Periodic Economic Model Predictive Control with Nonlinear-Constraint Relaxation for Water Distribution Networks

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***Abstract*— In this paper, a periodic economic model predictive control (EMPC) strategy with nonlinear algebraic constraint relaxations for water distribution networks (WDNs) is presented. A WDN is usually modeled by a series of differential-algebraic equations. When the hydraulic pressure/head and flow relations in the interconnected pipes are considered, the nonlinear algebraic equations will appear in the control-oriented model of WDNs. Specifically, two types of nonlinear algebraic equations are studied in terms of unidirectional and bidirectional flows in pipes. These nonlinear algebraic constraints are iteratively relaxed by a series of linear constraints. Therefore, the proposed EMPC strategy can be implemented by solving an optimization problem using the linear programming technique. Finally, the EMPC strategy with nonlinear algebraic constraint relaxations is verified in the Richmond water network. The comparison results of applying nonlinear EMPC strategy are also provided. The proposed nonlinear-constraint relaxation technique turns out to be much faster than the one obtained by a standard nonlinear optimization solver.**

I. INTRODUCTION

Water distribution networks (WDNs) are designed to supply water to consumers. They generally contain water tanks, a set of pressurized pipes, pumping stations or booster pumps, pressure reducing valves and consumption points. The configuration of the network may have a complex meshed layout to achieve service with a convenient geographical and topological distribution. Moreover, these networks have been built incrementally with urban development and may contain pipes of different material, length and diameter.

Basically, the dynamics of WDNs can be described by differential-algebraic-equation (DAE) systems that are also known as singular or descriptor systems [1]. A DAE system has two types of equations: differential and algebraic equations, where differential equations mainly describe the system dynamics and algebraic equations include the static relations. As presented in [1], the control-oriented model of WDNs is built by DAEs, where the algebraic equations contain linear and nonlinear ones. The nonlinear algebraic equations express the hydraulic pressure/head-flow relations of the interconnected pipes inside the WDNs. According to the pipe usage, there are two types: the unidirectional pipe in which the water is always transported in one direction and the bidirectional pipe in which the water flow may be reversed.

Model predictive control (MPC) [2], [3] offers a flexible and effective framework for a large number of engineering applications and has been investigated for the operational management of WDNs [4], [5], [6], [7]. Among different MPC strategies, economic MPC (EMPC) has attracted a lot of attention in recent years. It differs from the classical MPC strategy based on tracking a given reference, in that an economic performance index is optimized. The optimal control actions of EMPC are often found by means of an economic cost function that measures the performance in a control horizon. Hence, the cost function of EMPC is usually not set in a quadratic form but in a time-varying manner usually depending on an exogenous price signal.

The main contribution of this paper is to provide an iterative algorithm for relaxing the nonlinear algebraic equality constraints of WDNs and subsequently the relaxed algebraic constraints are implemented in the EMPC controller design with periodic operation. In order to guarantee the stability of EMPC, the terminal equality constraint is employed with the optimal steady states found by a nonlinear EMPC planner. Finally, the proposed iterative algorithm and the EMPC strategy are tested in the Richmond water network case study. The EMPC strategy with nonlinear algebraic constraint relaxations is also compared with the nonlinear EMPC strategy and the performance of relaxed constraints is discussed through some defined key performance indicators (KPIs).

The remainder of this paper is structured as follows. In Section II, the problem formulation is presented. The iterative algorithm of nonlinear constraint relaxations is introduced in Section III. The periodic EMPC strategy with relaxed constraints is designed in Section IV. In Section V, the Richmond water network case study is chosen to test the proposed EMPC strategy with nonlinear constraint relaxations. Finally, some conclusions are addressed in Section VI.

II. PROBLEM FORMULATION

The WDN may be modeled as a discrete-time DAE system defined by [1]

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}_u\mathbf{u}(k) + \mathbf{B}_v\mathbf{v}(k) + \mathbf{B}_d\mathbf{d}(k), \quad (1a)$$

$$\mathbf{0} = \mathbf{E}_x\mathbf{x}(k) + \mathbf{E}_u\mathbf{u}(k) + \mathbf{E}_v\mathbf{v}(k) + \mathbf{E}_d\mathbf{d}(k), \quad (1b)$$

$$\mathbf{0} = \mathbf{P}_x\mathbf{x}(k) + \mathbf{P}_z\mathbf{z}(k) + \psi\left(\mathbf{v}(k)\right), \quad (1c)$$

