State observer design for quadratic parameter varying (QPV) systems

Damiano Rotondo, Tor Arne Johansen

Abstract— **This paper addresses the problem of state observation in quadratic parameter varying (QPV) systems. In particular, a state observer is designed in such a way that the estimation error converges to zero with a desired rate of convergence in a given polytopic region of the error space. Under some assumptions, it is shown that design conditions can be given in the form of a set of bilinear matrix inequalities (BMIs), which can be reduced to linear matrix inequalities (LMIs), which are computationally more tractable. The main characteristics of the proposed approach are illustrated by means of an example, which confirms the validity of the theoretical results.**

Keywords: Quadratic systems, Lyapunov stability, observer design, gain-scheduling, linear matrix inequalities.

I. Introduction

The problem of state observation for nonlinear systems is of main importance in control systems theory, due to the fact that in many situations, some states are not accessible or state measurements are difficult and/or expensive to obtain. For this reason, since the last century, many researchers have worked on the extension of linear approaches for the design of state observers, as the one developed by [1], to nonlinear systems, see e.g. [2]–[4].

Among the approaches that have received attention from the control community in recent years, there are the gain-scheduling ones [5]. Gain-scheduling refers to changes in the control loop depending on the specific operation point about which the system is working. Successful gain-scheduling paradigms are the fuzzy Takagi-Sugeno (TS) [6] and the linear parameter-varying (LPV) [7] frameworks, for which several results concerning state observers have been obtained recently, see e.g. [8]–[12]. The main strength of these frameworks is that they have proved to be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters, that will depend on some endogenous signals, e.g. states, inputs or outputs [13]. In this case, the handled system is usually referred to as *quasi-LPV*, to make a further distinction with respect to *pure* LPV systems, for which the varying parameters depend only on exogenous signals [14].

Some recent research has been devoted to study quadratic systems, i.e. systems whose state equation contains cross-products between state variables, which can explain the dynamic behavior of several phenomena in a wide range of applications [15], [16]. As a matter of example, quadratic systems describe the dynamics of electric power systems [17], chemical reactors [18], robots [19], hydrodynamic flows [20], as well as the interaction between tumors and immune systems [21]. The problem of state observation for quadratic systems has been analyzed by a few works, see, e.g., [22], [23].

Most of the gain-scheduling approaches found in the literature have been developed for systems with a linear structure. However, in the last few years, some results about quadratic parameter varying (QPV) systems have appeared [24]–[26]. The QPV framework can be used to characterize some nonlinear systems, such as robotic manipulators [27] and inverted pendula [28], which are described typically by dynamic equations where quadratic terms depending on state variables appear due to centrifugal and Coriolis forces. Moreover, QPV systems can be obtained by calculating the first and second order terms of the Taylor expansion of a nonlinear plant about a family of operating points (this approach is the quadratic equivalent of the *linearization scheduling* for LPV systems [29]). The main advantage of QPV systems, when compared to LPV systems, comes from the fact that a bigger class of nonlinearities can be represented exactly by the former, without resorting to partially embedding the state as a parameter, which would lead to *quasi-LPV* models, hence introducing conservativeness since the dynamics of the state is not taken into account properly.

More specifically, [24] have considered the problem of local stabilization for QPV systems using parameter-dependent quadratic Lyapunov functions and an S-procedure approach. On the other hand, [25] have considered the enforcement of a given polytopic region of the state space to belong to the region of attraction of the QPV system. Finally, from a more applicative point of view, [26] have shown that, by employing a QPV suspension system, the suspension shock performance of a vehicle can be improved without degrading road holding and ride comfort. In contrast to the previously mentioned works, which addressed the control of QPV systems, the goal of this paper is to address the observer design problem for QPV systems. In particular, the contribution consists in extending the results previously

This work has been partially funded by the Spanish State Research Agency (AEI) and the European Regional Development Fund (ERFD) through the projects SCAV (ref. MINECO DPI2017-88403-R) and DEOCS (ref. MINECO DPI2016-76493). This work has been also supported by the AEI through the Maria de Maeztu Seal of Excellence to IRI (MDM-2016-0656) and the grant Juan de la Cierva-Formacion (FJCI-2016-29019), and by the Research Council of Norway through the AMOS Center, grant number 223254.

