Partitioning of Large-Scale Systems using Game-Theoretic Coalitional Methods*

F. J. Muros†, J. M. Maestre†, C. Ocampo-Martinez‡, E. Algaba§ and E. F. Camacho†

***Abstract*— In this paper, tools from cooperative game theory are combined with predictive control to perform the partitioning of large-scale systems (LSS). More specifically, a partitioning algorithm based on the Shapley value to rank the links by using a cooperative cost game is proposed. To this end, coalitional model predictive control, which offers a trade-off between control performance and communication burden, is considered to assess the value of the coalitions in the game. Also, combinatorial explosion issues are relieved by means of an attribution of value to the links based on the nodes they connect. The proposed method is implemented in the Barcelona drinking water network as a real LSS case study, showing the effectiveness of the proposed approach.**

I. INTRODUCTION

Model predictive control (MPC) has become the accepted standard for complex constrained multivariable control problems in the processes industry [1]. It corresponds to a complete control methodology that uses a model to predict the future evolution of the process starting from the current system state along a receding horizon. MPC has also distributed formulations to enjoy well-known advantages such as scalability, modularity, and the capacity of controlling large-scale systems (LSS) [2]. The key concept is to decompose the overall control problem into smaller pieces assigned to local controllers or *agents*, which use partial system information and are able to communicate with each other. Applications of these schemes include network systems, such as traffic, water or power networks [3], among many others.

The aforementioned pieces are generally grouped into neighborhoods or *coalitions* that are traditionally kept fixed along time [4]. In contrast to these static approaches, different schemes that explicitly consider a dynamic evolution of the agents that belong to each coalition have recently appeared. These so-called *coalitional approaches* offer a reduction of the communication burden without compromising the control system performance [5]–[12]. More specifically, this work follows the line of research of [10]–[12], where several tools from cooperative game theory were integrated in coalitional schemes to gain knowledge about the relevance of the communication links and the local controllers. In these schemes, the objective function to be optimized in a networked control structure was interpreted as the characteristic function of a cooperative cost game with restricted cooperation [13]. Certainly, it was shown that solution concepts of this game provide information about the relevance of the agents and their connections involved in the distributed control problem. In this sense, the well-studied one-point solution concept Shapley value [14], which measures an averaged contribution of each player into the game, will be considered in this work.

Beyond its dynamic rationale, coalitional control can also be used to determine static neighborhoods [15]. In fact, it can be profitable for the overall system that several agents permanently share their information, and some others do not communicate at all. This natural procedure, necessary to determine the structure that should be considered for the control network, is known as the system decomposition or *partitioning* [16]–[20]. In this way, the main contribution of this work is to perform the partitioning of an LSS by means of a Shapley value-based partitioning algorithm, which represents an enhancement of a preliminar algorithm – only suitable for small networks – introduced in [15]. To this end, a cooperative game defined in the set of agents is introduced here. A coalitional MPC scheme, which allows for including constraints on the states and the inputs, is also used in this paper to measure the cost of the different coalitions enabled at each time instant. Likewise, a way to redistribute the game-theory results from the agents to the links is proposed in this work. The straightforward consequence is a mitigation of combinatorial explosion issues with respect to [15]. To test the proposed partitioning algorithm, the Barcelona drinking water network, which was reported in [21], [22], is chosen here as the case study.

The remainder of this paper is organized as follows. In Section II, the problem setting is stated in a coalitional networked framework. Next, in Section III, a connection between the fields of coalitional control and cooperative game theory is given and characteristic indices for the Shapley value of the links are proposed as well. In Section IV, a Shapley value-based partitioning algorithm is presented. In Section V, the Barcelona drinking water network is introduced as the case study to illustrate the effectiveness of the proposed partitioning approach. Finally, conclusions and lines of future work are presented in Section VI.

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II. COALITIONAL MPC

Consider the class of distributed linear systems constituted by $\mathcal{N} = \{1, 2, \ldots, |\mathcal{N}|\}$ interconnected subsystems or agents. The dynamics of subsystem $i \in \mathcal{N}$ can be mathematically described as

$$\begin{aligned} \mathbf{x}_i(k+1) &= \mathbf{A}_{ii}\mathbf{x}_i(k) + \mathbf{B}_{ii}\mathbf{u}_i(k) + \mathbf{w}_i(k), \\ \mathbf{w}_i(k) &= \sum_{j \neq i} [\mathbf{A}_{ij}\mathbf{x}_j(k) + \mathbf{B}_{ij}\mathbf{u}_j(k)] + \mathbf{B}_{\mathbf{p}_i}\mathbf{d}_i(k), \end{aligned} \tag{1}$$

with $\mathbf{x}_i(k) \in \mathbb{R}^{n_{\mathrm{x}_i}}$ being the state vector of agent i, $\mathbf{u}_i(k) \in \mathbb{R}^{n_{\mathrm{u}_i}}$ its input vector, and $\mathbf{w}_i(k) \in \mathbb{R}^{n_{\mathrm{x}_i}}$ the related disturbances, which can be either external to the whole system, denoted by $\mathbf{d}_i(k) \in \mathbb{R}^{n_{\mathrm{d}_i}}$, or be caused by the neighbors as well. Likewise, $\mathbf{A}_{ii} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{x}_i}}$, $\mathbf{B}_{ii} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{u}_i}}$, $\mathbf{A}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{x}_j}}$, $\mathbf{B}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{u}_j}}$ and $\mathbf{B}_{\mathbf{p}_i} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{d}_i}}$ are system matrices of suitable dimensions. It is also assumed that states and inputs are constrained into an independent set defined by a collection of linear inequalities, i.e.,

$$\mathbf{x}_i(k) \in \mathcal{X}_i, \quad \mathcal{X}_i \subseteq \mathbb{R}^{n_{\mathrm{x}_i}}, \qquad \mathbf{u}_i(k) \in \mathcal{U}_i, \quad \mathcal{U}_i \subseteq \mathbb{R}^{n_{\mathrm{u}_i}}. \tag{2}$$

