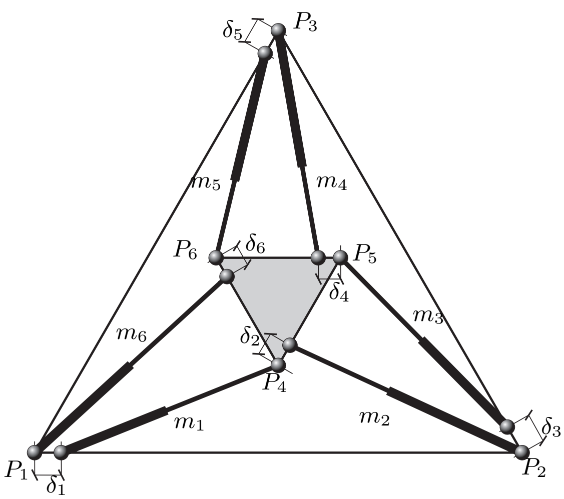

Fig. 5. Contrarily to what happens to the Stoughton-Arai approximation, the proposed modification leads to a 6-6 platform kinematically equivalent to the octahedral manipulator.

In this case, the equation system obtained by applying Gaussian Elimination on the corresponding matrix $\mathbf{P}$ results in :

$$\left.\begin{aligned} 2t - y + yz + xt &= 0 \\ (\sqrt{3}z + t - \sqrt{3})y &= 0 \\ -2\sqrt{3}z + 4t + \sqrt{3}x - y + \sqrt{3}xz + 3yz - 2\sqrt{3} &= 0 \end{aligned}\right\} \tag{7}$$

The resolution of this system gives correspondences between base and platform attachments that leave singularities invariant. The base and platform cubic curves, in this case, factorize into the 3 lines:

$$(\sqrt{3}z - t + \sqrt{3})(\sqrt{3}z + t - \sqrt{3})t = 0,$$

and

$$(-3x + \sqrt{3}y - 6)(3x + \sqrt{3}y - 6)y = 0,$$

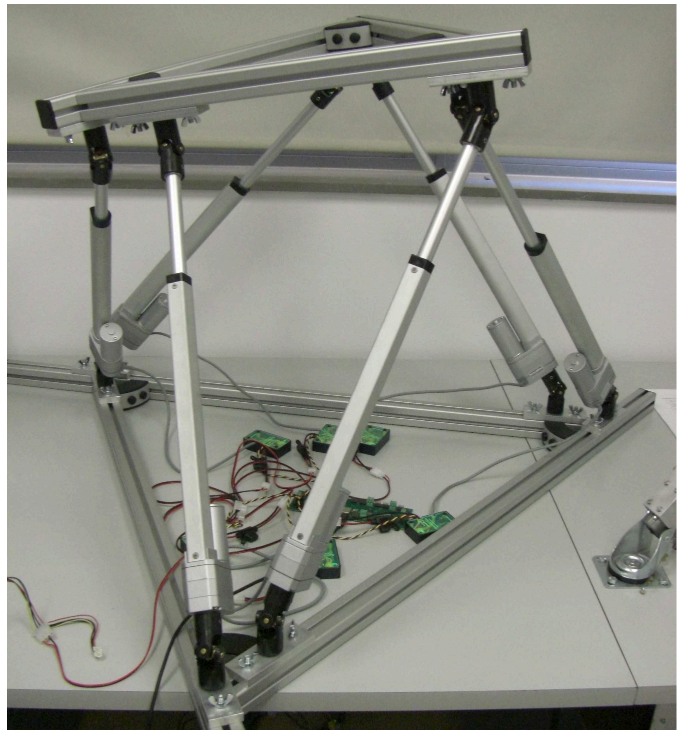

Fig. 6. This platform consists of six extensible legs connecting a moving platform to a fixed base. We avoid the use of multiple spherical joints (that is, spherical joints sharing the same center) without losing the properties of the celebrated octahedral architecture.

respectively.