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obtained by [23] for quadratic time invariant (QTI) systems, to the QPV case. Hence, the paper provides a set of design conditions for ensuring the convergence to zero of the state estimation error in a given polytopic region of the error space, under the assumptions that: i) the varying parameter vector is known in real-time; and ii) the trajectory of the state is inside an ellipsoidal set, which contains the above-mentioned polytope.

The paper is structured as follows. Section II presents the QPV state observer and the main assumptions made in this work. The design conditions for calculating a suitable observer gain are provided in Section III. Section IV presents a numerical example that shows the main features of the developed approach. Finally, the conclusions are discussed in Section V.

Notation: Given a symmetric matrix M, the notation $M \succ 0$ ($M \prec 0$) means that M is positive (negative) definite. On the other hand, $M \succeq 0$ ($M \preceq 0$) denotes that M is positive (negative) semidefinite. The shorthand notation $He\{M\}$ will be used for $M + M^T$.

II. Problem formulation

Let us consider the following quadratic parameter varying (QPV) system:

$$\dot{x}(t) = A(\theta(t))x(t) + N(\theta(t), x(t))x(t) + B(\theta(t))u(t) \quad (1)$$

$$y(t) = C(\theta(t))x(t) + M(\theta(t), x(t))x(t) \quad (2)$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the known input, $y \in \mathbb{R}^{n_y}$ is the system output, and $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the varying parameter vector that schedules both the matrix functions $A(\theta(t)), B(\theta(t)), C(\theta(t))$ and the nonlinear functions $N(\theta(t), x(t))$, $M(\theta(t), x(t))$, defined as:

$$N(\theta(t), x(t)) = \begin{pmatrix} x(t)^T N_1(\theta(t)) \\ x(t)^T N_2(\theta(t)) \\ \vdots \\ x(t)^T N_{n_x}(\theta(t)) \end{pmatrix} \quad (3)$$

$$M(\theta(t), x(t)) = \begin{pmatrix} x(t)^T M_1(\theta(t)) \\ x(t)^T M_2(\theta(t)) \\ \vdots \\ x(t)^T M_{n_y}(\theta(t)) \end{pmatrix} \quad (4)$$

where $N_1(\theta(t)), \ldots, N_{n_x}(\theta(t)), M_1(\theta(t)), \ldots, M_{n_y}(\theta(t))$ are matrix functions of appropriate dimensions.

Given a suitable number of points $x_{(i)}$, $i = 1, \ldots, p$, let us denote as $\mathcal{P} \subset \mathbb{R}^{n_x}$, with $0 \in \mathcal{P}$, the polytope obtained as:

$$\mathcal{P} = Co\left\{x_{(1)}, x_{(2)}, \ldots, x_{(p)}\right\} \quad (5)$$

where $Co\{\cdot\}$ denotes the convex hull operation. The representation (5) is usually called *vertex (V-) representation*, and is equivalent to the so-called *half-space (H-) representation*:

$$\mathcal{P} = \left\{x \in \mathbb{R}^{n_x} : a_k^T x \leq 1, k = 1, \ldots, q\right\} \quad (6)$$

where q is an appropriate integer number [30].

Given the QPV system (1)-(2), the polytope $\mathcal{P}$ and a scalar $\alpha > 0$, we want to find a gain-scheduled dynamical system (state observer):

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), u(t), y(t), \theta(t)), \quad \hat{x}(0) = 0 \quad (7)$$

where $\hat{x} \in \mathbb{R}^{n_x}$ denotes the observed state, such that the state estimation error $e(t) \triangleq x(t) - \hat{x}(t)$ converges to zero with rate of convergence α in $\mathcal{P}$. Note that this problem is akin to the one considered for QTI systems in [23].

The following assumptions will be made throughout the paper.

Assumption 1: The varying parameter vector $\theta(t)$ is assumed to be known, and can vary arbitrarily fast in Θ.