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where $\mathbf{x}(k) \in \mathbb{R}^{n_x}$ denotes the vector of hydraulic heads of the storage water tanks as system states at time instant k. $\mathbf{u}(k) \in \mathbb{R}^{n_u}$ and $\mathbf{v}(k) \in \mathbb{R}^{n_v}$ denote the vectors of manipulated and non-manipulated flows through actuators and interconnected pipes at time instant k, respectively. $\mathbf{d}(k) \in \mathbb{R}^{n_d}$ denotes the vector of water demands at time instant k. $\mathbf{z}(k) \in \mathbb{R}^{n_z}$ denotes the vector of hydraulic heads of the non-storage nodes at time instant k. $\mathbf{A}$, $\mathbf{B}_u$, $\mathbf{B}_v$, $\mathbf{B}_d$, $\mathbf{E}_x$, $\mathbf{E}_u$, $\mathbf{E}_v$, $\mathbf{E}_d$, $\mathbf{P}_x$ and $\mathbf{P}_z$ are time-invariant system matrices of appropriate dimensions. $\psi(\cdot) \in \mathbb{R}^{n_e}$ represents the collection with n_e-dimension of nonlinear mapping functions describing the relationship between the hydraulic head and the water flow through the interconnected pipes.

Assumption 1: From the predictive control viewpoint, a short-term demand forecasts are required along the prediction horizon. In this paper, the predicted water demands are assumed to be known by using a suitable forecasting algorithm, for instance the demand forecasting algorithm proposed in [8].

Definition 1: The DAE system in (1) is called under the T-periodic operation with respect to the known algebraic variable $\mathbf{d}(k) = \mathbf{d}(k+T)$ and exogenous signal in economic cost function $\mathbf{p}(k) = \mathbf{p}(k+T)$ if there exist a period $T \in \mathbb{Z}_{\geq 1}$ such that for all $k \in \mathbb{N}$ it holds that $\mathbf{x}(k) = \mathbf{x}(k+T)$.

In terms of the nonlinear algebraic equations in the vector of (1c), the head-flow relationship for a pipe can be written as [9]

$$\alpha_i v_i \left|v_i\right|^{\beta-1} + \Delta h_i = 0, \tag{2}$$

where $\alpha_i \in \mathbb{R}_+$ denotes the known parameter for the i-th nonlinear equation, β denotes the power factor with the condition of $\beta > 1$, $v_i \subset \mathbf{v}$ denotes a non-manipulated water flow and Δh_i denotes the i-th row in the combination of $\mathbf{P}_x\mathbf{x} + \mathbf{P}_z\mathbf{z}$.

As discussed in the introduction, there are pipes in WDNs that are used in a unidirectional or bidirectional way. For the unidirectional pipe, (2) can be reformulated as

$$\alpha_i v_i^{\beta} + \Delta h_i = 0, \tag{3}$$

with the constraint on v_i of

$$0 \leq v_i \leq v_{i,\max}. \tag{4}$$

In order to relax the nonlinear algebraic equations of WDNs in (2) as a series of linear inequality constraints by an iterative over-bounding algorithm, the linear inequality constraints can be written as follows:

$$\widetilde{\mathbf{P}}_x^r\mathbf{x}(k) + \widetilde{\mathbf{P}}_z^r\mathbf{z}(k) + \widetilde{\mathbf{P}}_v^r\mathbf{v}(k) + \widetilde{\mathbf{P}}_b^r \geq 0, \tag{5a}$$
$$\widetilde{\mathbf{P}}_x^l\mathbf{x}(k) + \widetilde{\mathbf{P}}_z^l\mathbf{z}(k) + \widetilde{\mathbf{P}}_v^l\mathbf{v}(k) + \widetilde{\mathbf{P}}_b^l \leq 0, \tag{5b}$$

where $\widetilde{\mathbf{P}}_x^r$, $\widetilde{\mathbf{P}}_z^r$, $\widetilde{\mathbf{P}}_v^r$, $\widetilde{\mathbf{P}}_b^r$, $\widetilde{\mathbf{P}}_x^l$, $\widetilde{\mathbf{P}}_z^l$, $\widetilde{\mathbf{P}}_v^l$ and $\widetilde{\mathbf{P}}_b^l$ denote the matrices of appropriate dimensions obtained by using the iterative algorithm presented in Section III.