Actually, it can be checked that system (7) has 6 sets of solutions

$$\begin{aligned} \Delta_{b1} = \{&(x, y, z, t) \mid \\ &x = \lambda, y = (\lambda_1 + 2)\sqrt{3}, z = 0, t = \sqrt{3}; \lambda_1 \in \mathbb{R}\}, \\ \Delta_{b2} = \{&(x, y, z, t) \mid \\ &x = \lambda_2, y = (2 - \lambda_2)\sqrt{3}, z = 1, t = 0; \lambda_2 \in \mathbb{R}\}, \\ \Delta_{b3} = \{&(x, y, z, t) \mid \\ &x = \lambda_3, y = 0, z = -1, t = 0; \lambda_3 \in \mathbb{R}\}, \\ \Delta_{p1} = \{&(x, y, z, t) \mid \\ &x = -2, y = 0, z = \lambda_4, t = \sqrt{3}(\lambda_4 + 1); \lambda_4 \in \mathbb{R}\}, \\ \Delta_{p2} = \{&(x, y, z, t) \mid \\ &x = 0, y = 2\sqrt{3}, z = \lambda_5, t = \sqrt{3}(1 - \lambda_5); \lambda_5 \in \mathbb{R}\}, \\ \Delta_{p3} = \{&(x, y, z, t) \mid \\ &x = 2, y = 0, z = \lambda_6, t = 0; \lambda_6 \in \mathbb{R}\}. \end{aligned}$$

These are 6 point-line correspondences, that is, to each vertex of the base (platform) triangle corresponds a line on the platform (base) triangle. This means that, for the Griffis-Duffy type manipulator, we can fix the attachments at the vertexes of the platform (base), and then rearrange the opposite attachments along a line in the base (platform) without modifying the kinematics of the platform.

As a result, by moving the six midpoint attachments along their supporting lines, the manipulator can be rearranged into the manipulator depicted in Fig. 4-(right), which is the widely known octahedral manipulator. This is an interesting result, because we can avoid the use of multiple spherical joints (that is, spherical joints sharing the same center) without losing the properties of the celebrated octahedral architecture [25].

Following the design in Fig. 5, a manipulator has been constructed in the Laboratory of Parallel Robots, at the Institut de Robòtica i Informàtica Industrial [43] (Fig. 6). Its advantage is that it is a 6-6 manipulator with the same kinematics and singularities as the widely studied octahedral manipulator. We computed the relationship between the legs lengths before and after the rearrangement in Fig. 5, in the form of equation (10) in the appendix, *i. e.*, we obtained $\mathbf{A}$ and $\mathbf{b}$. Given a configuration of the manufactured manipulator, its leg lengths are used to compute the leg lengths of a virtual octahedral using (10). With the new legs lengths, we can solve the kinematics of the octahedral, whose solution will be the pose of the platform (see more details in [44], [45]). The manipulator in Fig. 6 is a practical proof that, indeed, such rearrangement does not change the kinematic solution of the octahedral.

Most, if not all, of the Stewart-Gough platform practical implementations are based on an approximation to the octahedral manipulator, locating the spherical joints close together but avoiding the double spherical joints, and thus, resulting in a different manipulator with a complex kinematic solution. Here we propose a design that is not an approximation, but has the same kinematic properties as the octahedral without any double-spherical joint. This has applications ranging from the well known flight simulators to micro-positioning devices.

V. Case study II: Pentapods

A pentapod is usually defined as a 5-degree-of-freedom fully-parallel manipulator with an axial spindle as moving platform. This kind of manipulators have revealed as an interesting alternative to serial robots handling axisymmetric tools. The moving platform can freely rotate around the axis defined by the five aligned revolute joints, but if this rotation axis is made coincident with the symmetry axis of the tool, the uncontrolled motion becomes irrelevant in most cases. The particular geometry of pentapods permits that, in one tool axis, large inclination angles are possible thus overcoming the orientation limits of the classical Stewart-Gough platform.

A pentapod involves only 5 of the 6 legs of the Stewart-Gough platform, with the platform attachments collinear. This 5 legs form a rigid component by itself that can be studied separately. In addition to the platform attachments collinearity, if we consider all the base attachments coplanar, then we can write the coordinates of the attachments as $\mathbf{a}_i = (x_i, y_i, 0)^T$ and $\tilde{\mathbf{b}}_i = (z_i, 0, 0)^T$ for $i = 1..5$ and the corresponding matrix $\mathbf{P}$ after some simplifications reads

$$\mathbf{P} = \begin{pmatrix} z_1 & x_1 & y_1 & x_1 z_1 & y_1 z_1 & 1 \\ z_2 & x_2 & y_2 & x_2 z_2 & y_2 z_2 & 1 \\ z_3 & x_3 & y_3 & x_3 z_3 & y_3 z_3 & 1 \\ z_4 & x_4 & y_4 & x_4 z_4 & y_4 z_4 & 1 \\ z_5 & x_5 & y_5 & x_5 z_5 & y_5 z_5 & 1 \\ z & x & y & xz & yz & 1 \end{pmatrix}. \quad (8)$$