The goal of each local controller is to steer a sequence of future states over a prediction horizon N_{p}, that is, $\mathbf{X}_i(k+1 : k+N_{\mathrm{p}}) = \{\mathbf{x}_i(k+1), \ldots, \mathbf{x}_i(k+N_{\mathrm{p}})\}$. To this end, the controller must solve the following open-loop finite-horizon optimization problem:

$$\mathbf{U}_i^*(k : k+N_{\mathrm{p}}-1) = \arg\min_{\mathbf{U}_i(k:k+N_{\mathrm{p}}-1)} \sum_{r=0}^{N_{\mathrm{p}}-1} \ell_i\big(\mathbf{x}_i(k+r), \mathbf{u}_i(k+r)\big), \tag{3}$$

subject to (1), (2), a forecast of the expected disturbances $\hat{\mathbf{W}}_i(k : k+N_{\mathrm{p}}-1) = \{\hat{\mathbf{w}}_i(k), \ldots, \hat{\mathbf{w}}_i(k+N_{\mathrm{p}}-1)\}$, and a measured initial state $\hat{\mathbf{x}}_i(k)$. Likewise, $\ell_i\big(\mathbf{x}_i(k), \mathbf{u}_i(k)\big)$ is related to a certain convex stage cost that is minimized along N_{p} at each time step. As a result, the sequence of the optimal control inputs over N_{p}, that is, $\mathbf{U}_i^*(k : k+N_{\mathrm{p}}-1) = \{\mathbf{u}_i^*(k), \ldots, \mathbf{u}_i^*(k+N_{\mathrm{p}}-1)\}$ is obtained.

A. Networked Coalitional Structure

In coalitional control, the agents are merged at each time instant into several $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n_c}$ disjoint neighborhoods or coalitions, with $\bigcup_{r=1}^{n_c} \mathcal{C}_r = \mathcal{N}$, which behave as a single agent and evolve *dynamically* with time. Conversely, this work deals with coalitional control in a static sense, being the objective here to find a *time-independent* set $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n_c}$. It is possible to manage this approach from a graph-theory viewpoint, considering that agents in $\mathcal{N}$ are initially connected by a network characterized by an undirected graph $(\mathcal{N}, \mathcal{E})$, where $\mathcal{E} = \mathcal{N} \times \mathcal{N}$ is the set of edges or links corresponding to all possible communication connections among the agents. Notice that the number of elements in both sets are related by

$$|\mathcal{E}| = ½\ |\mathcal{N}||\mathcal{N} - 1|. \tag{4}$$

Each link $l \in \mathcal{E}$ can be classified according to its relevance from a control viewpoint. In fact, it can be more profitable for the overall system performance to fix/disconnect some links permanently. This way, the partitioning objective will correspond to find the best configuration of links, named *network topology* and denoted by Λ, according to several partitioning requirements, as it will be shown in Section IV. Notice that the condition for a coalition $\mathcal{C} \subseteq \mathcal{N}$ to be formed is to be connected by topology Λ. When it does happen, a model analogous to (1) is calculated at a coalition level, i.e.,

$$\begin{aligned} \mathbf{x}_{\mathcal{C}}(k+1) &= \mathbf{A}_{\mathcal{C}\mathcal{C}}\mathbf{x}_{\mathcal{C}}(k) + \mathbf{B}_{\mathcal{C}\mathcal{C}}\mathbf{u}_{\mathcal{C}}(k) + \mathbf{w}_{\mathcal{C}}(k), \\ \mathbf{w}_{\mathcal{C}}(k) &= \sum_{j \notin \mathcal{C}} [\mathbf{A}_{\mathcal{C}j}\mathbf{x}_j(k) + \mathbf{B}_{\mathcal{C}j}\mathbf{u}_j(k)] + \mathbf{B}_{\mathbf{p}_{\mathcal{C}}}\mathbf{d}_{\mathcal{C}}(k), \end{aligned} \tag{5}$$

