Distributed Zonotopic Set-Membership State Estimation based on Optimization Methods with Partial Projection ⋆

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Abstract: A distributed set-membership approach is proposed for the state estimation of large-scale systems. The uncertain system states are bounded in a sequence of the distributed set-membership estimators considering unknown-but-bounded system disturbances and measurement noise. In the framework of the set-membership approach, the measurement consistency test is implemented by finding parameterized intersection zonotopes. The size of the intersection zonotope is minimized by solving an optimization problem including a sequence of linear/bilinear matrix inequalities based on the weighted 2-norm criterion of the generator matrix. Meanwhile, for the distributed set-membership estimators, the partial projection method is considered to correct the estimation of the neighbor state. On the other hand, an on-line method is also provided. Finally, the proposed distributed set-membership approach is verified in a case study based on a urban drainage network.

Keywords: Distributed set-membership approach, state estimation, partial projection method, zonotopes, large-scale systems, urban drainage networks

1. INTRODUCTION

The set-membership state estimation for uncertain dynamic systems as an iterative algorithm has been well-discussed in the last decades (Jaulin et al., 2001; Puig et al., 2001; Alamo et al., 2005; Puig, 2010), which can be also applied into fault diagnosis and fault-tolerant control (see, e.g. Fagarasan et al. (2004), Puig (2010), Olaru et al. (2010), Blesa et al. (2011), Blesa et al. (2016)). The set-membership approach can deal with unknown-but-bounded system state disturbances and measurement noise. The uncertain system states are bounded in a set with a specific geometrical characteristic, for instance a zonotope (considering its simple computational complexity). And then, with the measurement outputs, the system consistency test can be implemented by computing the intersection between the predicted uncertain state set (by using the direct image with the state model) and the measurement state set (by using the inverse image with the output function). The intersection set is usually over-approximated by also using a zonotope (Alamo et al., 2005; Le et al., 2013b).

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The intersection zonotope can be formulated with respect to a correction matrix (Alamo et al., 2005). This correction matrix can be found by a number of different methods. Among them, the optimal correction matrix can be computed by solving an optimization problem to minimize the P-radius of the zonotope (Le et al., 2013a). In fact, the P-radius of a zonotope represents a corresponding ellipsoid. In terms of large-scale systems, the challenge is to solve an optimization problem in a centralized way with a large amount of decision variables.

The main limitation of the zonotopic set-membership state estimation is the dimension of the system model. For a high dimensional model, it is difficult to provide an outer approximation of the intersection zonotope. Naturally, the distributed set-membership approach is quite suitable especially for large-scale systems. From the literature, only a few papers have discussed the distributed set-membership approach (see, e.g. García et al. (2016)). In principle, a distributed set-membership estimator is designed by using a sequence of sets for bounding the uncertain system states in different subsystems.

This paper proposes a distributed set-membership approach for state estimation in large-scale systems. The system consistency test with the available measurements is implemented by finding a sequence of the intersection zonotopes between the prediction zonotopes and the measurement zonotopes to be used by a set of distributed estimators. The sizes of these intersection zonotopes are minimized by using an optimization-based method with a series of linear matrix inequalities (LMIs). The weighted 2-norm of the generator matrix of the zonotope is used as the size criterion. Besides, the bounds of the neighbor states from two sub-systems can be corrected applying the projection method. On the other hand, an on-line method is also discussed for updating the correction matrices method. Finally, an urban drainage network is used for testing the effectivenesses of the proposed approaches.

The remainder of this paper is organized as follows: Problem statement is formulated in Section 2. The distributed set-membership approach based on the optimization method with partial projections is proposed in Section 3. Results of applying the proposed distributed set-membership approach into the case study of an urban drainage network are shown in Section 4. Finally, conclusion is drawn in Section 5.