Besides, the defined variables in (1) are constrained with their minimums and maximums as follows:

$$\mathbf{x}_{\min} \leq \mathbf{x}(k) \leq \mathbf{x}_{\max}, \tag{6a}$$
$$\mathbf{u}_{\min} \leq \mathbf{u}(k) \leq \mathbf{u}_{\max}, \tag{6b}$$
$$\mathbf{v}_{\min} \leq \mathbf{v}(k) \leq \mathbf{v}_{\max}, \tag{6c}$$
$$\mathbf{z}_{\min} \leq \mathbf{z}(k) \leq \mathbf{z}_{\max}. \tag{6d}$$

The objective function of the EMPC strategy defined in (1) is given by $\ell(\mathbf{x}(k), \mathbf{u}(k), \mathbf{p}(k))$ with respect to an exogenous signal $\mathbf{p}(k)$ describing the operational cost. This function measures the economic performance of WDNs. For a WDN, the water demands $\mathbf{d}(k)$ are usually considered periodic depending on the daily consumptions and the electricity price signals $\mathbf{p}(k)$ also behave daily periodic. Hence, the WDN is desired to be operated by a periodic control strategy.

Generally speaking, the periodic EMPC strategy for WDNs can be implemented by solving the following H_p-horizon optimization problem $\mathbb{P}_T^g(k)$:

$$\min_{\mathbf{x}^*(k),\mathbf{u}^*(k)} \sum_{k=0}^{Hp-1} \ell(\mathbf{x}(k+1), \mathbf{u}(k), \mathbf{p}(k)), \tag{7}$$

subject to (1a), (1b), (5), (6) and a periodic terminal equality constraint defined by

$$\mathbf{x}(k+H_p) = \mathbf{x}_s(\langle k+H_p\rangle_T), \tag{8}$$

where $\mathbf{x}_s(\langle k+H_p\rangle_T)$ denotes the T-periodic optimal steady states that will be found by an optimal EMPC planner.

III. The Iterative Algorithm of Nonlinear Algebraic Constraint Relaxations

A. Nonlinear Algebraic Constraint Relaxation for Unidirectional Pipes

The nonlinear algebraic equation of unidirectional pipes (3) is equivalent to the satisfaction of the two following inequalities:

$$\alpha_i v_i^{\beta} + \Delta h_i \geq 0, \tag{9a}$$
$$\alpha_i v_i^{\beta} + \Delta h_i \leq 0, \tag{9b}$$

in which v_i^{β} is a convex function due to $\beta > 1$. Therefore, (9a) can be relaxed with (4) as

$$\alpha_i v_{i,\max}^{\beta-1} v_i + \Delta h_i \geq 0. \tag{10}$$

Besides, the constraint (9b) can be replaced by N_a sampled operating points $v_{i,j}^{\star}$ for $j = 1, 2, \ldots, N_a$ defined by the following inequality:

$$\alpha_i v_i^{\beta} + \Delta h_i \leq \alpha_i(a_j v_i + b_j) + \Delta h_i \leq 0, \tag{11}$$

in which parameters of a_j and b_j are given by

$$a_j = \beta {v_{i,j}^{\star}}^{\beta-1}, \tag{12a}$$
$$b_j = (1-\beta){v_{i,j}^{\star}}^{\beta}. \tag{12b}$$

In general, the nonlinear algebraic equation for a unidirectional pipe (3) can be relaxed by using N_a+1 inequality

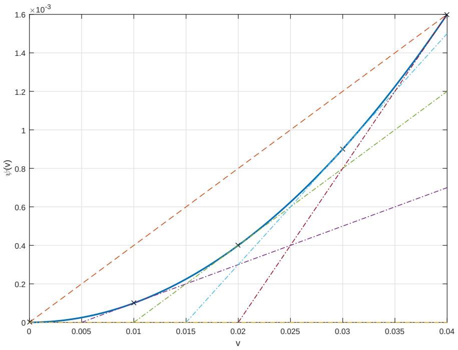


Fig. 1. Nonlinear algebraic constraint relaxation for unidirectional pipes: original constraint is plotted in blue bold line, upper bound is shown in dashed line and lower bounds are shown in dashed dotted lines.

constraints as presented in (10) and (11). Fig. 1 shows the computational result of an example of v^β.