with $\mathbf{x}_{\mathcal{C}}(k) = [\mathbf{x}_i(k)]_{i\in\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}}}$, $\mathbf{u}_{\mathcal{C}}(k) = [\mathbf{u}_i(k)]_{i\in\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{u}_\mathcal{C}}}$, $\mathbf{w}_{\mathcal{C}}(k) = [\mathbf{w}_i(k)]_{i\in\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}}}$ and $\mathbf{d}_{\mathcal{C}}(k) = [\mathbf{d}_i(k)]_{i\in\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{d}_\mathcal{C}}}$ being respectively the coalitional states, inputs, overall disturbances and external disturbances that aggregate the corresponding vectors, and $\mathbf{A}_{\mathcal{C}\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}} \times n_{\mathrm{x}_\mathcal{C}}}$, $\mathbf{B}_{\mathcal{C}\mathcal{C}} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}} \times n_{\mathrm{u}_\mathcal{C}}}$, $\mathbf{A}_{\mathcal{C}j} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}} \times n_{\mathrm{x}_j}}$, $\mathbf{B}_{\mathcal{C}j} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}} \times n_{\mathrm{u}_j}}$ and $\mathbf{B}_{\mathbf{p}_{\mathcal{C}}} \in \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}} \times n_{\mathrm{d}_\mathcal{C}}}$ are obtained by aggregating the corresponding individual matrices. The coalitional constraints become

$$\begin{aligned} \mathbf{x}_{\mathcal{C}}(k) \in \mathcal{X}_{\mathcal{C}} \subseteq \mathbb{R}^{n_{\mathrm{x}_\mathcal{C}}}, &\quad \mathcal{X}_{\mathcal{C}} = \textstyle\prod_{i\in\mathcal{C}} \mathcal{X}_i, \\ \mathbf{u}_{\mathcal{C}}(k) \in \mathcal{U}_{\mathcal{C}} \subseteq \mathbb{R}^{n_{\mathrm{u}_\mathcal{C}}}, &\quad \mathcal{U}_{\mathcal{C}} = \textstyle\prod_{i\in\mathcal{C}} \mathcal{U}_i. \end{aligned} \tag{6}$$

Consequently, the coalitional MPC controller solves an optimization problem at each time instant k described by

$$\mathbf{U}_{\mathcal{C}}^*(k : k+N_{\mathrm{p}}-1) = \arg\min_{\mathbf{U}_{\mathcal{C}}(k:k+N_{\mathrm{p}}-1)} \sum_{r=0}^{N_{\mathrm{p}}-1} \ell_{\mathcal{C}}\big(\mathbf{x}_{\mathcal{C}}(k+r), \mathbf{u}_{\mathcal{C}}(k+r)\big), \tag{7}$$

subject to (5), (6), the aggregate forecast of the expected disturbances $\hat{\mathbf{W}}_{\mathcal{C}}(k : k+N_{\mathrm{p}}-1)$, and a measured coalitional initial state $\hat{\mathbf{x}}_{\mathcal{C}}(k)$. Also, $\ell_{\mathcal{C}}\big(\mathbf{x}_{\mathcal{C}}(k), \mathbf{u}_{\mathcal{C}}(k)\big)$ is the coalitional stage cost to be minimized, and $\mathbf{U}_{\mathcal{C}}^*(k : k+N_{\mathrm{p}}-1)$ refers to the optimal sequence of coalitional control inputs over N_{p}.

Finally, note that to compute the centralized MPC scheme in a distributed fashion, it is enough to calculate the optimal input sequence by taking $\mathcal{C} = \mathcal{N}$ and solving (7).

III. GAME-THEORY VIEWPOINT

In [10]–[12], [15], the key to integrate game-theory results into distributed control was to interpret the set of edges $\mathcal{E}$ as the set of players in a cooperative game by defining a certain characteristic function that assigns a value to each topology $\Lambda \subseteq \mathcal{E}$. In this work, this perspective is changed to working directly with a game defined over the set of agents $\mathcal{N}$. To this end, a cost function $\boldsymbol{v}$ that assigns a cost to each coalition of players $\mathcal{C} \subseteq \mathcal{N}$ is defined by

$$\begin{aligned} v(\mathcal{C}, \mathbf{x}_{\mathcal{N}}) = \sum_{r=0}^{N_{\mathrm{p}}-1} \Bigg[& \ell_{\mathcal{C}}\big(\mathbf{x}_{\mathcal{C}}(k+r+1), \mathbf{u}_{\mathcal{C}}^*(k+r)\big) \\ & + \sum_{i \notin \mathcal{C}} \ell_i\big(\mathbf{x}_i(k+r+1), \mathbf{u}_i^*(k+r)\big) \Bigg], \end{aligned} \tag{8}$$

with $\ell_i\big(\mathbf{x}_i(k), \mathbf{u}_i(k)\big)$ and $\ell_{\mathcal{C}}\big(\mathbf{x}_{\mathcal{C}}(k), \mathbf{u}_{\mathcal{C}}(k)\big)$ being the stage costs introduced in the previous section. This cost function is evaluated by computing the control sequence of coalition $\mathcal{C}$, i.e., $\mathbf{u}_{\mathcal{C}}^*(k)$, which is obtained by solving (7). The rest of the agents calculate their input sequences $\mathbf{u}_i^*(k)$ by solving (3) independently. This choice avoids undefinition issues because it allows evaluating (8) with information from *all* the agents that take part in game $(\mathcal{N}, \boldsymbol{v})$, independently if they are either in or out coalition $\mathcal{C}$. This way, the use of (8) has a clear advantage with respect to the approach in [15], where linear feedback gains $\mathbf{K}_\Lambda$ were designed for each Λ. Here, each coalition $\mathcal{C}$ solves its own optimization problem, which is decoupled from the rest of the network. Hence, only $2^{|\mathcal{N}|}$ optimization problems should be solved, far less than the $2^{|\mathcal{E}|}$ problems required in [15].