2. PROBLEM STATEMENT

Consider a discrete-time linear system as shown in Fig. 1 partitioned into subsystems:

$$\sum_i : x_{k+1}^i = A_i x_k^i + B_i u_k^i + \omega_k^i, \tag{1a}$$

$$\sum_i : y_k^i = C_i x_k^i + \upsilon_k^i, \tag{1b}$$

where $i = 1, 2, \ldots, s$ denotes the index of the subsystem. $x^i \in \mathbb{R}^{n_{x_i}}$ denotes the vector of system states. $u^i \in \mathbb{R}^{n_{u_i}}$ denotes the vector of inputs. $y_k^i \in \mathbb{R}^{m_i}$ represents the vector of measurement outputs. A_i, B_i and C_i are linear system matrices of appropriate dimensions. $\omega_k^i \in \mathbb{R}^{n_{w^i}}$ and $\upsilon_k^i \in \mathbb{R}^{m_i}$ denote the unknown system disturbance vector and measurement noise vector, respectively.

In general, some system states might be shared into different subsystems based on the chosen partitioning approach in the considered distributed model. Hence, there may have overlapped parts of some subsystems.

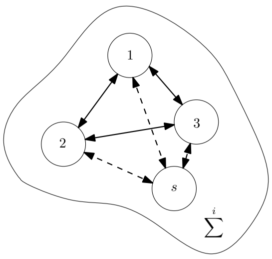

Fig. 1. Distributed linear system model

A zonotope $\mathcal{Z} \in \mathbb{R}^n$ $(m \geq n)$ is defined by an unitary hypercube $\mathbf{B}^m = [-1, +1]^m$ affine projection with the center $p \in \mathbb{R}^n$ and a generator matrix $H \in \mathbb{R}^{n \times m}$ as $\mathcal{Z} = p \oplus H\mathbf{B}^m$. For simplicity, the zonotope $\mathcal{Z}$ is simply denoted as$\langle p, H \rangle$ with the following form:

$$\mathcal{Z} = \langle p, H \rangle = \left\{ p + Hz, z \in \mathbf{B}^m, \|z\|_\infty \leq 1 \right\}. \tag{2}$$

The operators $\oplus$ and $\odot$ denote the Minkowski sum and the linear image, respectively. Meanwhile, the following properties hold:

$$\langle p_1, H_1 \rangle \oplus \langle p_2, H_2 \rangle = \langle p_1 + p_2, [H_1 \;\; H_2] \rangle, \tag{3a}$$

$$L \odot \langle p, H \rangle = \langle Lp, LH \rangle. \tag{3b}$$

The interval hull $rs(H) \in \mathbb{R}^{n \times n}$ of the zonotope $\langle p, H \rangle$ can be regarded as an aligned minimum box. Therefore, the following condition holds:

$$\langle p, H \rangle \subset \langle p, rs(H) \rangle, \tag{4}$$

where $rs(H)$ is a diagonal matrix such that $rs(H)_{i,i} = \sum_{j=1}^m |H_{i,j}|$ for $i = 1, 2, \ldots, n$.

Assumption 1. ω_k^i and υ_k^i for $i = 1, 2, \ldots, s$ are unknown but bounded in centered zonotopic sets:

$$\omega_k^i \in \mathcal{W}_i = \langle 0, F_i \rangle, \;\; \upsilon_k^i \in \mathcal{V}_i = \langle 0, \Sigma_i \rangle, \tag{5}$$

where $\Sigma_i = \mathrm{diag}(\sigma_i)$ and σ_i is a vector of independent bounds for each measurement noise.

Assumption 2. The initial state vector x_0^i is also bounded into a known zonotopic set:

$$x_0^i \in \mathcal{X}_0^i = \langle p_0, H_0 \rangle. \tag{6}$$

As the set-membership approach firstly proposed in (Alamo et al., 2005), before introducing the distributed approach steps, some set definitions are recalled as follows:

Definition 1. Consider the model of the i-th subsystem (1) and the corresponding state $x_{k-1}^i \in \hat{\mathcal{X}}_{k-1}^i = \langle \hat{p}_{k-1}^i, \hat{H}_{k-1}^i \rangle, i = 1, 2, \ldots, s$ at time instant $k-1$, the uncertain state set $\mathcal{X}_k^i$ for $i = 1, 2, \ldots, s$ at time instant k can be propagated as $\mathcal{X}_k^i = A\mathcal{X}_{k-1}^i \oplus \mathcal{W}_i$.

Definition 2. Consider the model of the i-th subsystem (1) and the measurement outputs y_k^i, the corresponding consistent state set $\mathcal{X}_{y_k}^i$ for $i = 1, 2, \ldots, s$ at time instant k is defined as $\mathcal{X}_{y_k}^i = \left\{ x_k^i \in \mathbb{R}^{n_{x_i}} \mid y_k^i \in C_i x_k^i \oplus \mathcal{V}_i \right\}$.