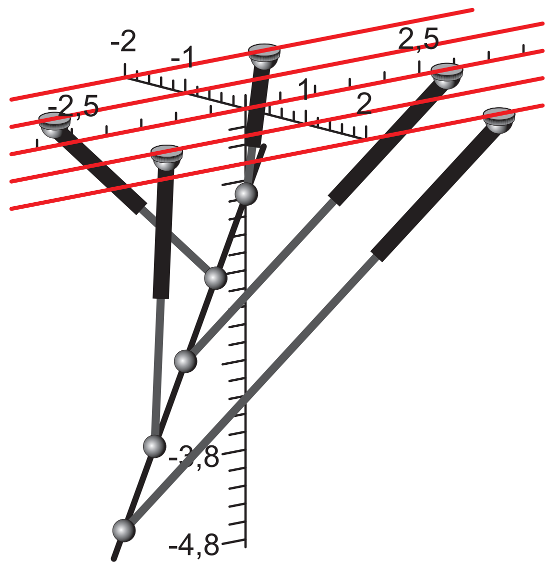

Fig. 7. Pentapod analyzed in Section V. Note that it is in an upside-down configuration, so that the platform is located under the base.

In this case, $\mathbf{P}$ is a square matrix, so its rank deficiency is characterized only by the equation $\det(\mathbf{P}) = 0$. In [48] it was shown that such condition defines a one-to-one correspondence between the platform attachments and the lines of a pencil attached to the base. The center of this pencil, called $\mathcal{B}$-point in [48], [49], plays an important role in the geometric characterization of the manipulator singularities.

Consider the example with numerical coordinates appearing in table V.

TABLE III
Attachments $\mathbf{a}_i = (x_i, y_i, 0)$ and $\tilde{\mathbf{b}}_i = (z_i, 0, 0)$

i	x_i	y_i	z_i
1	-2	2	-2
2	-1	-2	-1
3	0	3	0
4	1	-2	1
5	2	2	2

After substituting the numerical values in $\mathbf{P}$, we get that the condition for singularity invariance is

$$\det(\mathbf{P}) = x - z = 0. \quad (9)$$

This means that any leg can be rearranged to a leg going from the base attachment $\mathbf{a} = (\lambda, y, 0)^T$ to $\tilde{\mathbf{b}} = (\lambda, 0, 0)^T$ without modifying the singularity locus (where for a fixed λ, the y coordinate can take any value). This corresponds to the rearrangements plotted in Fig. 7, that is, a one-to-one correspondence between the attachments at the platform and a pencil of parallel lines attached at the base. In this case, the center of the pencil lies at infinity.

This particular architecture was proven to be quadratically solvable in [50], [48], that is, its forward kinematics can be obtained by solving only 2 quadratic polynomials. If we fix the attachments of the platform, the corresponding base attachments can be relocated to any point of the red

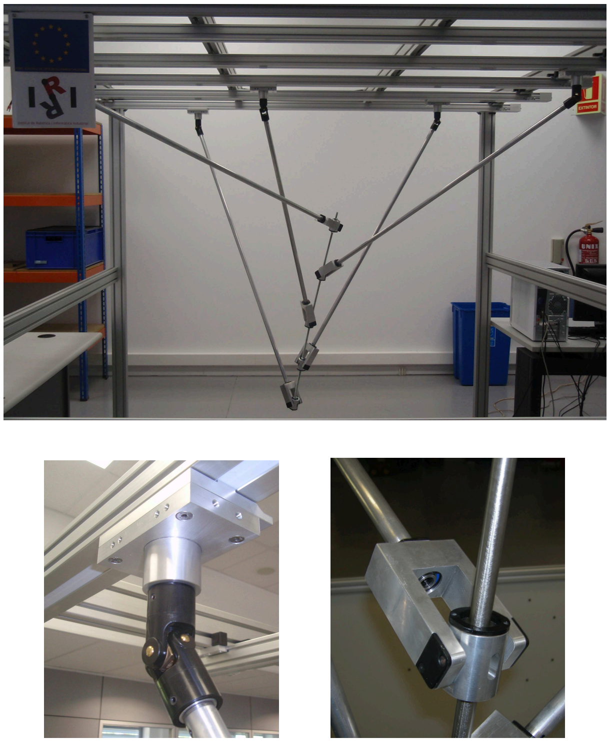

Fig. 8. Prototype of the reconfigurable quadratically-solvable pentapod and its joint implementations.