Once the game is defined, the next step is to choose a payoff rule to get the corresponding cost or benefit that each player expects from the game. In this work the Shapley value [14] is considered. It assigns to game $(\mathcal{N}, \boldsymbol{v})$ vector $\boldsymbol{\phi}(\mathcal{N}, \boldsymbol{v})$, defined $\forall i \in \mathcal{N}$ as

$$\phi_i(\mathcal{N}, \boldsymbol{v}) = \sum_{\mathcal{C} \subseteq \mathcal{N}, i \notin \mathcal{C}} \frac{|\mathcal{C}|!(|\mathcal{N}| - |\mathcal{C}| - 1)!}{|\mathcal{N}|!} [v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})], \tag{9}$$

that is, the marginal contribution of each agent i is averaged for all the possible coalition permutations it can be part of.

Given that the partitioning procedure proposed in this work will be performed by enabling/disabling links among the different agents, a measure of the relevance of the links is required. This way, note that given a link $l = \{i, j\} \in \mathcal{E}$, it is possible to redistribute the Shapley value of the agents that are the end-points of this link, i.e., i and j, by means of the following expression:

$$\xi_l(\mathcal{N}, \boldsymbol{v}) = \frac{1}{|\mathcal{E}_i|}\phi_i(\mathcal{N}, \boldsymbol{v}) + \frac{1}{|\mathcal{E}_j|}\phi_j(\mathcal{N}, \boldsymbol{v}), \quad l = \{i, j\}, \tag{10}$$

with $\mathcal{E}_i$ and $\mathcal{E}_j$ being, respectively, the set of links connected to agents i and j.

A. Measure Indices based on the Shapley Value

In this section, two indices related to the Shapley value are provided to supply information regarding the relevance of the links. To this end, the following procedure is considered:

<u>*Procedure*</u>

Let L be a given number of samples. Do, for each sample:

a) Obtain a measurement of the initial state $\hat{\mathbf{x}}_{\mathcal{N}}(k)$ and the expected disturbances $\hat{\mathbf{W}}_{\mathcal{N}}(k : k + N_{\mathrm{p}} - 1)$.
b) Calculate $v(\mathcal{C})$ by using (8), $\forall \mathcal{C} \subseteq \mathcal{N}$, where $\mathbf{u}_{\mathcal{C}}^*(k)$ and $\mathbf{u}_i^*(k)$ are obtained, respectively, by solving (7) for coalition $\mathcal{C}$ and (3), for the agents out of $\mathcal{C}$.
c) Evaluate $\phi_i(\mathcal{N}, \boldsymbol{v}), \forall i \in \mathcal{N}$, by means of (9).

Note that, it is possible to calculate the mean value and the standard deviation from the obtained results. Their discrete-time equations, considering L samples and equiprobable elements, can be found in [23], and reproduced below.

$$\mu_i = \frac{1}{L}\sum_{h=1}^{L} \phi_i^h(\mathcal{N}, \boldsymbol{v}), \quad \sigma_i = \sqrt{\frac{1}{L}\sum_{h=1}^{L}(\phi_i^h(\mathcal{N}, \boldsymbol{v}) - \mu_i)^2}. \tag{11}$$

Finally, the redistribution of the aforementioned values among the links is provided by (10), i.e., with $l = \{i, j\}$

$$\mu_l^\xi = \frac{1}{|\mathcal{E}_i|}\mu_i + \frac{1}{|\mathcal{E}_j|}\mu_j, \tag{12}$$

$$\sigma_l^\xi = \frac{1}{|\mathcal{E}_i|}\sigma_i + \frac{1}{|\mathcal{E}_j|}\sigma_j. \tag{13}$$

Note that (12) provides a way to arrange and compare the links according to their relevance from a control-performance perspective. This way, the lower the mean value of a link is, the more useful the link becomes. Likewise, (13) represents a measure of how tightly the samples are clustered around the mean value. Both indices will be taken into account in the partitioning algorithm proposed in the following section.

IV. PARTITIONING ALGORITHM

In [15], a partitioning algorithm to group the atomic components of a distributed system into agents was provided. Here, this viewpoint is enhanced by considering that the agents stem from the constraints imposed on the system by the node equations, i.e., there is a pre-partitioning stage in which some atomic components are grouped due to the junction nodes. Once the agents are defined, the goal in this paper is to find what agents should cooperate to improve the overall system performance. To this end, as commented before, a communication link between each pair of agents is initially assumed, with the total number of links given by (4). Then, not only the mean value but also the standard deviation are considered in this work to classify the links through their Shapley value, by means of the computation of indices (12) and (13). Considering this ranking, the following partitioning algorithm is proposed:

<u>*Partitioning Algorithm*</u>

Let $(\mathcal{N}, \mathcal{E})$ be an undirected graph that describes a set $\mathcal{N}$ of agents connected by links $l \in \mathcal{E}$. Let $\mathscr{L}_{\mathrm{c}}, \mathscr{L}_{\mathrm{e}} \in \mathbb{R}$, $\kappa \in \mathbb{R}^+$ be given thresholds, with $\mathscr{L}_{\mathrm{c}} < \mathscr{L}_{\mathrm{e}}$. Then,