If the solution of the relaxed linear constraints does not satisfy the original nonlinear equations, the relaxations can be refined in an iterative way. The iterative algorithm of the constraint relaxations can be realized in two ways: adding a penalty term or refining the region of v_i. The upper bound can be moved using a small positive value τ and the argument region is refined as $\left[v^a, v^b\right] \subseteq [0, v_{\max}]$. In order to improve the relaxation performance, one of the following ways can be considered:

1) Adding a Penalty Term: Consider a slack variable τ_i, (10) can be replaced by

$$\alpha_i v_{i,\max}^{\beta-1} v_i + \Delta h_i - \tau_i \geq 0, \tag{13a}$$
$$\tau_i \geq 0, \tag{13b}$$

where the objective is to find a minimum τ_i at each time step such that the relaxation constraints have a solution. Thus, the penalty cost function for $\tau_i(k+j)$ at time step $k+j$ along the MPC prediction horizon H_p can be written as

$$\ell^e(\tau_i(k+j)) \triangleq \lambda^e(j)\tau_i(k+j), \tag{14}$$

where $\lambda^e(j)$ denotes the weight of this objective for $j = 1, 2, \ldots, H_p$. Note that this weight can be set as a forgetting (monotonically decreasing) factor along H_p.

2) Refining the Argument Region: Since the relaxed constraints will be implemented into the MPC open loop, it can be regarded as an iterative algorithm. Along the prediction horizon, the previous solution $v_i(k-1)$ can be obtained. Therefore, the bounds of the refined region can be found by

$$v_i^a(k) = v_i(k-1) - \delta_i, \tag{15a}$$
$$v_i^b(k) = v_i(k-1) + \delta_i, \tag{15b}$$

where δ_i denotes the parameter that decides the size of the refined region.

B. Nonlinear Algebraic Constraint Relaxation for Bidirectional Pipes

The nonlinear algebraic equation of bidirectional pipes (2) is equivalent to

$$\alpha_i v_i \left|v_i\right|^{\beta-1} + \Delta h_i \leq 0, \tag{16a}$$
$$\alpha_i v_i \left|v_i\right|^{\beta-1} + \Delta h_i \geq 0, \tag{16b}$$

where these two inequality constraints are not convex for $v_{i,\min} \leq v_i \leq v_{i,\max}$. Hence, they cannot be relaxed directly.

To deal with (16a), the following inequality is required:

$$a_j^l v_i + b_j^l \leq v_i \left|v_i\right|^{\beta-1}, \tag{17}$$

where $v_{i,\min} \leq v_i \leq v_{i,\max}$. For a given a_j, b_j should satisfy the following inequality:

$$b_j^l \leq \min_{v_{\min} \leq v_i \leq v_{\max}} (v_i \left|v_i\right|^{\beta-1} - a_j^l v_i). \tag{18}$$

The minimum a_1^l for $v_{i,\min} \leq v_i \leq v_{i,\max}$ can be determined by $a_1^l = \beta {v_i^\star}^{\beta-1}$ and $v_i^\star$ is required to satisfy the following condition:

$$\beta {v_i^\star}^{\beta-1} = \frac{{v_i^\star}^2 + {v_{i,\min}}^2}{v_i^\star - v_{i,\min}}, \tag{19}$$

and the parameter b_j^l can be obtained by

$$b_1^l = {v_i^\star}^\beta - a_j^l v_i^\star. \tag{20}$$

Besides, for $v_i^\star \leq v_i \leq v_{i,\max}$, the constraints can be complemented by N_b linearized inequality with sampled operating points. Hence, there are $N_b + 1$ linear constraints for bounding the nonlinear constraint (16a).

In terms of (16b), it is also necessary to add the following inequality:

$$a_j^r v_i + b_j^r \geq v_i \left|v_i\right|^{\beta-1}, \tag{21}$$

Because $v_i \left|v_i\right|^{\beta-1}$ is symmetric with respect to its augment region, a_1^r can be computed as (19) and b_1^r can be computed as

$$b_1^r = v_i^\star(-v_i^\star)^{\beta-1} + a_j^r v_i^\star. \tag{22}$$

The other N_b constraints for $v_{i,\min} \leq v_i \leq v_i^\star$ with $v_i^\star < 0$ can be also computed with the assigned operating points.