lines plotted in Fig. 7. Taking advantage of this idea, at the Laboratory of Parallel Robots of IRI we have developed a reconfigurable manipulator prototype based on this structure. Its base attachments can be reconfigured along actuated guides, without modifying the nature of its forward kinematics nor the singularities of the manipulator, and thus increasing the versatility of the manipulator, since for each task, the legs can be reconfigured to equally distribute the forces among its legs (Fig. 8).

This pentapod design can be useful to enlarge the workspace of robots handling axisymmetric tools, with applications such as 5-axis milling, laser-engraving, spray-based paintins and water-jet cutting.

VI. Case study III: A decoupled Stewart-Gough platform

Consider the manipulator in Fig. 9. It contains a tripod and 3 more legs, with all the base attachments coplanar. Thus, without loss of generality, we can write the coordinates of the attachments as $\mathbf{a}_i = (x_i, y_i, 0)^T$ and $\tilde{\mathbf{b}}_i = (r_i, s_i, t_i)^T$. This manipulator is said to be decoupled because the three legs forming the tripod give the position of the platform, while the three remaining ones orient it [23]. When the tripod is rigid, *i. e.*, fixed at a position, this manipulator is also known as spherical [46], [47].

Consider the example with numeric coordinates appearing in Table IV. After performing Gaussian Elimination on the corresponding matrix $\mathbf{P}$, only six non-zero elements remain at the last row. That is, a leg rearrangement will be singularity-

TABLE IV

Attachment coordinates $\mathbf{a}_i = (x_i, y_i, 0)$ and $\mathbf{b}_i = \mathbf{p} + \mathbf{R}(r_i, s_i, t_i)^T$

i	x_i	y_i	r_i	s_i	t_i
1	2	-1	2	2	0
2	5	4	2	2	0
3	-1	4	2	2	0
4	7	-2	5	0	1
5	2	7	2	5	1
6	-3	-2	-1	0	1

invariant if it fulfills the following 6 conditions

$$\begin{aligned}
-2xr + yr + 4x - 2y + 6r - 6s + 18t &= 0,\\
-4xr/3 + xs + 2x/3 + 6r - 6s + 12t &= 0,\\
1/5(17xr + ys - 34x - 10y - 34r + 34s - 207t) &= 0,\\
5xr/3 + xt - 10x/3 - 5r + 5s - 17t &= 0,\\
9xr/5 + yt - 18x/5 - 18r/5 + 18s/5 - 89t/5 &= 0,\\
-1xr/2 + x + r - 3s/2 + 9t/2 + 1 &= 0.
\end{aligned}$$

This system of equations has 4 sets of solutions:

$$\begin{aligned}
\mathcal{T} = \{&(x, y), (r, s, t) \mid\\
&x = \lambda, y = \mu; r = 2, s = 2, t = 0, \lambda, \mu \in \mathbb{R}\},\\
\Delta_1 = \{&(x, y), (r, s, t) \mid\\
&x = 2, y = 7; r = 2, s = 2 + 3\lambda, t = \lambda, \lambda \in \mathbb{R}\},\\
\Delta_2 = \{&(x, y), (r, s, t) \mid x = 7, y = -2;\\
&r = 5 - 3\lambda/2, s = \lambda, t = 1 - \lambda/2, \lambda \in \mathbb{R}\},\\
\Delta_3 = \{&(x, y), (r, s, t) \mid x = -3, y = -2;\\
&r = 2 - 3\lambda, s = 2 - 2\lambda, t = \lambda, \lambda \in \mathbb{R}\}.
\end{aligned}$$

The first one corresponds to the tripod component and it means that base attachments can be rearranged to any point of the base plane as long as its corresponding platform attachment is the vertex of the tripod. The other 3 sets correspond to point-line correspondences as before, depicted as red lines in Fig. 9. This means that $\mathbf{b}_4$, $\mathbf{b}_5$ and $\mathbf{b}_6$ can be relocated to any other point of the red lines (as long as their corresponding base attachments remain the same).