1) By using (12) and (13), respectively, obtain indices μ_l^ξ and σ_l^ξ, $\forall l \in \mathcal{E}$.
2) Keep fixed the links belonging to set $\mathcal{E}_{\mathrm{c}}$ with

$$l \in \mathcal{E}_{\mathrm{c}} \longleftrightarrow \begin{cases} \mu_l^\xi < \mathscr{L}_{\mathrm{c}}, \\ \frac{|\mu_l^\xi|}{\sigma_l^\xi} > \kappa. \end{cases} \tag{14}$$

3) Disconnect the links in set $\mathcal{E}_{\mathrm{e}}$, where

$$l \in \mathcal{E}_{\mathrm{e}} \longleftrightarrow \begin{cases} \mu_l^\xi > \mathscr{L}_{\mathrm{e}}, \\ \frac{|\mu_l^\xi|}{\sigma_l^\xi} > \kappa. \end{cases} \tag{15}$$

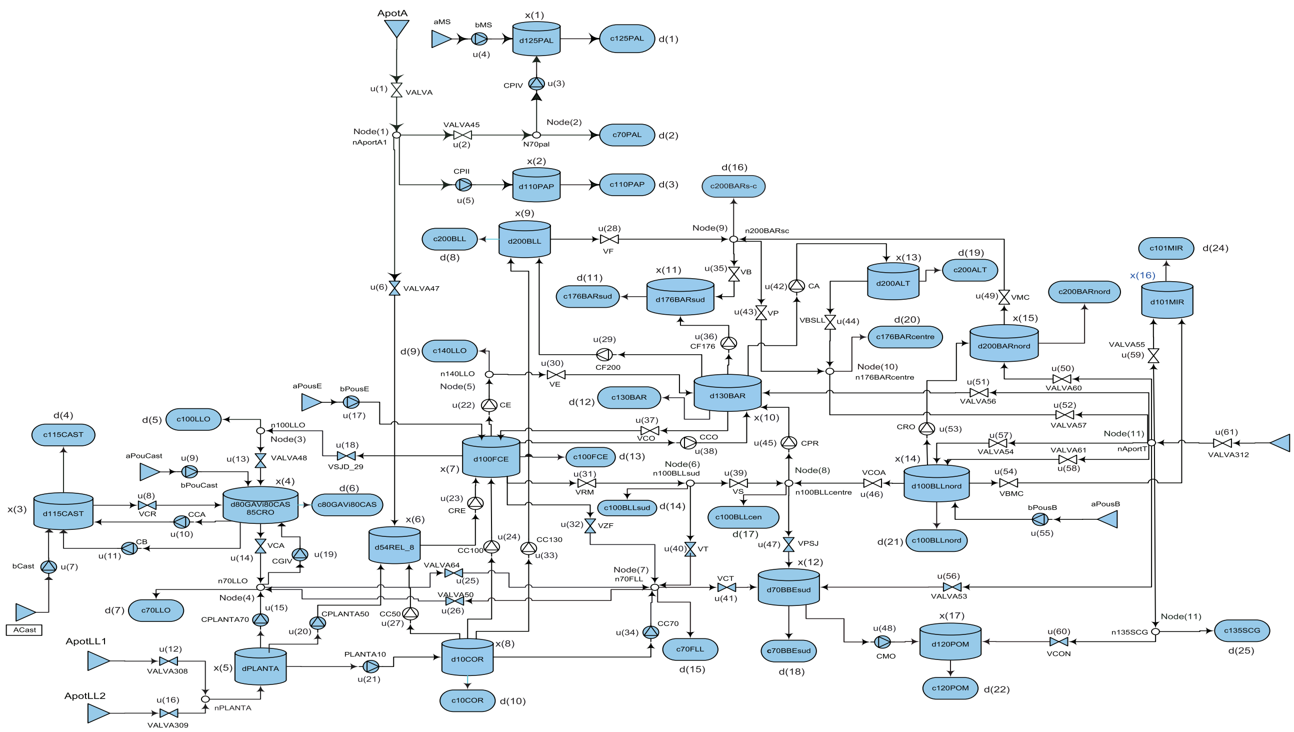

Fig. 1. Aggregate model of the Barcelona DWN

Therefore, links $l \in \mathcal{E}_c$ are economical in control terms and fixed, while links $l \in \mathcal{E}_e$ are removed since they are too costly for the system. In other words, the configuration of the system will be described after the partitioning by the following network:

$$(\mathcal{N}, \mathcal{E} \backslash (\mathcal{E}_c \cup \mathcal{E}_e)), \tag{16}$$

where the links in $\mathcal{E} \backslash (\mathcal{E}_c \cup \mathcal{E}_e)$ may be dynamically enabled or disabled at each time instant by means of a coalitional control approach, in the line of [10], [11]. A diagram that summarizes the partitioning method is given in Fig. 2.

Remark 1 *Term* $|\mu_l^\xi|/\sigma_l^\xi$ *decreases with the dispersion of the data set. Hence, by imposing* $|\mu_l^\xi|/\sigma_l^\xi > \kappa$*, the maximum dispersion for a link to be a suitable candidate for being always either fixed or disconnected is limited.*

V. CASE STUDY

The proposed partitioning scheme has been implemented in the Barcelona drinking water network (DWN), which is managed by Aguas de Barcelona, S.A. (AGBAR). The Barcelona DWN distributes the water supplied by the Ter and Llobregat rivers, which are regulated at their head by dams with an overall capacity of 600 hm^3, to the whole Barcelona metropolitan area. Besides the rivers, some additional underground wells also contribute to an overall flow of around 7 m^3/s, which is become potable by four drinking water treatment plants. Given the limits in the water flow provided by each source, there exist different water prices depending on water treatments and legal extraction canons.