Fig. 2 shows the results obtained applying the proposed nonlinear constraint relaxation method. In general, the nonlinear algebraic equation of bidirectional pipes (2) can be relaxed by $2N_b + 2$ linear inequality constraints.

C. Iterative Algorithm

The relaxed nonlinear algebraic equations for both unidirectional and bidirectional pipes will be implemented into the EMPC controller design. Therefore, the nonlinear constraint relaxation approach is considered as an iterative algorithm along the MPC prediction horizon H_p.

For nonlinear constraint relaxation of unidirectional pipes, adding the penalty term is selected as an additional slack decision variable while refining the argument region is required to find a suitable value δ_i for each v_i that can be found by using a heuristic method in a line search.

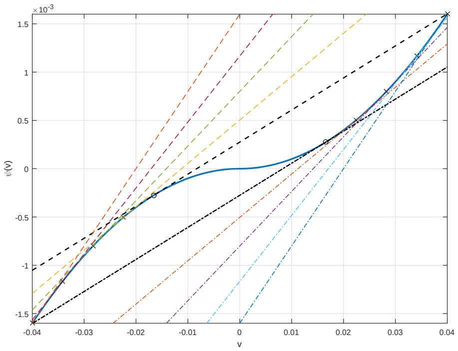


Fig. 2. Nonlinear algebraic constraint relaxation for bidirectional pipes: original constraint is plotted in blue bold line, upper bounds are shown in dashed line and lower bounds are shown in dashed dotted lines.

As a result, the relaxed constraints along the MPC prediction horizon can be formulated as follows:

$$\widetilde{\mathbf{P}}^r_x(j)\mathbf{x}(k+j) + \widetilde{\mathbf{P}}^r_z(j)\mathbf{z}(k+j) + \widetilde{\mathbf{P}}^r_v(j)\mathbf{v}(k+j) + \widetilde{\mathbf{P}}^r_b(j) \geq 0, \tag{23a}$$

$$\widetilde{\mathbf{P}}^l_x(j)\mathbf{x}(k+j) + \widetilde{\mathbf{P}}^l_z(j)\mathbf{z}(k+j) + \widetilde{\mathbf{P}}^l_v(j)\mathbf{v}(k+j) + \widetilde{\mathbf{P}}^l_b(j) \leq 0, \tag{23b}$$

in which $j = 1, \ldots, H_p$.

IV. Periodic Economic Model Predictive Control with Nonlinear Constraint Relaxations

Given a WDN described by the DAE model (1), the economic performance of this system is evaluated by the economic cost function $\ell(\mathbf{x}(k), \mathbf{u}(k), \mathbf{p}(k))$ that considers the system state $\mathbf{x}$, control input $\mathbf{u}$ and an exogenous time-varying signal $\mathbf{p}$.

The periodic operation implies that the time-varying economic cost function is T-periodic described by $\ell(\mathbf{x}(k), \mathbf{u}(k), \mathbf{p}(k)) = \ell(\mathbf{x}(k+T), \mathbf{u}(k+T), \mathbf{p}(k+T))$.

In this paper, the optimal steady states of EMPC can be found by a nonlinear EMPC planner under the periodic operation. This planner can be implemented by solving the EMPC optimization as a steady-state problem $\mathbb{P}^s_{\mathrm{T}}(k)$:

$$\min_{\mathbf{x}^*(i), \mathbf{u}^*(i)} V^s_T(\mathbf{x}, \mathbf{u}, \mathbf{p}) \triangleq \sum_{i=0}^{T-1} \ell(\mathbf{x}(i+1), \mathbf{u}(i), \mathbf{p}(i)), \tag{24}$$

subject to (1), (6) and

$$\mathbf{x}(0) = \mathbf{x}(T). \tag{25}$$

By solving the steady-state optimization problem $\mathbb{P}^s_{\mathrm{T}}(k)$ once, the feasible solutions $\mathbf{x}^*(i)$ and $\mathbf{u}^*(i)$ for $i = 0, 1, \ldots, T-1$ are chosen as the periodic steady states $\mathbf{x}_s(\langle k \rangle_T)$ and inputs $\mathbf{u}_s(\langle k \rangle_T)$ associated with the period of T.