In Fig. 10 we show two possible singularity-invariant leg rearrangements of the manipulator at hand. For all of them, the decoupling properties remain the same as they are all equivalent manipulators.

Note that with this strategy we cannot completely eliminate all the multiple spherical joints. But we can design a decoupled manipulator with only single spherical joints by imposing extra alignments on the attachments. For example, consider the manipulator in Fig. 11. It is still decoupled but, in this case, it contains a Line-Plane component. As mentioned in the preceding section, for the 5 legs forming the Line-Plane component, there exists a one-to-one relationship between the collinear attachments (the Line) and a pencil of lines (at the Plane), whose center is called $\mathcal{B}$-point. In this case, the $\mathcal{B}$-point is made coincident with the attachment $\mathbf{b}_1$ and the pencil

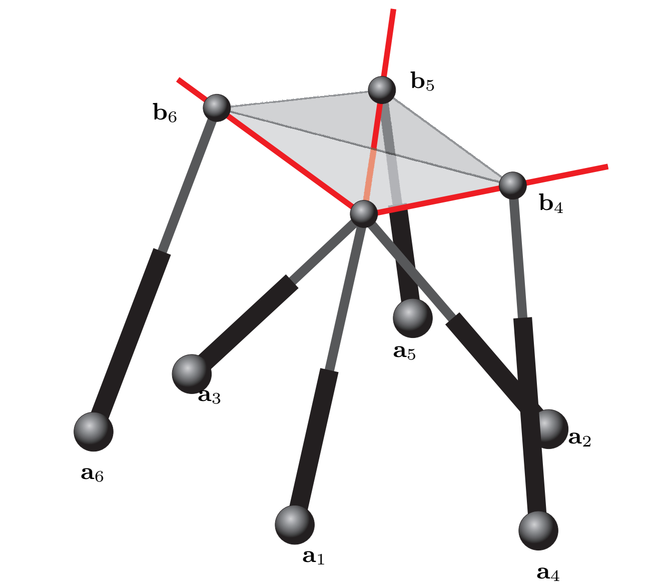

Fig. 9. A decoupled manipulator with non-planar platform. In blue, its singularity-invariant leg rearrangement lines.

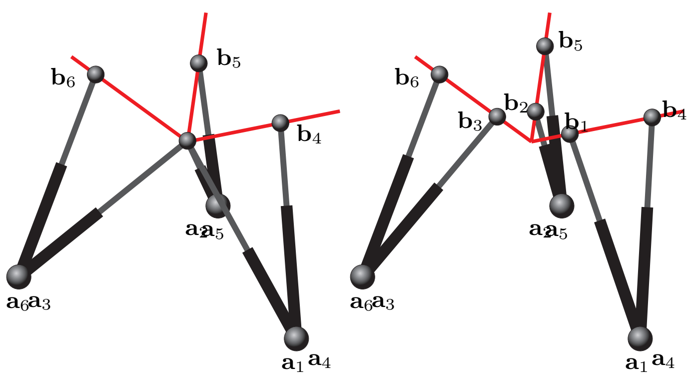

Fig. 10. Singularity-invariant leg rearrangements from the example in Fig. 9.