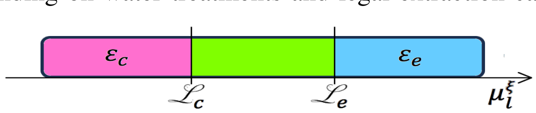

Fig. 2. Partitioning algorithm diagram

A. Coalitional Control Model

Control-oriented schemes for DWNs have been widely analyzed in the literature [24]. In particular, several control approaches of the Barcelona DWN are discussed in [21], [22]. In this paper, an aggregate version of the entire Barcelona DWN analyzed in [22] is considered and depicted in Fig. 1. This model contains 17 tanks, 61 actuators – divided in 26 pumps and 35 valves – and 25 sectors of consume that represent the external disturbances. As seen in Fig. 1, water volumes (in m^3) are indicated by x, flows (in m^3/s) by u and sectors of consume (also in m^3/s) by d, according to the notation introduced in Section II. The number that follows between parentheses identifies the corresponding variable.

As mentioned in the previous section, a pre-partitioning into agents will be performed due to the nodes that appear in Fig. 1. For example, the node equations

$$\begin{aligned} u(1) - u(2) - \boldsymbol{u(5)} - \boldsymbol{u(6)} &= 0, \\ u(2) - \boldsymbol{u(3)} - d(2) &= 0, \end{aligned} \tag{17}$$

physically connect flows $u(1)$, $u(2)$, $u(3)$, $u(5)$ and $u(6)$. Hence, the values of these flows must be determined at the same time. For this reason, all states that comprise *incoming* flows involved in (17) should belong to the same agent, i.e.,

$$\begin{aligned} x^+(1) &= x(1) + \boldsymbol{u(3)} + u(4) - d(1), \\ x^+(2) &= x(2) + \boldsymbol{u(5)} - d(3), \\ x^+(6) &= x(6) + \boldsymbol{u(6)} + u(20) + u(27) - u(23), \end{aligned} \tag{18}$$

where superindex $^+$ refers to the successors state. Following this approach, nine agents have been defined, where the criterion of considering *outgoing* flows, e.g., $u(23)$, as disturbances has been assumed. For instance, agent 1 is fully described by (17) and (18).

Note that the coupling among the subsystems is given through their inputs. Therefore, in the case study, $\mathbf{A}_{ij} = 0$ in (1), and equivalently, $\mathbf{A}_{\mathcal{C}j} = 0$ in (5). From an overall centralized viewpoint, the following equations are satisfied:

$$\mathbf{x}_{\mathcal{N}}(k+1) = \mathbf{A}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{B}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k) + \mathbf{B}_{\mathbf{p}_{\mathcal{N}}}\mathbf{d}_{\mathcal{N}}(k), \quad (19a)$$

$$0 = \mathbf{E}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k) + \mathbf{E}_{\mathbf{d}\mathcal{N}}\mathbf{d}_{\mathcal{N}}(k), \quad (19b)$$

with $\mathbf{x}_{\mathcal{N}}(k) \in \mathbb{R}^{17}$, $\mathbf{u}_{\mathcal{N}}(k) \in \mathbb{R}^{61}$ and $\mathbf{d}_{\mathcal{N}}(k) \in \mathbb{R}^{25}$. This way, (19a) corresponds with the dynamics of the storage tanks, and (19b) describes the network static relations due to the mass balance at junction nodes. Notice that, from a centralized viewpoint, $\mathbf{w}_{\mathcal{N}}(k)$ is only composed of the external disturbances $\mathbf{B}_{\mathbf{p}_{\mathcal{N}}}\mathbf{d}_{\mathcal{N}}(k)$.

Finally, consider the main physical constraints of the DWN given by the variables related to the tank volumes and manipulated flows, i.e., $\forall k$

$$\mathbf{x}_{\mathcal{N}}^{\min} \leq \mathbf{x}_{\mathcal{N}}(k) \leq \mathbf{x}_{\mathcal{N}}^{\max}, \qquad \mathbf{u}_{\mathcal{N}}^{\min} \leq \mathbf{u}_{\mathcal{N}}(k) \leq \mathbf{u}_{\mathcal{N}}^{\max}. \quad (20)$$

B. System Management Criteria

The following management policies for the Barcelona DWN are considered given the knowledge of the system and the performance objectives to be reached (see [21], [22]):

- Minimizing drinking water production and transport costs due to chemicals, legal canons and electricity costs, which are expressed as

$$f_{1,i}(k) = \mathbf{W_e}(\boldsymbol{\alpha_1} + \boldsymbol{\alpha_2}(k))^{\mathrm{T}}\mathbf{u}_i(k), \quad (21)$$

  where vector $\boldsymbol{\alpha_1} \in \mathbb{R}^{n_{\mathrm{u}_i}}$ corresponds to water costs, vector $\boldsymbol{\alpha_2}(k) \in \mathbb{R}^{n_{\mathrm{u}_i}}$ considers time-dependent electricity costs, and matrix $\mathbf{W_e} \in \mathbb{R}^{n_{\mathrm{u}_i} \times n_{\mathrm{u}_i}}$ adds the corresponding prioritization to the aforementioned costs.