Definition 3. Consider the model of the i-th subsystem (1), the corresponding uncertain state set $\mathcal{X}_k^i$ and consistent state set $\mathcal{X}_{y_k}^i$ for $i = 1, 2, \ldots, s$ in *Definition 1 and 2*, the distributed exact uncertain state set $\hat{x}_k^i \in \hat{\mathcal{X}}_k^i$ for $i = 1, 2, \ldots, s$ at time instant k can be computed by means of the intersection between $\mathcal{X}_k^i$ and $\mathcal{X}_{y_k}^i$ as $\hat{\mathcal{X}}_k^i = \mathcal{X}_k^i \cap \mathcal{X}_{y_k}^i$.

Considering that the set-membership approach is an iterative method, the outer approximations of the exact uncertain state set $\hat{\mathcal{X}}_k^i$ for $i = 1, 2, \ldots, s$ at time instant k are built by means of the zonotopes as well.

Proposition 1. Given the estimated state $\hat{x}_{k-1}^i$ bounded in the zonotope $\langle \hat{p}_{k-1}^i, \hat{H}_{k-1}^i \rangle$ and the consistent state set $\mathcal{X}_{y_k}^i$ expressed in a polytopic representation as

$$\mathcal{X}_{y_k}^i = \left\{ x_k^i \in \mathbb{R}^{n_{x_i}} \mid \left| C_i x_k^i - y_k^i \right| \leq \sigma_i \right\}, \tag{7}$$

Then, there exists a correction matrix $\Lambda_i \in \mathbb{R}^{n_{x_i} \times n_{m_i}}$ at time instant k such that the corresponding exact uncertain states $\hat{x}_k^i$ can be enclosed in the following guaranteed intersection zonotope:

$$\hat{x}_k^i \in \hat{\mathcal{X}}_k^i(\Lambda_i) = \langle \hat{p}_k^i(\Lambda_i), \hat{H}_k^i(\Lambda_i) \rangle, \tag{8}$$

with

$$\hat{p}_k^i(\Lambda_i) = (I - \Lambda_i C_i) A_i \hat{p}_{k-1}^i + (I - \Lambda_i C_i) B_i u_k^i + \Lambda_i y_k^i, \tag{9a}$$

$$\hat{H}_k^i(\Lambda_i) = \Big[(I - \Lambda_i C_i) A_i \downarrow_{q,W} \hat{H}_{k-1}^i \;\;\; (I - \Lambda_i C_i) F_i \;\;\; \Lambda_i \Sigma_i \Big], \tag{9b}$$

where $\downarrow_{q,W} \hat{H}_{k-1}^i$ is the reduced-order generator matrix at the time instant $k-1$ with a suitable weight W.

Proof. According to the distributed model in (1) and $\hat{x}^i_{k-1} \in \langle \hat{p}^i_{k-1}, \hat{H}^i_{k-1} \rangle$, there exists a vector $s_1 \in \mathbf{B}^r$ such that the uncertain state x^i_k can be propagated by

$$x^i_k = A\hat{p}^i_{k-1} + B_i u^i_k + \left[A\hat{H}^i_{k-1} \; F_i \right] s_1. \tag{10}$$

There exists a correction matrix $\Lambda_i \in \mathbb{R}^{n_{x_i} \times n_{m_i}}$ such that by adding and subtracting a term $\Lambda_i C_i \left[A\hat{H}^i_{k-1} \; F_i \right] s_1$ to (10), we have

$$\begin{aligned} x^i_k = & A\hat{p}^i_{k-1} + B_i u^i_k + \Lambda_i C_i \left[A\hat{H}^i_{k-1} \; F_i \right] s_1 \\ & + \left[(I - \Lambda_i C_i) A\hat{H}^i_{k-1} \; (I - \Lambda_i C_i) F_i \right] s_1. \end{aligned} \tag{11}$$