The economic cost function might only consider the economic performance, where the optimal state may be not stable. Following the shifted economic cost function defined in [10] and [11], if the steady-state operation is desired, the pure economic cost function can be modified as a trade-off economic cost function:

$$\begin{aligned}\bar{\ell}(\mathbf{x}(k), \mathbf{u}(k), \mathbf{p}(k)) &\triangleq \theta \ell(\mathbf{x}(k), \mathbf{u}(k), \mathbf{p}(k)) \\ &\quad + (1-\theta)\gamma(\mathbf{x}_s(\langle k \rangle_T), \mathbf{u}_s(\langle k \rangle_T)),\end{aligned} \tag{26}$$

where θ denotes the trade-off parameter that indicates the importance of pure economic performance and convergence to the optimal periodically steady states. If $\theta = 1$, then the cost function only measures the pure economic performance. $\alpha(\cdot)$ is chosen to be positive definite with respect to $\mathbf{x}_s(\langle k \rangle_T)$ and $\mathbf{u}_s(\langle k \rangle_T)$. Normally, $\gamma(\cdot)$ can be chosen with weight matrices Q and S as follows:

$$\begin{aligned}\gamma(\mathbf{x}_s(\langle k \rangle_T), \mathbf{u}_s(\langle k \rangle_T)) &\triangleq \frac{1}{2}\Big(\|\mathbf{x}(k) - \mathbf{x}_s(\langle k \rangle_T)\|_{2,Q} \\ &\quad + \|\mathbf{u}(k) - \mathbf{u}_s(\langle k \rangle_T)\|_{2,S}\Big),\end{aligned} \tag{27}$$

where $\|\cdot\|_{2,W}$ denotes the weighted 2-norm by the weighing matrix W.

According to [11], the total economic cost function over the prediction horizon H_p can be formulated in the average form as

$$V^t(\mathbf{x}, \mathbf{u}, \mathbf{p}) \triangleq \tfrac{1}{H_p} \sum_{i=0}^{H_p-1} \bar{\ell}(\mathbf{x}(k+i+1 \mid k), \mathbf{u}(k+i \mid k), \mathbf{p}(k+i \mid k)). \tag{28}$$

Because the WDN is assumed to be strictly dissipative, the periodic terminal constraint is set as equality as

$$\mathbf{x}(i+H_p) = \mathbf{x}_s(\langle i+H_p \rangle_T). \tag{29}$$

In general, the periodic EMPC strategy with nonlinear constraint relaxations (PEMPC-NCR) can be implemented by solving the following optimization problem $\mathbb{P}^o_T(k)$:

$$\min_{\mathbf{x}^*(i), \mathbf{u}^*(i)} V^t(\mathbf{x}, \mathbf{u}, \mathbf{p}), \tag{30}$$

subject to (1a), (1b), (6), the relaxed nonlinear constraints (23) and the terminal equality constraint (29).

V. Case Study: Richmond Water Network

In order to test the proposed nonlinear constraint relaxation approach and periodic EMPC strategy, the Richmond water network[1] is used as the case study. The topology of the Richmond water network is shown in Fig. 3. In this network, there are 6 tanks, 7 actuators (pumps), 11 water demand sectors, 41 non-storage nodes and 41 interconnected pipes.

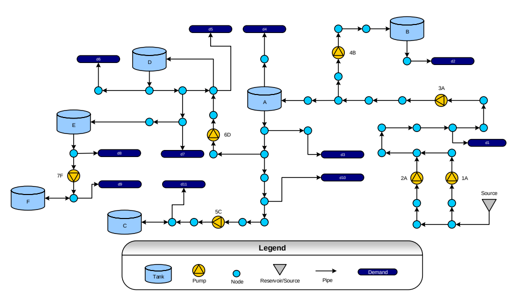


Fig. 3. Topology of the Richmond water network

[1] http://emps.exeter.ac.uk/engineering/research/cws/resources/benchmarks

The control-oriented model of the Richmond water network including the flow and pressure/hydraulic head variables can be characterized in the form of (1). The system dynamics in (1a) can be obtained by means of the mass balance at each storage tank. The linear algebraic equation in (1b) can be found by also using the mass balance at each non-storage node. The nonlinear algebraic equation in (1c) can be written by using the flow-head formula, for instance the *Chezy-Manning* formula:

$$z_i - z_j = R_{i,j} v_{i,j} \left| v_{i,j} \right|, \tag{31}$$

where z_i and z_j denote the hydraulic heads at two adjacent nodes. $v_{i,j}$ denotes the water flow through the pipe between these two adjacent nodes and $R_{i,j}$ is the *Chezy-Manny* parameter [9].