- Maintaining the stored volume around a given safety value in case of emergency, which is achieved by minimizing

$$f_{2,i}(k) = (\mathbf{x}_i(k) - \beta\mathbf{x}_i^{\max})^{\mathrm{T}}\mathbf{W_x}(\mathbf{x}_i(k) - \beta\mathbf{x}_i^{\max}), \quad (22)$$

  with $\beta \in \mathbb{R}^+$ being a safety volumen parameter, where $\mathbf{x}_i^{\max} \in \mathbb{R}^{n_{\mathrm{x}_i}}$ takes from (20) the corresponding states, and with $\mathbf{W_x} \in \mathbb{R}^{n_{\mathrm{x}_i} \times n_{\mathrm{x}_i}}$ being a weighting matrix.

- Penalizing sudden variations of the control inputs by minimizing

$$f_{3,i}(k) = \Delta\mathbf{u}_i^{\mathrm{T}}(k)\mathbf{W_{\Delta u}}\Delta\mathbf{u}_i(k), \quad (23)$$

  with $\Delta\mathbf{u}_i(k) = \mathbf{u}_i(k) - \mathbf{u}_i(k-1)$, and where $\mathbf{W_{\Delta u}} \in \mathbb{R}^{n_{\mathrm{u}_i} \times n_{\mathrm{u}_i}}$ is also a weighting matrix.

Therefore, the following individual cost related to agent $i \in \mathcal{N}$ is considered in this work:

$$\ell_i(k) = f_{1,i}(k) + f_{2,i}(k) + f_{3,i}(k), \quad (24)$$

being the aggregate cost of a certain coalition $\mathcal{C}$, defined by

$$\ell_{\mathcal{C}}(k) = \sum_{i \in \mathcal{C}} \ell_i(k). \quad (25)$$

TABLE I
MEAN VALUES AND STANDARD DEVIATIONS, $l = \{i, j\}$

$\mu_{ij}^{\xi}(\times 10^9)$		$\sigma_{ij}^{\xi}(\times 10^8)$	
$\mu_{12} = -0.1829$	$\mu_{37} = 0.2281$	$\sigma_{12} = 1.6701$	$\sigma_{37} = 1.3613$
$\mu_{13} = 0.2460$	$\mu_{38} = -1.5507$	$\sigma_{13} = 1.3651$	$\sigma_{38} = 1.2035$
$\mu_{14} = 2.1817$	$\mu_{39} = -1.0447$	$\sigma_{14} = 1.4616$	$\sigma_{39} = 0.9295$
$\mu_{15} = 2.1430$	$\mu_{45} = 2.1085$	$\sigma_{15} = 1.4321$	$\sigma_{45} = 1.4245$
$\mu_{16} = -0.1615$	$\mu_{46} = -0.1960$	$\sigma_{16} = 1.6026$	$\sigma_{46} = 1.5949$
$\mu_{17} = 2.1983$	$\mu_{47} = 2.1638$	$\sigma_{17} = 1.4655$	$\sigma_{47} = 1.4578$
$\mu_{18} = 0.4194$	$\mu_{48} = 0.3850$	$\sigma_{18} = 1.3077$	$\sigma_{48} = 1.3000$
$\mu_{19} = 0.9254$	$\mu_{49} = 0.8910$	$\sigma_{19} = 1.0337$	$\sigma_{49} = 1.0260$
$\mu_{23} = -2.1530$	$\mu_{56} = -0.2346$	$\sigma_{23} = 1.5659$	$\sigma_{56} = 1.5654$
$\mu_{24} = -0.2174$	$\mu_{57} = 2.1251$	$\sigma_{24} = 1.6624$	$\sigma_{57} = 1.4283$
$\mu_{25} = -0.2560$	$\mu_{58} = 0.3463$	$\sigma_{25} = 1.6329$	$\sigma_{58} = 1.2706$
$\mu_{26} = -2.5605$	$\mu_{59} = 0.8523$	$\sigma_{26} = 1.8034$	$\sigma_{59} = 0.9966$
$\mu_{27} = -0.2008$	$\mu_{67} = -0.1794$	$\sigma_{27} = 1.6663$	$\sigma_{67} = 1.5988$
$\mu_{28} = -1.9796$	$\mu_{68} = -1.9582$	$\sigma_{28} = 1.5085$	$\sigma_{68} = 1.4410$
$\mu_{29} = -1.4736$	$\mu_{69} = -1.4522$	$\sigma_{29} = 1.2345$	$\sigma_{69} = 1.1670$
$\mu_{34} = 0.2115$	$\mu_{78} = 0.4016$	$\sigma_{34} = 1.3574$	$\sigma_{78} = 1.3039$
$\mu_{35} = 0.1729$	$\mu_{79} = 0.9076$	$\sigma_{35} = 1.3279$	$\sigma_{79} = 1.0299$
$\mu_{36} = -2.1316$	$\mu_{89} = -0.8713$	$\sigma_{36} = 1.4984$	$\sigma_{89} = 0.8721$