From (7), there exists a vector $s_2 \in \mathbf{B}^{n_{m_i}}$ such that

$$C_i x^i_k - y^i_k = \Sigma_i s_2. \tag{12}$$

Replacing x^i_k in (12) by (10), we obtain

$$C_i \left[A\hat{H}^i_{k-1} \; F_i \right] s_1 = y^i_k + \Sigma_i s_2 - C_i \left(A\hat{p}^i_{k-1} + B_i u^i_k \right). \tag{13}$$

With (11) and (13), the exact uncertain state $\hat{x}^i_k$ can be found by

$$\begin{aligned} \hat{x}^i_k = & A\hat{p}^i_{k-1} + B_i u^i_k + \Lambda_i y^i_k + \Lambda_i \Sigma_i s_2 \\ & - \Lambda_i C \left(A\hat{p}^i_{k-1} + B_i u^i_k \right) \\ & + \left[(I - \Lambda_i C_i) A\hat{H}^i_{k-1} \; (I - \Lambda_i C_i) F_i \right] s_1 \\ = & (I - \Lambda_i C_i) A_i \hat{p}^i_{k-1} + (I - \Lambda_i C_i) B_i u^i_k + \Lambda_i y^i_k \\ & + \left[(I - \Lambda_i C_i) A_i \hat{H}^i_{k-1} \quad (I - \Lambda_i C_i) F_i \quad \Lambda_i \Sigma_i \right] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \end{aligned} \tag{14}$$

In order to reduce the complexity of the generator matrix of the zonotope, the reduction operator $\downarrow_{q,W}$ (Combastel, 2005) is used. Hence, the proof is completed. $\square$

According to (Le et al., 2013a,b), the size of the intersection zonotope can be measured by using the weighted 2-norm, which is defined as the P-radius of the intersection zonotope as

$$\begin{aligned} \ell^i_k \triangleq & \max_{x^i_k \in \hat{\mathcal{X}}^i_k} \left\| x^i_k - \hat{p}^i_k(\Lambda_i) \right\|^2_{P_i} \\ = & \max_{z \in \mathbf{B}^r} \left\| \hat{H}^i_k(\Lambda_i) z \right\|^2_{P_i}, \end{aligned} \tag{15}$$

where $z \in \mathbf{B}^r$ denotes the unitary vector of the appropriate dimension depending on $\hat{H}^i_k(\Lambda_i)$. P_i denotes the weighting matrix of appropriate dimension for the i-th subsystem. Note that this weighting matrix P_i can be also used as W in (9b).

In fact, the P-radius can be regarded as a corresponding ellipsoid in terms of the zonotope. The design of the correction matrix Λ_i is required to guarantee that the P-radius is not increasing, that is to find a minimum ellipsoid enclosing the intersection zonotope. Hence, we obtain the correction matrix Λ_i in such a way that there is a scalar β_i such that the following condition can be satisfied:

$$\ell^i_k \leq \beta_i \ell^i_{k-1} + \max_{s_1 \in \mathbf{B}^{n_{w_i}}} \| F_i s_1 \|^2_2 + \max_{s_2 \in \mathbf{B}^{n_{v_i}}} \| \Sigma_i s_2 \|^2_2, \tag{16}$$

where s_1 and s_2 denote the unitary vectors of appropriate dimensions depending on the system disturbances and the measurement noise.

To determine the unknown variables including Λ_i, β_i and P_i for the i-th intersection zonotope, the objectives of the minimization of the zonotope together with satisfying the condition (16) are required.

The main steps of the distributed set-membership state estimation are written as follows:

- Find a sequence of the intersection zonotopes $\hat{\mathcal{X}}^i_k$ with $i = 1, 2, \ldots, s$ for all the subsystem with the suitable correction matrices Λ_i.
- For the neighbor states (shared states in two subsystems) $x^{i,j}_k$, the projection zonotopes used for state estimation can be found by using a projection method. Therefore, the neighbor state estimation can be implemented by computing another intersection between two projection zonotopes:

  $$x^{i,j}_k \in \hat{\mathcal{X}}^{i,j}_k \left(\bar{\Lambda}_{i,j} \right) \cap \hat{\mathcal{X}}^{j,i}_k \left(\bar{\Lambda}_{j,i} \right), j = 1, \ldots, N_i, \tag{17}$$

  where $\hat{\mathcal{X}}^{i,j}_k \left(\bar{\Lambda}_{i,j} \right)$ denotes the zonotope for the i-th intersection zonotope projected to the neighbor states shared with the j-th subsystem at time instant k. N_i denotes the number of neighborhoods for the i-th subsystem.