Assumption 2: The trajectory of the state $x(t)$ is contained in the ellipsoidal set:

$$\mathcal{F} = \left\{x \in \mathbb{R}^{n_x} : x^T Q^{-1} x \leq 1\right\} \quad (8)$$

with $Q \succ 0$ and $\mathcal{P} \subset \mathcal{F}$.

Remark 1: By recalling that $\mathcal{S} \subset \mathbb{R}^{n_x}$ is an invariant set for an autonomous nonlinear system $\dot{x}(t) = f(x(t))$ if, for all $x(0) \in \mathcal{S}$, the solution $x(t) \in \mathcal{S}$ for $t > 0$ [31], then Assumption 2 holds when $u(t)$ is a state-dependent control law which makes the origin an asymptotically stable equilibrium point of (1) *with invariant set $\mathcal{F}$, as in [25].*

Following [32] and [23], an observer that is based on the *linear output error injection principle* is proposed, as follows:

$$\begin{aligned} \dot{\hat{x}}(t) = {} & A(\theta(t))\hat{x}(t) + N(\theta(t), \hat{x}(t))\hat{x}(t) + B(\theta(t))u(t) \\ & + L(\theta(t))\left[y(t) - C(\theta(t))\hat{x}(t) - M(\theta(t), \hat{x}(t))\hat{x}(t)\right] \end{aligned} \quad (9)$$

Then, the dynamical system that describes the behavior of the state estimation error is a QPV system subject to an external input (the unknown system state $x(t)$):

$$\begin{aligned} \dot{e}(t) = {} & \left[A(\theta(t)) - L(\theta(t))C(\theta(t))\right]e(t) \\ & - N(\theta(t), e(t))e(t) + L(\theta(t))M(\theta(t), e(t))e(t) \\ & + \tilde{N}(\theta(t), e(t))x(t) - L(\theta(t))\tilde{M}(\theta(t), e(t))x(t) \end{aligned} \quad (10)$$

where:

$$\tilde{N}(\theta(t), e(t)) = \begin{pmatrix} e(t)^T He\{N_1(\theta(t))\} \\ e(t)^T He\{N_2(\theta(t))\} \\ \vdots \\ e(t)^T He\{N_{n_x}(\theta(t))\} \end{pmatrix} \quad (11)$$

$$\tilde{M}(\theta(t), e(t)) = \begin{pmatrix} e(t)^T He\{M_1(\theta(t))\} \\ e(t)^T He\{M_2(\theta(t))\} \\ \vdots \\ e(t)^T He\{M_{n_y}(\theta(t))\} \end{pmatrix} \quad (12)$$

III. Observer design

This section provides the design conditions that allow obtaining the matrix $L(\theta(t))$ of the observer (9) such that $e(t)$ converges to zero with rate of convergence α in $\mathcal{P}$. The observer design approach proposed hereafter is an extension to QPV systems of the method developed by [23] for QTI systems.

Theorem 1: Let $P \succ 0$, $0 < \gamma < 1$, and the matrix function $\Gamma(\theta) \in \mathbb{R}^{n_x \times n_y}$ be such that $\forall i \in \{1,\ldots,p\}$, $\forall j \in \{1,\ldots,p\}$, $\forall k \in \{1,\ldots,q\}$ and $\forall \theta \in \Theta$:

$$\begin{pmatrix} 1 & x_{(i)}^T \\ x_{(i)} & P \end{pmatrix} \succeq 0 \tag{13}$$

$$\begin{pmatrix} 1 & \gamma a_k^T P \\ \gamma P a_k & P \end{pmatrix} \succeq 0 \tag{14}$$

$$\begin{pmatrix} 1 & \gamma a_k^T Q \\ \gamma Q a_k & Q \end{pmatrix} \succeq 0 \tag{15}$$

$$He\left\{ \gamma[PA(\theta) - \Gamma(\theta)C(\theta)] - P \begin{pmatrix} x_{(j)}^T He\{N_1(\theta)\} - x_{(i)}^T N_1(\theta) \\ x_{(j)}^T He\{N_2(\theta)\} - x_{(i)}^T N_2(\theta) \\ \vdots \\ x_{(j)}^T He\{N_{n_x}(\theta)\} - x_{(i)}^T N_{n_x}(\theta) \end{pmatrix} \right. \tag{16}$$
$$\left. + \Gamma(\theta) \begin{pmatrix} x_{(i)}^T M_1(\theta) - x_{(j)}^T He\{M_1(\theta)\} \\ x_{(i)}^T M_2(\theta) - x_{(j)}^T He\{M_2(\theta)\} \\ \vdots \\ x_{(i)}^T M_{n_y}(\theta) - x_{(j)}^T He\{M_{n_y}(\theta)\} \end{pmatrix} \right\} + \gamma\alpha P \prec 0$$