Considering the structure of the Richmond water network shown in Fig. 3, there are 41 nonlinear algebraic equations associated with 41 pipes. Among them, there are 2 bidirectional pipes in the form of (2) and 39 unidirectional pipes in the form of (3).

The cost function for the Richmond water network includes mainly includes three parts [1]: the economic term, the safety term and the smoothness term, which can be formulated as follows:

$$\begin{aligned} V^t \triangleq \frac{1}{H_p} \sum_{j=0}^{H_p-1} \Big(& \lambda_1 \ell^e(k+j) + \lambda_2 \ell^s(k+j) \\ & + \lambda_3 \ell^m(k+j) + \lambda^e(j) \tau_i(k+j) \Big), \end{aligned} \tag{32}$$

where $\ell^e(k+j)$, $\ell^s(k+j)$ and $\ell^m(k+j)$ denote the economic, safety and smoothness terms, respectively. The detailed definitions of these functions can be found in [1]. λ_1, λ_2 and λ_3 are fixed weights for each objective. $\lambda^e(j)$ denotes the weight for the penalty term of the relaxed constraints, which is set as a forgetting factor:

$$\lambda^e(j) = \lambda^e(j-1) - \epsilon, \tag{33a}$$

$$\lambda^e(0) = \lambda^e, \tag{33b}$$

where ϵ denote the relaxed step and λ^e is the initial value of this weight.

For a WDN, the economic performance is the most important. Therefore, the weights are selected as $\lambda_1 = 10, \lambda_2 = 1$, $\lambda_3 = 0.1$, $\lambda^e = 1$ and $\epsilon = 0.01$. A WDN behaves as a periodic system with a period of 24 hours because of the periodicity of the demand and electric tariffs. Hence, the period of the EMPC strategy T is chosen as 24. The prediction horizon of this simulation H_p is also chosen as 24. Furthermore, both N_a and N_b are chosen as 10. Therefore, there are 11 relaxed constraints for replacing each (3) and 22 relaxed constraints for replacing each (2).

In order to investigate the effectiveness and efficiency of the PEMPC-NCR strategy, the nonlinear periodic EMPC (NPEMPC) strategy is used by solving the following optimization problem $\mathbb{P}_T^n(k)$:

$$\min_{\mathbf{x}^*(i), \mathbf{u}^*(i)} V^t(\mathbf{x}, \mathbf{u}, \mathbf{p}), \tag{34}$$

subject to (1), (6) and the terminal equality constraint (29).

The simulation is carried out in a PC of Intel Core i7-5500U CPU and 12GB RAM with *MATLAB* R2015a. The PEMPC-NCR optimization problem is solved by means of *YALMIP* toolbox [12] and *CPLEX* solver. The NPEMPC optimization problem is solved by using the nonlinear programming technique using *YALMIP* toolbox [12] and *IPOPT* solver implemented in *OPTI* toolbox [13].

A. Key Performance Indicators

In order to compare the control performances of two MPC strategies, the following KPIs are used [6]:

$$KPI_E \triangleq \frac{1}{n_s} \sum_{k=1}^{n_s} \Big(\mathbf{p}(k) \mathbf{u}(k) \Big), \tag{35a}$$

$$KPI_S \triangleq \frac{1}{n_s} \sum_{k=1}^{n_s} \sum_{i=1}^{n_x} \max \Big\{ 0, \big(x_i^t - x_i(k) \big) \Big\}, \tag{35b}$$

$$KPI_M \triangleq \frac{1}{n_s} \sum_{k=1}^{n_s} \sum_{i=1}^{n_x} \Big(x_i(k) - x_i^t \Big), \tag{35c}$$

$$KPI_U \triangleq \frac{1}{n_s} \sum_{k=1}^{n_s} \sum_{i=1}^{n_u} \Big(u_i(k) - u_i(k-1) \Big)^2, \tag{35d}$$