Remark 1. The projection zonotopes can be found by adding the new objective with the projection constraints and satisfying the condition (16). The P-radius is minimized mainly regarding to the projected state.

3. DISTRIBUTED SET-MEMBERSHIP STATE ESTIMATION

The distributed set-membership approach is used for finding a sequence of the intersection zonotopes. The system states x^i_k in the i-th subsystem can be estimated by using the interval hull of the intersection zonotope $\hat{\mathcal{X}}^i_k$ at time instant k. Besides, the neighbor states $x^{i,j}_k$ shared with the j-th subsystem can be updated by computing the intersection between the projection zonotopes $\hat{\mathcal{X}}^{i,j}_k$ and $\hat{\mathcal{X}}^{j,i}_k$ at time instant k.

The correction matrix Λ_i and the weighting matrix P_i with $i = 1, 2, \ldots, s$ can be firstly computed off-line by solving a BMI/LMI optimization problem. By selecting different objective, the projection zonotopes can be also found off-line. On the other hand, a new on-line method for updating the correction matrix Λ_i is proposed in this paper.

3.1 Computing the Intersection Zonotopes

According to (Le et al., 2013a), the convergence condition (16) of the P-radius of the zonotope can be reformulated as a LMI. Therefore, there exists a vector $Y_i \in \mathbb{R}^{n_{x_i}}$, a matrix P_i and a scalar β_i such that the following LMI can be satisfied:

$$\begin{bmatrix} \beta_i P_i & 0 & 0 & A_i^T P_i - A_i^T C_i^T Y_i^T \\ * & F_i^T F_i & 0 & F_i^T P_i - F_i^T C_i^T Y_i^T \\ * & * & \Sigma_i^T \Sigma_i & \Sigma_i Y_i^T \\ * & * & * & P_i \end{bmatrix} \succeq 0. \tag{18}$$

Let $\epsilon_i = \max_{s_1 \in \mathbf{B}^{n_{w_i}}} \| F_i s_1 \|^2_2 + \max_{s_2 \in \mathbf{B}^{n_{v_i}}} \| \Sigma_i s_2 \|^2_2$ and the time instant $k \to \infty$ such that

$$\ell^i_\infty = \beta_i \ell^i_\infty + \epsilon_i. \tag{19}$$

From (19), we have

$$\ell_{\infty}^{i} = \frac{\epsilon_i}{1-\beta_i}. \tag{20}$$

Therefore, the uncertain trajectories are ultimately bounded in the corresponding ellipsoid $\mathcal{E}_i \triangleq \left\{ x^i \mid {x^i}^T P_i x^i \leq \frac{\epsilon_i}{1-\beta_i} \right\}$. The ellipsoid of the smallest diameter can be found by solving an eigenvalue problem (Boyd et al., 1994). Therefore, there exists a maximum positive scalar τ_i such that

$$\frac{(1-\beta_i)P_i}{\epsilon_i} \succeq \tau_i I, \tag{21a}$$

$$\tau_i > 0. \tag{21b}$$

Hence, Λ_i is computed by solving the following BMI/LMI problem:

$$\max_{\tau_i,\beta_i,P_i,Y_i} \tau_i \tag{22}$$

and subject to (18) and (21).

After solving the above BMI optimization problem, Λ_i can be obtained off-line as follows:

$$\Lambda_i = P_i^{-1} Y_i. \tag{23}$$

It is noticed that the BMI optimization problem (22) can be solved as a LMI one by using the linear search algorithm for finding $\beta_i \in (0, 1]$. Therefore, the LMI optimization problem can be implemented by searching the maximum $\tau_i > 0$ from the smallest β_i with a desirable accuracy.

3.2 Computing the Projection Zonotopes

The objective of the optimization problem (22) is determined by finding the ellipsoid of the smallest diameter. For the projection zonotope, this objective can be moved into finding the smallest ellipsoid only for the neighbor states.