Then, the observer (9) with $L(\theta(t)) = P^{-1}\Gamma(\theta(t))$ is such that the estimation error $e(t)$ converges to zero with rate of convergence α in $\mathcal{P}$.

Proof: Let us consider the Lyapunov function:

$$V(e(t)) = e(t)^T P e(t) \tag{17}$$

Due to (13), the following is true [33]:

$$\mathcal{P} \subset \mathcal{E} = \left\{ e \in R^{n_x} : e^T P^{-1} e \leq 1 \right\} \tag{18}$$

where $\mathcal{E}$ corresponds to a level curve of $V(e)$.

Let us define $\tilde{\mathcal{P}}$ as an enlarged version of $\mathcal{P}$ obtained by multiplying all the coordinates of its vertices by $\rho = \gamma^{-1} > 1$:

$$\begin{aligned} \tilde{\mathcal{P}} &= Co\left\{ \rho x_{(1)}, \rho x_{(2)}, \ldots, \rho x_{(p)} \right\} \\ &= \left\{ x \in \mathbb{R}^{n_x} : \gamma a_k^T x = \frac{a_k^T}{\rho} x \leq 1, k = 1,\ldots,q \right\} \end{aligned} \tag{19}$$

By means of Schur complements, (14)-(15) are equivalent to:

$$\begin{matrix} \gamma a_k^T P \gamma a_k \leq 1 \\ \gamma a_k^T Q \gamma a_k \leq 1 \end{matrix} \qquad k = 1,\ldots,q \tag{20}$$

which guarantee that $\mathcal{E} \subset \tilde{\mathcal{P}}$ and $\mathcal{F} \subset \tilde{\mathcal{P}}$.

Following the reasoning provided by [23], if $\mathcal{E}$ is an invariant set for (10), and the Lyapunov function (17) satisfies:

$$\dot{V}(e(t)) + \alpha V(e(t)) < 0 \tag{21}$$

for all $e \in \mathcal{E}$ and $x \in \mathcal{F}$, then $e(t)$ converges exponentially to zero with rate of convergence α in $\mathcal{E}$. If (21) holds for all $e, x \in \tilde{\mathcal{P}}$, then it follows from (20) that it holds for all $e \in \mathcal{E}$ and $x \in \mathcal{F}$.

Taking into account (10), and through the change of variable $\Gamma(\theta) = PL(\theta)$, (21) is equivalent to the following matrix inequality:

$$\begin{aligned} &He\{PA(\theta) - \Gamma(\theta)C(\theta) - PN(\theta,e) + \Gamma(\theta)M(\theta,e) \\ &+ P \begin{pmatrix} x^T He\{N_1(\theta)\} \\ x^T He\{N_2(\theta)\} \\ \vdots \\ x^T He\{N_{n_x}(\theta)\} \end{pmatrix} - \Gamma(\theta) \begin{pmatrix} x^T He\{M_1(\theta)\} \\ x^T He\{M_2(\theta)\} \\ \vdots \\ x^T He\{M_{n_y}(\theta)\} \end{pmatrix} \Bigg\} \\ &+ \alpha P \prec 0 \end{aligned} \tag{22}$$

which is affine w.r.t. e and x, such that it can be rewritten at the vertices of the polytope $\tilde{\mathcal{P}}$, obtaining (16).