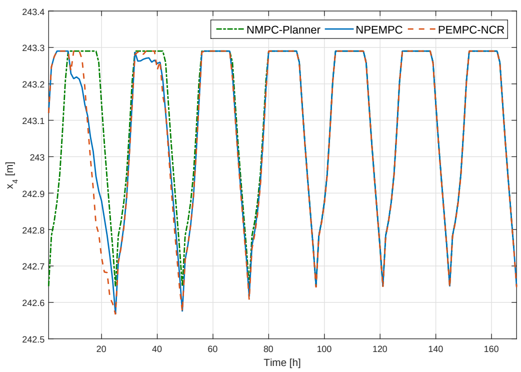
where KPI_E denotes the economic KPI that evaluates the hourly operational costs. KPI_S denotes the safety KPI that measures the average deviations of the accumulated water is below the safety level in the storage tanks. KPI_M represents the measured safety KPI that accumulates how much water has been reserved above the safety level in the network and KPI_U addresses the smoothness KPI that computes the collected slew rates of control inputs. Moreover, x_i^t denotes the safe hydraulic head at i-th tank satisfying its underlying unexpected demands and n_s represents the number of hours considered in the assessment.

B. Results

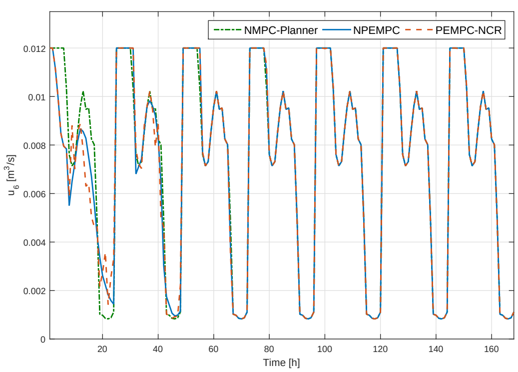
The KPI results to test control performances are shown in Table I. In general, performances of two MPC strategies are similar. Specifically, the operational costs of PEMPC-NCR is a little cheaper than NPEMPC but they are very close as shown in the KPI_E results. According to the KPI_S and KPI_M results, there are small deviation between the safety level and actual reserved water in the storage tanks for both MPC strategies and the accumulated water above safety levels of PEMPC-NCR is more than NPEMPC. It is because the weight for the safety term in the cost function is chosen as a medium value compared to the other terms. Finally, the KPI_U results show that the water network is controlled by applying the NPEMPC strategy under more smooth control inputs than the PEMPC-NCR strategy.

TABLE I
PERFORMANCE COMPARISONS OF TWO MPC STRATEGIES

MPC Strategy	KPI_E	KPI_S	KPI_M	KPI_U
NPEMPC	0.7028	0.1914	6.5249	0.00005
PEMPC-NCR	0.6992	0.2604	6.7078	0.0013



(a) x_4



(b) u_6

Fig. 4. Simulation result of selected state and input by the NPEMPC and PEMPC-NCR strategies: the optimal steady states are shown in dashed dotted lines, system states operated by the NPEMPC strategy are shown in solid lines and system states operated by the PEMPC-NCR strategy are shown in dashed lines

The simulation results of some system states and a control input are shown in Fig. 4. The steady states are obtained by using the nonlinear planner built in $\mathbb{P}^s_{\mathrm{T}}(k)$. It is clear that the actual states by applying the NPEMPC and PEMPC-NCR strategies are approaching the optimal steady states, which also proves that performances of two strategies are similar.

The total simulation time of NPEMPC is approximately 62.86 minutes while the computation time of PEMPC-NCR is about 1.63 minutes. Hence, the PEMPC-NCR strategy has a significant improvement of the reduction of computational complexity and is able to obtain the similar performance as the NPEMPC strategy.

VI. Conclusions

This paper studied a periodic EMPC strategy with nonlinear algebraic constraint relaxations for the operational management of WDNs. The proposed iterative algorithm can deal with the nonlinear algebraic equations of the interconnected pipes. Therefore, the nonlinear algebraic equations are relaxed by a series of the linear inequality constraints. As shown in the performance KPIs, the economic performances of the NPEMPC and PEMPC-NCR strategies are similar. Meanwhile, through the simulation results of system state and control input evolutions with two MPC strategies, it is clearly shown that they can obtain the similar results. Thus, the proposed iterative algorithm for the nonlinear algebraic equation is effective. Besides, the computation time of the PEMPC-NCR strategy is significantly faster than the NPEMPC strategy by using the nonlinear programming technique. Hence, the proposed EMPC strategy with nonlinear-constraint relaxation is efficient and effective. As a future work, the stability will be discussed with the proposed control strategy.

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