The ellipsoid is defined by

$$\begin{bmatrix} x_a - c_a \\ x_b - c_b \end{bmatrix}^T \begin{bmatrix} P_{aa} & P_{ab}^T \\ P_{ab} & P_{bb} \end{bmatrix} \begin{bmatrix} x_a - c_a \\ x_b - c_b \end{bmatrix} \leq 1. \tag{24}$$

where $\begin{bmatrix} c_a^T & c_b^T \end{bmatrix}^T$ is the center of the ellipsoid.

The projection on the x_a space of the ellipsoid (24) is given by

$$(x_a - c_a)^T \left(P_{aa} - P_{ab}^T P_{bb}^{-1} P_{ab} \right) (x_a - c_a) \leq 1. \tag{25}$$

From (25), it is clear that this projection set is still an ellipsoid. In order to minimize the size of this new ellipsoid, we can minimize the maximum eigenvalue of new matrix of the ellipsoid. Therefore, by using the Schur complement lemma, the constraints (21) in the optimization problem (22) is rewritten as

$$\begin{bmatrix} \frac{(1-\beta_j)}{\epsilon} P_{aa}^{i,j} - \tau_i I & \frac{(1-\beta_j)}{\epsilon} {P_{ab}^{i,j}}^T \\ \frac{(1-\beta_j)}{\epsilon} P_{ab}^{i,j} & \frac{(1-\beta_j)}{\epsilon} P_{bb}^{i,j} \end{bmatrix} \succ 0, \tag{26a}$$

$$\tau_j > 0, \tag{26b}$$

with $\bar{P}_{i,j}$ can be divided into $\bar{P}_{i,j} = \begin{bmatrix} P_{aa}^{i,j} & {P_{ab}^{i,j}}^T \\ P_{ab}^{i,j} & P_{bb}^{i,j} \end{bmatrix}$.

Then, the correction matrix $\bar{\Lambda}_{i,j}$ for the i-th subsystem with the j-th projection zonotope can be found by solving the following optimization problem:

$$\max_{\tau_j,\beta_j,P_{aa}^{i,j},P_{ab}^{i,j},P_{bb}^{i,j},Y_j} \tau_j \tag{27}$$

and subject to (18) and (26).

As a result, $\bar{\Lambda}_{i,j}$ can be computed off-line by

$$\bar{\Lambda}_{i,j} = \bar{P}_{i,j}^{-1} Y_j. \tag{28}$$

Remark 2. In some specific cases, the projection method could not improve so much performance since the state bounding results are satisfactory.

3.3 On-line Updating Correction Matrices

By implementing the optimization problems (22) and (27), the weighting matrices P_i and $\bar{P}_{i,j}$ of the intersection and projection zonotope for $i = 1, 2, \ldots, s$ and $j = 1, \ldots, N_i$ can be obtained.

Consider the weighting matrix P_i for the i-th intersection zonotope obtained by implementing the optimization (22), the P-radius in (15) at time instant k can be reformulated with known P_i as

$$\begin{aligned} \tilde{\ell}_k &= \max_{z \in \mathbf{B}^r} \left\| \hat{H}_k^i(\tilde{\Lambda}_k^i) \cdot z \right\|_{P_i}^2 \\ &= \tilde{z}^T \cdot \hat{H}_k^i(\tilde{\Lambda}_k^i)^T P_i \hat{H}_k^i(\tilde{\Lambda}_k^i) \cdot \tilde{z}, \end{aligned} \tag{29}$$

where $\tilde{z}$ denotes the unitary vector in order to find the maximum radius of the zonotope. $\tilde{\Lambda}_k^i$ denotes the updated correction matrix for the i-th intersection zonotope at time instant k.

In order to deal with the infinite vertexes of the ellipsoid and find $\tilde{\Lambda}_k^i$, there always exists a diagonal matrix D_k at time instant k such that the following inequality holds:

$$D_k \geq \hat{H}_k^i(\tilde{\Lambda}_k^i)^T P_i \hat{H}_k^i(\tilde{\Lambda}_k^i). \tag{30}$$

Applying Schur complement lemma to (30), we have the following LMI:

$$\begin{bmatrix} D_k & \hat{H}_k^i(\tilde{\Lambda}_k^i)^T P_i \\ * & P_i \end{bmatrix} \succeq 0. \tag{31}$$

Since the trace of the diagonal matrix D_k is a measure of the size of the uncertain set, the on-line updated correction matrix $\tilde{\Lambda}_k^i$ and the diagonal matrix D_k can be found by solving the following optimization problem:

$$\min_{\tilde{\Lambda}_k^i, D_k} tr(D_k) \tag{32}$$

subject to (31), where $tr(\cdot)$ denotes the trace of a given matrix.