Since $\mathcal{E}$ corresponds to a level curve of $V(e)$, then it is an invariant set for (10), and from (18), it follows that $e(t)$ converges to zero with rate of convergence α in $\mathcal{P}$, which completes the proof. □

Remark 2: Note that Theorem 1 has been developed using a Lyapunov function (17) *with constant Lyapunov matrix P. Similarly to the LPV case [34], a Lyapunov function with parameter-dependent matrix* $P(\theta(t))$ *could be used in order to take into account bounds on the rate of variation of* $\theta(t)$*, although at the expense of increasing the overall complexity. However, such an extension goes beyond the scope of this paper and will be considered in future work.*

The main difficulty with using (16) from a practical perspective is that it represents an infinite number of conditions that should be satisfied. Similarly to the case of LPV systems [35], we will consider the class of QPV with constant output matrices, i.e. $C(\theta(t)) = C$ and $M_i(\theta(t)) = M_i$ for $i = 1,\ldots,n_y$, and polytopic state matrices:

$$\begin{pmatrix} A(\theta(t)) \\ N_1(\theta(t)) \\ \vdots \\ N_{n_x}(\theta(t)) \end{pmatrix} = \sum_{l=1}^{N} \mu_l(\theta(t)) \begin{pmatrix} A_l \\ N_{1,l} \\ \vdots \\ N_{n_x,l} \end{pmatrix} \tag{23}$$

with some finite N and:

$$\sum_{l=1}^{N} \mu_l(\theta) = 1, \quad \mu_l(\theta) \geq 0 \qquad \begin{matrix} \forall l = 1,\ldots,N \\ \forall \theta \in \Theta \end{matrix} \tag{24}$$

In this case, by choosing the state observer gain as follows:

$$L(\theta(t)) = \sum_{l=1}^{N} \mu_l(\theta(t)) L_l \tag{25}$$

(16) can be reduced to a finite set of conditions, as stated by the following corollary.

Corollary 1: Let $P \succ 0$, $0 < \gamma < 1$ and the matrices $\Gamma_j \in \mathbb{R}^{n_x \times n_y}$ be such that $\forall i \in \{1,\ldots,p\}$, $\forall j \in \{1,\ldots,p\}$, $\forall k \in$

$\{1,\ldots,q\}$ and $\forall l \in \{1,\ldots,N\}$:

$$He\left\{\gamma[PA_l-\Gamma_l C]-P\begin{pmatrix} x_{(j)}^T He\{N_{1,l}\}-x_{(i)}^T N_{1,l} \\ x_{(j)}^T He\{N_{2,l}\}-x_{(i)}^T N_{2,l} \\ \vdots \\ x_{(j)}^T He\{N_{n_x,l}\}-x_{(i)}^T N_{n_x,l}\end{pmatrix}\right. \quad (26)$$
$$\left.+\Gamma_l\begin{pmatrix} x_{(i)}^T M_1-x_{(j)}^T He\{M_1\} \\ x_{(i)}^T M_2-x_{(j)}^T He\{M_2\} \\ \vdots \\ x_{(i)}^T M_{n_x}-x_{(j)}^T He\{M_{n_x}\}\end{pmatrix}\right\}+\gamma\alpha P \prec 0$$

and (13)-(15) hold. Then, the observer (9) with $L(\theta(t))$ as in (25) and $L_j = P^{-1}\Gamma_j$ is such that the estimation error, w.r.t. the system (1)-(4) with constant output matrices and state matrices satisfying (23)-(24), converges to zero with rate of convergence α in $\mathcal{P}$.

Proof: Taking into account the property of matrices [36] that any linear combination of (26) with non-negative coefficients, of which at least one different from zero, is negative definite, using the coefficients $\mu_l(\theta(t))$, and taking into account (23)-(25), then (16) is obtained. □

Note that (13)-(15) and (26) represent a set of bilinear matrix inequalities (BMIs), due to the product γP. However, it is possible to grid the interval of admissible values for the scalar γ, and apply Corollary 1 for each fixed γ. In this way, (26) becomes a set of linear matrix inequalities (LMIs), which can be solved efficiently using available toolboxes/solvers, e.g. YALMIP [37]/SeDuMi [38].