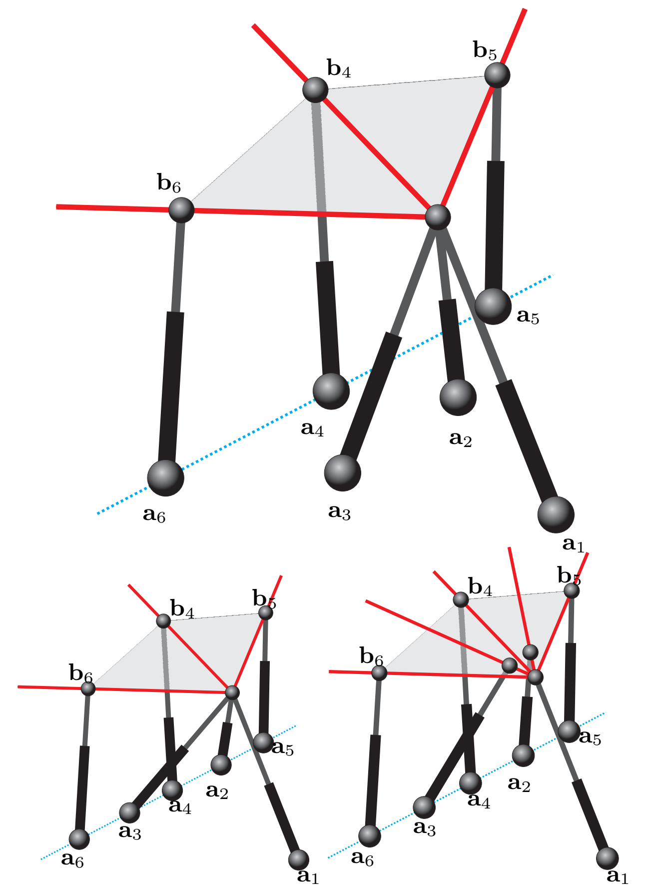

Fig. 11. A decoupled manipulator with a collinearity at the base and coplanar platform and base. Below, the rearrangement to an equivalent manipulator with only single spherical joints.

of lines is located attached to the platform. We can split the triple spherical joint by placing two more attachments collinear with $\mathbf{a}_4$, $\mathbf{a}_5$ and $\mathbf{a}_6$ and then moving the platform attachment along their corresponding lines of the pencil (see [48] for more details).

A manipulator with decoupled position and orientation has many advantages. For example, the calibration becomes simpler because the translation and rotation become 3-dimensional independent functions instead of a complex 6-dimensional one. It also simplifies path planning, for example, in cooperation tasks between manipulators. As a drawback, the designs presented to date have a complex implementation and small range of motion due to joint limits. Our proposed architecture is simpler than any other decoupled manipulator presented before, in the sense that it only contains single spherical joints. But it can benefit from the kinematic decoupling properties, as all the computations can be performed for the original decoupled manipulator and be used for the rearranged one. In other words, we have designed a decoupled parallel manipulator easier to implement that avoids the common drawbacks of multiple spherical joints.

VII. Conclusions

The present work shows how the application of singularity-invariant leg rearrangements provides a new geometric approach to the study of Stewart-Gough platform singularities. Indeed, three case studies have been provided that illustrate several new results. The background theory is based on the mathematical invariance of type 2 singularities, as no other type of singularities appear for Stewart-Gough platforms. Nevertheless, it is also extensible to type 1 singularities. To apply it to lower mobility parallel platforms, a more thoughtful mathematical definition of singularity invariance may be needed to consider constrain singularities [51].

We have presented a tool to detect equivalences between manipulators, allowing the application of previously known geometric interpretations of singularities to new architectures. This is the case of the Griffis-Duffy platform in Section IV. The 6-6 Stewart-Gough platform prototype shown in Fig. 6 has the same kinematic properties as the octahedral manipulator, that is, the same geometric interpretation for its singularities applies, as well as all other kinematic properties studied in the extensive literature about the octahedral manipulator, with a crucial advantage: the new architecture contains only single spherical joints.

We have also shown how decoupled manipulators can be rearranged to equivalent and apparently non-decoupled manipulators, with different configurations of their spherical joints that might be easier to implement.

Moreover, the hidden geometric structure revealed by the curves of singularity-invariant leg rearrangements can be of help in the simplification of the forward kinematics resolution. For example, in the case study II, we show a manipulator that is quadratically solvable.

Finally, new geometric interpretations of singularities have been found thanks to singularity-invariant leg rearrangements. For example, for pentapods with planar bases, the identified pencil of lines at the base of the manipulator reveals to be crucial for the geometric interpretation of its singularities. Similar interpretations represent a challenge for future work.

In conclusion, this indirect approach to the analysis of Stewart-Gough platform singularities has succeeded in finding new results in a topic with an extensive previous literature.

APPENDIX

In this appendix we mathematically define the notion of singularity-invariance and we give the derivation of matrix $\mathbf{P}$ in equation (1).

Singularities are defined as the zeros of the determinant of the Jacobian matrix. The Jacobian matrix relates the velocities of the joints with the twist of the platform $\mathcal{T}$ in the well known equation $\mathbf{J}\mathcal{T} = \dot{\Theta}$ [17].