C. Simulation Results

The results presented in this work have been tested for the Barcelona DWN by using the Matlab® solver `quadprog` in a 2.6 GHz Intel Core® i5, 8 GB RAM computer. This way, a coalitional MPC scheme has been implemented in open loop with a prediction horizon $N_{\mathrm{p}} = 24$, and the following values for the performance parameters: $\mathbf{W_e} = 0.7\mathbf{I}$, $\beta = 0.8$, $\mathbf{W_x} = 0.2\mathbf{I}$, $\mathbf{W_{\Delta u}} = 0.1\mathbf{I}$, with $\mathbf{I}$ being the identity matrix of suitable dimensions. According to (4), the nine agents obtained due to the node equations are related to 36 possible communication links. The mean values and standard deviations for all these links have been calculated by means of (12) and (13), considering $L = 100$ measurements of the initial state and expected disturbances, which are taken from historical data. Relevant statistics are represented in Table I.

At this point, the partitioning algorithm proposed in Section IV has been implemented with $\mathscr{L}_{\mathrm{c}} = -1.8 \times 10^9$, $\mathscr{L}_{\mathrm{e}} = 0.8 \times 10^9$ and $\kappa = 10$, set by trial and error. As a result, sets $\mathcal{E}_{\mathrm{c}}$ and $\mathcal{E}_{\mathrm{e}}$ are given by

$$\mathcal{E}_{\mathrm{c}} = \{\{2,3\},\{2,6\},\{2,8\},\{3,6\},\{6,8\}\}, \quad (26)$$

$$\mathcal{E}_{\mathrm{e}} = \{\{1,4\},\{1,5\},\{1,7\},\{4,5\},\{4,7\},\{5,7\}\}, \quad (27)$$

where links $\{1,9\}$, $\{4,9\}$, $\{5,9\}$, and $\{7,9\}$ were not included in set $\mathcal{E}_{\mathrm{e}}$ since they do not verify condition $\frac{|\mu_l^{\xi}|}{\sigma_l^{\xi}} > \kappa$. As extracted from (26), it can be concluded that agents 2, 3, 6 and 8 should be merged in a new single agent. Likewise, according to (27), there should not be cooperation among agents 1, 4, 5 and 7. Nevertheless, agent 9 is free to cooperate with anyone given its large dispersion. An overview of the results provided by the proposed partitioning algorithm is shown in Fig. 3, where the links have been drawn in a color scale between green and yellow, with darkest links representing the useful ones.

Note that it is possible to verify that the performance of the proposed algorithm is good, by examining the best coalitions of agents according to the mean cost they obtained

TABLE II

AGENT OCCURRENCES IN THE 20 BEST-PERFORMANCE COALITIONS

Agent	1	2	3	4	5	6	7	8	9
Ocurrences	11	20	20	12	11	20	11	20	16

in our experiments. In Table II, it is shown how many times each agent appears in the 20 coalitions with minimum mean cost. As can be seen, agents 2, 3, 6 and 8 appear in all of these coalitions, fact that shows their relevance. Agents 1, 4, 5 and 7 appear only in 11 or 12 out of these coalitions, which again is aligned with the outcome of the proposed approach. Finally, agent 9 appears in 16 out of the 20 best coalitions, which makes it a worthy candidate for occasional information exchanges.

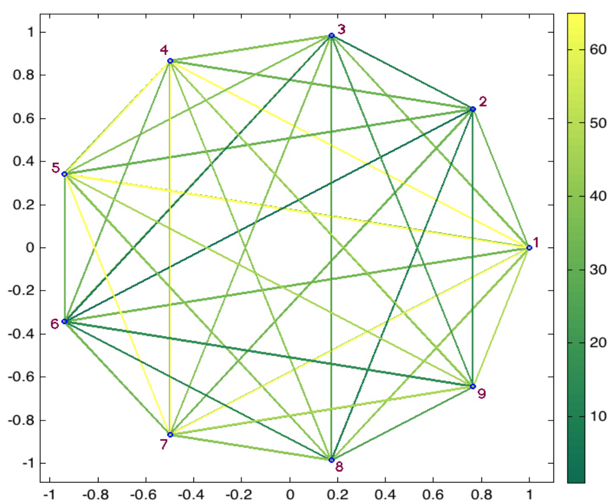

Fig. 3. Case study partitioning overview

VI. CONCLUSIONS AND FUTURE WORK

In this work, a heuristic partitioning algorithm that classifies the communication links inside a network from a control-performance perspective has been introduced. A game over agents has been defined and coalitional model predictive control has been considered in the optimization procedure. A way to redistribute the Shapley value of the agents among the links has been presented as well. The choice of fixing/removing links is then based on indices related to their Shapley value's mean and standard deviation. The proposed algorithm has been tested with the Barcelona drinking water network as an LSS case study.

The partitioning performed is aligned with the results obtained by examining the best coalitions, which illustrates the feasibility of the proposed approach. For this reason, current work includes the use of randomized methods to apply the proposed partitioning algorithm to larger networks, where the exhaustive computing of every coalition is not possible. Also, the current static nature of the proposed partitioning approach motivates to explore its application to dynamic partitioning to improve the system performance. Likewise, a stability analysis of the proposed algorithm, performance comparisons with other partitioning methods, and alternative control formulations instead of (8) could also be taken into account in further works.

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