Remark 3: In the case of a QPV system with parameter varying output matrices $C(\theta(t)), M_1(\theta(t)), \ldots, M_{n_x}(\theta(t))$, it is possible to obtain a system with constant output matrices by post-filtering the output vector $y(t)$, as proposed by [35]. More specifically, let us define a new output vector $\tilde{y}(t)$ such that:

$$\dot{x}_y(t) = A_y x_y(t) + B_y y(t) \quad (27)$$

$$\tilde{y}(t) = C_y x_y(t) \quad (28)$$

with A_y stable and $B_y = diag(b_y^{(1)},\ldots,b_y^{(n_y)})$. Then, the QPV system resulting from the connection between (1)-(2) and (27)-(28) is:

$$\begin{pmatrix}\dot{x}(t)\\ \dot{x}_y(t)\end{pmatrix} = \begin{pmatrix} A(\theta(t)) & 0 \\ B_y C(\theta(t)) & A_y\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix} \quad (29)$$
$$+\begin{pmatrix} \begin{pmatrix} x(t)^T & x_y(t)^T\end{pmatrix}\begin{pmatrix} N_1(\theta(t)) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix} \\ \vdots \\ \begin{pmatrix} x(t)^T & x_y(t)^T\end{pmatrix}\begin{pmatrix} N_{n_x}(\theta(t)) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix} \\ \begin{pmatrix} x(t)^T & x_y(t)^T\end{pmatrix}\begin{pmatrix} b_y^{(1)} M_1(\theta(t)) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix} \\ \vdots \\ \begin{pmatrix} x(t)^T & x_y(t)^T\end{pmatrix}\begin{pmatrix} b_y^{(n_y)} M_{n_y}(\theta(t)) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix}\end{pmatrix}$$

$$\tilde{y}(t) = \begin{pmatrix} 0 & C_y\end{pmatrix}\begin{pmatrix} x(t)\\ x_y(t)\end{pmatrix} \quad (30)$$

which has a suitable structure for performing the design.

IV. Example

Let us consider a QPV system as in (1)-(2), with $u(t)=0$ and:

$$A(\theta(t)) = \begin{pmatrix} -4-\theta_1(t) & 10 & 2+2\theta_2(t) \\ -1 & -1-\theta_2(t) & 1.5+2\theta_1(t) \\ 1 & 1 & -4-3\theta_1(t)\end{pmatrix}$$

$$N_1(\theta(t)) = \begin{pmatrix} 0.5 & 1+\theta_1(t) & 0 \\ 0 & 0 & -\theta_2(t) \\ 0 & 1+\theta_2(t) & 0\end{pmatrix}$$

$$N_2(\theta(t)) = \begin{pmatrix} -0.4 & 0 & 1-\theta_1(t) \\ 1.5 & 0 & 1+\theta_2(t) \\ 2+\theta_2(t) & 0 & 0\end{pmatrix}$$

$$N_3(\theta(t)) = \begin{pmatrix} 1.5 & -0.5-\theta_1(t) & 0 \\ 3+\theta_1(t) & 0 & 0 \\ 0 & 2-\theta_2(t) & 0\end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \quad M_3 = 0_{3\times 3}$$

with $\theta_1, \theta_2 \in [0,1]$. By considering all the possible combinations of minimum and maximum values for the scheduling variables θ_1 and θ_2, the matrix functions $A(\theta(t)), N_1(\theta(t)), N_2(\theta(t)), N_3(\theta(t))$ can be expressed in the polytopic form (23) with $N=4$ and:

$$\mu_1(\theta(t)) = (1-\theta_1(t))(1-\theta_2(t))$$
$$\mu_2(\theta(t)) = (1-\theta_1(t))\theta_2(t)$$
$$\mu_3(\theta(t)) = \theta_1(t)(1-\theta_2(t))$$
$$\mu_4(\theta(t)) = \theta_1(t)\theta_2(t)$$

$$A_1 = \begin{pmatrix} -4 & 10 & 2 \\ -1 & -1 & 1.5 \\ 1 & 1 & -4\end{pmatrix} \quad A_2 = \begin{pmatrix} -4 & 10 & 4 \\ -1 & -2 & 1.5 \\ 1 & 1 & -4\end{pmatrix}$$
$$A_3 = \begin{pmatrix} -5 & 10 & 2 \\ -1 & -1 & 3.5 \\ 1 & 1 & -7\end{pmatrix} \quad A_4 = \begin{pmatrix} -5 & 10 & 4 \\ -1 & -2 & 3.5 \\ 1 & 1 & -7\end{pmatrix}$$