We define a leg rearrangement as singularity-invariant if, and only if, there is an affine one-to-one relationship between the leg lengths before and after such rearrangement. Such relationship can always be computed independently of the pose parameters, *i.e.*, it is constant with respect to time. More formally, if $\Lambda = (d_1, ..., d_6)$ are the lengths of the legs after the rearrangement and $\boldsymbol{\Theta} = (l_1, \ldots, l_6)$ the original ones, we can write

$$\Lambda = \mathbf{A}\Theta + \mathbf{b}, \tag{10}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ a constant vector. Indeed, differentiating with respect to time the above equation gives a linear relationship between the joint velocities before and after the rearrangement. Substituting such linear relationship in the equation $\mathbf{J}\mathcal{T} = \dot{\Theta}$ leads to $\mathbf{A}\mathbf{J}\mathcal{T} = \dot{\Delta}$. In other words, the Jacobian matrix of the rearranged platform is $\mathbf{A}\mathbf{J}$, whose determinant has the same zeros as $\mathbf{J}$, *i.e.*, the same singularities as the original platform. See [34] for details.

In practice, we perform rearrangements of only one leg at a time, as any sequence of singularity-invariant leg rearrangements is also singularity-invariant. In Fig. 1 we show the rearrangement of the leg j, that is, the relocation of the attachments $\mathbf{a}_j$ and $\tilde{\mathbf{b}}_j$ to the new coordinates $\mathbf{a} = (x, y, z)^T$ and $\tilde{\mathbf{b}} = (r, s, t)^T$. Such rearrangement will be singularity invariant if, and only if, the length of the new relocated leg is uniquely determined by the geometry parameters and the joint parameters. In [34], it was shown that such rearrangement is singularity invariant if, and only if, the coordinates (x, y, z, r, s, t) make the matrix $\mathbf{P}$ in (1) to be rank defective. The key of the derivation of this matrix is based on the computation of the length of the new relocated leg, d, and on the consideration of under which conditions it can be computed independently of the pose parameters. For space reasons, we only give a sketch of the proof.

By definition, leg lengths satisfy

$$(\mathbf{b}_i - \mathbf{a}_i)^2 = l_i^2, \text{ for } i = 1, \ldots, 6 \text{ and}$$
$$(\mathbf{b} - \mathbf{a})^2 = d^2,$$

where the geometric and joint parameters are considered given and the unknowns are the pose parameters and d. This quadratic system of equations can be converted into a linear system by simplifying all the equations, using the properties of the orthogonality and determinant equal to 1 of the rotation matrix $\mathbf{R}$ and introducing new variables $u = \mathbf{p}\cdot\mathbf{i}$, $v = \mathbf{p}\cdot\mathbf{j}$ and $w = \mathbf{p}\cdot\mathbf{k}$. The only quadratic terms in the resultant system are the same 3 terms in all the equations, $p_x^2 + p_y^2 + p_z^2$, which can be eliminated subtracting the first equation from all the others. The result is a linear system of 6 equations in 16 unknowns: the 12 pose parameters, the 3 new variables $\{u, v, w\}$ and the length we want to compute, d. We define the vector of unknowns as $\chi = \{p_x, p_y, p_z, i_x, i_y, i_z, j_x, j_y, j_z, k_x, k_y, k_z, u, v, w\}$.

We want to compute d, so we choose 5 extra unknowns from the list χ to solve the linear system using Crammer's rule. As a result, we obtain an algebraic expression depending on the other 10 left unknowns in the form

$$d = \frac{1}{c_{02}}(c_{01} + \sum_{i=1}^{10} c_i \chi[i]) \tag{11}$$

where $\chi[i]$ is the ith element from the list of non-chosen unknowns χ. It can be shown that for a non architecturally singular manipulator, we can always choose a set of unknowns to solve the linear system so that $c_{02} \neq 0$.

To guarantee that such expression does not depend on any of the unknowns, we need to impose the 10 coefficients c_i to be zero. It can be checked that such 10 coefficients are 10 maximal minors[1] of the matrix $\mathbf{P}$. Depending on which are the variables for which one chooses to solve the system, different combinations of minors appear but all from the same matrix. Finally, using linear algebra properties, it can be seen that such minors will vanish if, and only if, the matrix $\mathbf{P}$ is rank defective.

The ten coefficients c_i are expressions depending only on geometric parameters and the coordinates of the new attachments. In fact, they are the same 10 polynomials that we obtain when we study the rank deficiency of the matrix $\mathbf{P}$ in equation (2).

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[1] A maximal minor of a matrix $\mathbf{A}$ is the determinant of a square sub-matrix cut down from $\mathbf{A}$, with the maximum size you can obtain removing one or more of its rows or columns.

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