Similar to the on-line method for intersection zonotope in (32), after finding the weighting matrix $\bar{P}_{i,j}$, the updated correction matrix $\check{\Lambda}_k^{i,j}$ at time instant k can be obtained by solving the following optimization problem:

$$\min_{\check{\Lambda}_k^{i,j}, \bar{D}_k} tr(\bar{D}_k) \tag{33a}$$

subject to

$$\begin{bmatrix} \bar{D}_k & \hat{H}_k^T(\check{\Lambda}_k^{i,j})\bar{P}_i \\ * & \bar{P}_i \end{bmatrix} \succeq 0. \tag{33b}$$

Remark 3. As mentioned in the projection method, the on-line method might produce the similar results as the off-line method without important improvements for some system states because there exists the trade-off to minimize the zonotope size for all the system states.

3.4 Distributed Set-Membership Algorithm

As presented before, the intersection zonotopes are found for observing the normal distributed states at each subsystem while the neighbor states can be observed by using the intersection between the projection zonotopes. In general, the distributed set-membership approach can be summarized as the following algorithm.

Algorithm 1 Distributed set-membership algorithm

Data: $\hat{\mathcal{X}}_0^i$, $\hat{\mathcal{X}}_0^{i,j}$ and $\hat{\mathcal{X}}_0^{j,i}$ known for $i = 1, \ldots, s$ and $j = 1, \ldots, N_i$;

Off-line compute P_i and $\bar{P}_{i,j}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, N_i$ by means of solving the optimization problems (22) and (27);

for k=1:t_{end} **do**

  Compute the correction matrix $\tilde{\Lambda}_k^i$ for the intersection zonotopes by solving the optimization problem (32) for $i = 1, \ldots, s$;

  Compute the correction matrix $\check{\Lambda}_k^{i,j}$ for the projection zonotopes by solving the optimization problem (33) for $i = 1, \ldots, s$ and $j = 1, \ldots, N_i$;

  Compute the intersection between the projection zonotopes: $x_k^{i,j} \in \hat{\mathcal{X}}_k^{i,j}\left(\bar{\Lambda}_{i,j}\right) \cap \hat{\mathcal{X}}_k^{j,i}\left(\bar{\Lambda}_{j,i}\right), j = 1, \ldots, N_i$,;

  The state estimation results can be found by using the interval hull (4);

end

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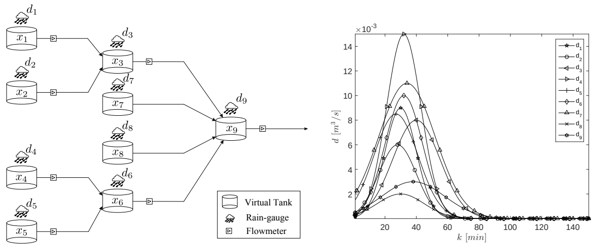
4. CASE STUDY: AN URBAN DRAINAGE NETWORK

4.1 Description

In order to verify the proposed distributed set-membership approach, a urban drainage network from a portion of the Bogotá (Colombia) urban drainage network (Barreiro-Gomez et al., 2015) is chosen and its topology is shown in Figure 2. The virtual tanks/reservoirs model is used as the mathematical model of this network. In general, there are 9 virtual tanks and 7 sensors placed in the network for measuring the flows through the sewer pipes. In this example, it is considered that for virtual tank, the nominal rain run-off can be regarded as an known input signal. The stochastic rain run-off are assumed to be unknown but bounded in known zonotopes as system disturbances.



(a) (b)

Fig. 2. Topology of the urban drainage network and the rain runoff

The mathematical model for the i-th virtual tank is given by $\dot{x}^i = q_{in}^i - q_{out}^i$, where q_{in}^i denotes the inflows to the tank, x^i denotes the volume of the i-th virtual tank and the relationship of the outflow and the volume is given by a linear function as $q_{out}^i = K_i x^i$. The parameters K_i for each virtual tank are given in Table 1.