$$N_{1,1} = \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} \quad N_{1,2} = \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0\end{pmatrix}$$
$$N_{1,3} = \begin{pmatrix} 0.5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} \quad N_{1,4} = \begin{pmatrix} 0.5 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0\end{pmatrix}$$

$$N_{2,1} = \begin{pmatrix} -0.4 & 0 & 1 \\ 1.5 & 0 & 1 \\ 2 & 0 & 0\end{pmatrix} \quad N_{2,2} = \begin{pmatrix} -0.4 & 0 & 1 \\ 1.5 & 0 & 2 \\ 3 & 0 & 0\end{pmatrix}$$
$$N_{2,3} = \begin{pmatrix} -0.4 & 0 & 0 \\ 1.5 & 0 & 1 \\ 2 & 0 & 0\end{pmatrix} \quad N_{2,4} = \begin{pmatrix} -0.4 & 0 & 0 \\ 1.5 & 0 & 2 \\ 3 & 0 & 0\end{pmatrix}$$

$$N_{3,1} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad N_{3,2} = \begin{pmatrix} 1.5 & -0.5 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$N_{3,3} = \begin{pmatrix} 1.5 & -1.5 & 0 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad N_{3,4} = \begin{pmatrix} 1.5 & -1.5 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Let us consider the polytope $\mathcal{P}$ defined as in (6), with a_k chosen as:

$$a_1 = -a_2 = (50,0,0)$$

$$a_3 = -a_4 = (0,50,0)$$

$$a_5 = -a_6 = (0,0,50)$$

which is equivalent to the vertex representation with 8 vertices obtained by considering all the possible combinations of $+$ and $-$ signs in:

$$x_{(i)} = (\pm 0.02, \pm 0.02, \pm 0.02)^T$$

Using the results in [25], it can be shown that the state trajectories starting from points inside $\mathcal{P}$ are contained within the ellipsoid (8) with:

$$Q = \begin{pmatrix} 0.0112 & 0.0017 & 0.0006 \\ 0.0017 & 0.0025 & -0.0022 \\ 0.0006 & -0.0022 & 0.0061 \end{pmatrix}$$

Then, applying Corollary 1, a feasible solution for the observer design can be found with $\alpha = 10$ and $\gamma = 0.4$, obtaining:

$$P = \begin{pmatrix} 0.0025 & 0.0013 & -0.0012 \\ 0.0013 & 0.0025 & -0.0012 \\ -0.0012 & -0.0012 & 0.0021 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 31.8975 & -11.0863 \\ 0.4483 & 48.4510 \\ 19.5239 & 19.0527 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 41.1880 & -13.6776 \\ -3.0040 & 50.5688 \\ 27.6840 & 18.7641 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 40.8048 & -13.4659 \\ -2.0010 & 53.5213 \\ 29.5071 & 22.5930 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 42.4335 & -11.8197 \\ -1.9010 & 52.8039 \\ 32.9168 & 24.9591 \end{pmatrix}$$

The trajectories of the state variables and the corresponding observed variables, starting from $x(0) = (-0.02,-0.02,-0.02)^T$ and $\hat{x}(0) = (0,0,0)^T$, are plotted in Figs. 1-3, for a simulation in which the varying parameters were defined as $\theta_1(t) = 0.5 + 0.5\sin(4\pi t)$ and $\theta_2(t) = 0.5 + 0.5\cos(\pi t/5 + \pi/6)$ (Fig. 4 shows the evolution of $\mu_i(\theta(t))$ throughout the simulation). It can be seen that the observer achieves convergence to zero of the estimation error.