Table 1. Parameters of the UDN model

Tank	Coefficient K_i	Tank	Coefficient K_i
1	0.002332	6	0.004764
2	0.003870	7	0.008975
3	0.002217	8	0.005185
4	0.003170	9	0.006147
5	0.008239		

The distributed model of this network is used in discrete-time and the system states are divided into the 3 subsystems as shown in Table 2.

Table 2. Distributed Model of the UDN

Subsystem	Virtual tanks	Neighbor state
S_1	x_1, x_2, x_3	x_3
S_2	x_4, x_5, x_6	x_6
S_3	x_3, x_6, x_7, x_8, x_9	-

The proposed optimization problems can be solved by using the Yalmip toolbox (Löfberg, 2004) and the commercial LMI solvers, for instance MOSEK (MOSEK ApS, 2015). All the simulations have been done in a PC of Intel i7-5500U CPU 2.40 GHz with 12GB RAM.

4.2 Simulation Results

The state estimation result of x_7 by using the off-line and on-line distributed methods is plotted in Fig. 3. All the uncertain real states are bounded in two bounds. From the result, it is clear that the bounds from the on-line method are similar to the ones from the off-line method. As shown in Fig. 4, the bounds of two neighbor states x_3 and x_6 are obtained by using the off-line and on-line methods. The on-line method is able to find more tightened bounds in an iteratively way.

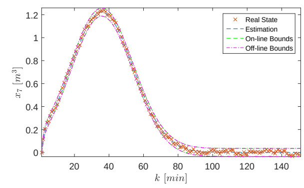
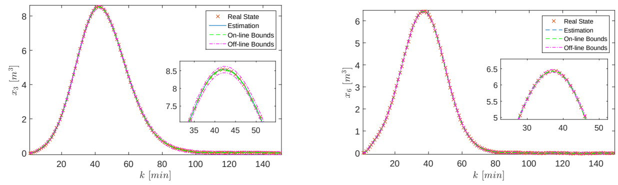


Fig. 3. State estimation result of x_7 by using the distributed set-membership approach



(a) x_3 (b) x_6

Fig. 4. State estimation results of the neighbor states by using the distributed set-membership approach

Moreover, the centralized set-membership approach is also applied to the general model of this network. In the centralized approach, only one intersection zonotope is obtained by using the off-line method without considering the projection constraints or using the on-line method. In order to compare the centralized and distributed approaches, the mean square error (MSE) of the state estimation and the mean square of the P-radius are computed. In Table 3, the comparison results for four approaches are presented. From the MSE, the centralized methods have bigger estimation error than the distributed one. Meanwhile, the on-line distributed approach is a little better than the others for this example. The results of the mean square of the P-radius for all the approaches are very close and the distributed one is a little larger than the centralized one. The simulation time for all the approaches are recorded in Table 4. From this table, it is noted that the recorded simulation time for the distributed approach corresponds to the worst-case time of a single subsystem since the distributed approach can be implemented using the parallel computing. Generally, the on-line approaches require longer time since the optimization problem is solved in an iterative way. The distributed approaches can reduce the required simulation time, which will be benefit for large-scale systems with a lot of variables.

Table 3. Comparisons between the centralized and distributed approaches

Approaches	MSE	$\|H\|_P^2$
Off-line Centralized	2.0357e-04	1.8149e-2
On-line Centralized	2.0357e-04	1.8149e-2
Off-line Distributed	1.3372e-04	1.8159e-2
On-line Distributed	1.0341e-04	1.8158e-2

Table 4. Simulation time of applying the centralized and distributed approaches

Approaches	Optimization time [s]	Total time [s]
Off-line Centralized	41.322	43.8
On-line Centralized	98.645	101.172
Off-line Distributed	35.8	36.5
On-line Distributed	87.4	88.3

5. CONCLUSION

In this paper, a distributed set-membership approach is proposed based on optimization methods. The P-radius criterion is used for evaluating the size of the zonotope. The distributed system states can be estimated by a sequence of the intersection zonotopes and projection zonotopes with respect to correction matrices. The correction matrices can be computed off-line and on-line. From the effective simulation results, the proposed distributed approach is effective. Meanwhile, we can see that the on-line distributed approach takes longer time for solving the optimization problem. Comparing to the centralized approach, the distributed set-membership approach is faster and also easier to apply to the large-scale systems.

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