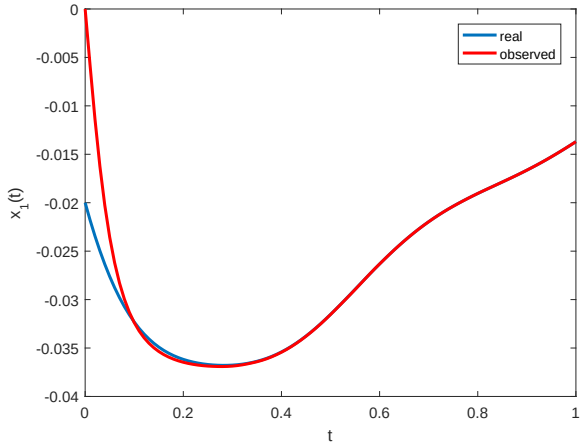

Fig. 1. Trajectory of the state variable $x_1(t)$ and its estimation $\hat{x}_1(t)$.

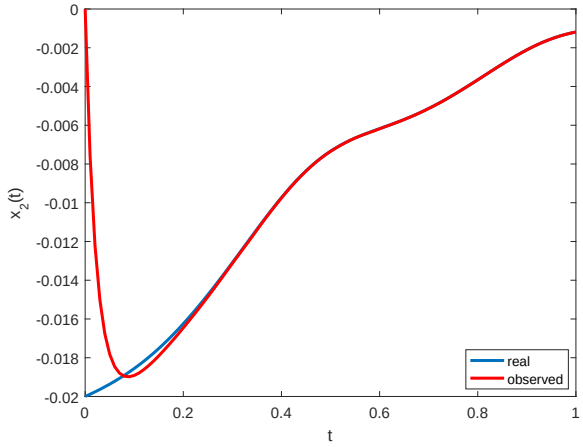

Fig. 2. Trajectory of the state variable $x_2(t)$ and its estimation $\hat{x}_2(t)$.

V. Conclusions

In this paper, the problem of state observation in QPV systems has been investigated. The proposed design, based on the reduction of a set of BMIs to LMIs, allows finding the observer gains such that the state estimation error converges to zero with a prescribed convergence rate, over a specified polytopic region of the error space. The validity of the theoretical results have been demonstrated by means of an academic example.

Future work will relax the assumption about the arbitrary time variation of the varying parameters $\theta(t)$ by using parameter-dependent Lyapunov matrices instead of the constant one considered in this paper.

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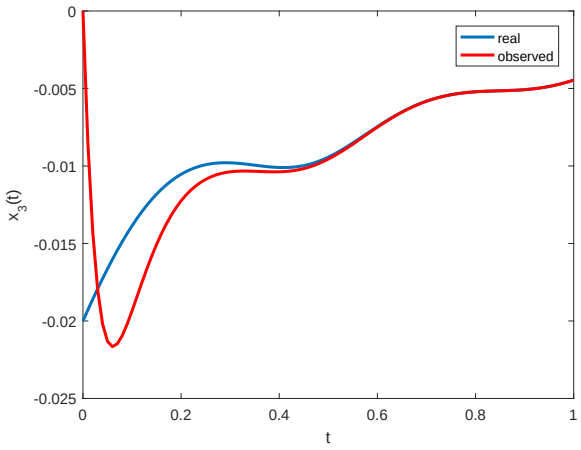

Fig. 3. Trajectory of the state variable $x_3(t)$ and its estimation $\hat{x}_3(t)$.

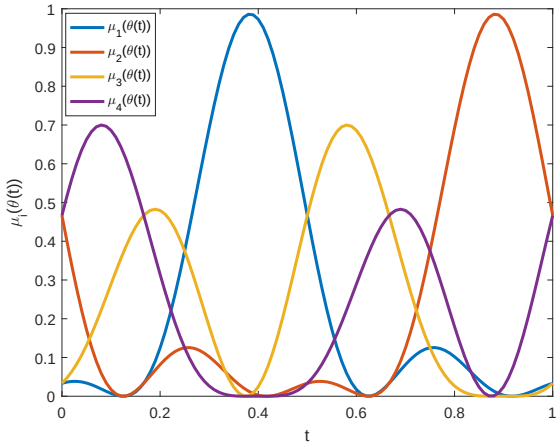

Fig. 4. Coefficients $\mu_l(\theta(t))$ of the polytopic decomposition (23).

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