$\mathcal{D}$-stable Controller Design for Lipschitz NLPV System ⋆

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Abstract: This paper addresses the design of a state-feedback controller for a class of nonlinear parameter varying (NLPV) systems in which the nonlinearity can be expressed as a parameter-varying Lipschitz term. The controller is designed to satisfy a $\mathcal{D}$-stability specification, which is akin to imposing constraints on the closed-loop pole location in the case of LTI and LPV systems. The design conditions,obtained using a quadratic Lyapunov function, are eventually expressed in terms of linear matrix inequalities (LMIs), which can be solved efficiently using available solvers. The effectiveness of the proposed method is demonstrated by means of a numerical example.

Keywords: Pole placement, $\mathcal{D}$-stability, gain-scheduling, linear matrix inequalities (LMIs), Lipschitz nonlinear systems, nonlinear parameter varying (NLPV) systems, controller design.

1. INTRODUCTION

Since its introduction, the linear parameter varying (LPV) paradigm has proved to be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters (Rugh and Shamma, 2000). In the last years, there has been an important progress in the development of analysis and design techniques for LPV systems (Rotondo, 2017). More recently, there has been a growing interest in extending these techniques to nonlinear parameter varying systems (NLPV), see e.g. Larimore (2013), Cai et al. (2015), Blesa et al. (2015), Rotondo and Johansen (2018), Lu et al. (2018), since in many practical applications the varying part appears in a nonlinear way.

This paper deals with a class of NLPV systems that contain a nonlinear parameter-varying Lipschitz term. Problems related to stabilization of Lipschitz nonlinear systems have been addressed by several works. For instance, Pagilla and Zhu (2004) considered full-state feedback controller design, observer design and output feedback controller design. On the other hand, Veselỳ and Körösi (2018) proposed an approach for the design of robust PID controllers for nonlinear Lipschitz systems based on the Bellman Lyapunov equation. Notably, design techniques based on linear matrix inequalities (LMIs) have been also proposed recently, see e.g. Zemouche et al. (2008), Zemouche and Boutayeb (2013), Gritli and Belghith (2018).

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Among the specifications that are of interest for LMI-based design, there is pole clustering in LMI regions, also known as $\mathcal{D}$-stability. Initially characterized by Chilali and Gahinet (1996) using a quadratic Lyapunov function with constant matrix, this idea was further developed by Peaucelle et al. (2000), who considered uncertain systems by means of a parameter-dependent Lyapunov function, and is still investigated nowadays, see e.g. the recent improvements brought by Nguyen et al. (2017), Chesi (2017).

Since the analysis and design of $\mathcal{D}$-stable controllers is based on LMIs, this performance was easily extended to LPV systems. For example, Ghersin and Sánchez-Peña (2010) showed that introducing regional pole placement into LPV controller design helped in improving the transient properties of the closed-loop response. This idea was exploited by Colmegna et al. (2016) to design a switched controller that could act *aggressively* when needed and by Cherifi et al. (2015), who proposed a $\mathcal{D}$-stabilizing controller under LMI conditions for quasi-LPV/Takagi-Sugeno model. Further development was brought by introducing shifting pole placement, which allowed online changes in the closed-loop transient behavior (Rotondo et al., 2015), which was recently extended to the design of PID controllers (Sánchez et al., 2018).

The main contribution of this paper is to propose a procedure for the design of $\mathcal{D}$-stable state-feedback controllers for a class of NLPV systems in which the nonlinearity can be expressed as a parameter-varying Lipschitz term.The design conditions obtained using a quadratic Lyapunov function are eventually expressed in terms of LMIs, which can be solved efficiently using available solvers. The effectiveness of the proposed method is demonstrated by means of a numerical example.

The structure of the paper is the following: in Section 2, the background results presented in the paper are recalled. Section

3 analyzes the $\mathcal{D}$-stability of Lipschitz NLPV systems. Section 4 presents the proposed design approach. Section 5 shows application results using a numerical example. Finally, Section 6 draws the main conclusions.

2. BACKGROUND

The idea of LMI regions was first introduced by Chilali and Gahinet (1996) in order to provide a Lyapunov-based characterization of pole clustering in stable subregions of the complex plane. Their formal definition is given as follows:

Definition 1. (LMI region) A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exist a matrix $\alpha = [\alpha_{kl}] \in \mathbb{S}^{m\times m}$ and a matrix $\beta = [\beta_{kl}] \in \mathbb{R}^{m\times m}$ such that:

$$\mathcal{D} = \{s \in \mathbb{C} : f_{\mathcal{D}} \prec 0\} \tag{1}$$

with:

$$f_{\mathcal{D}}(s) = \alpha + s\beta + \bar{s}\beta^T = [\alpha_{kl} + \beta_{kl}s + \beta_{lk}\bar{s}]_{1\leq k,l\leq m} \tag{2}$$

In other words, LMI regions are subsets of the complex plane that are represented by an LMI in s and $\bar{s}$. In Chilali and Gahinet (1996), it was shown that LMI regions include a wide variety of typical clustering regions, such as:

- Left-hand semiplanes $Re(s) < \lambda$:
$$\alpha = -2\lambda, \quad \beta = 1$$
- Disks of radius r and center $(-q, 0)$:
$$\alpha = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} \quad \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
- Conic sectors of angle ϕ:
$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix}$$

Based on the above definition, Chilali and Gahinet (1996) introduced the notion of $\mathcal{D}$-stability in order to describe whether the poles of a linear time invariant (LTI) system lie within a given LMI region or not. By means of a slight abuse of mathematical language, Ghersin and Sánchez Peña (2002) defined the poles of an autonomous LPV system:

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) \tag{3}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state, and $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the vector of varying parameters, as the set of all the LTI systems obtained by freezing $\theta(t)$ to all its possible values $\theta \in \Theta$. With such an extension, it is quite straightforward to define an LPV system (3) to be $\mathcal{D}$-stable if all its poles lie in $\mathcal{D}$. Then, by using the Lyapunov candidate function:

$$V\left(x(t)\right) = x(t)^T P x(t) \tag{4}$$

with $P \in \mathbb{S}_+^{n_x\times n_x}$, the following condition for assessing the quadratic stability of (3) can be obtained:

Proposition 1. (Quadratic $\mathcal{D}$-stability of LPV systems) Given an LMI region defined as in (1)-(2), the autonomous LPV system (3) is quadratically $\mathcal{D}$-stable if there exists $P \in \mathbb{S}_+^{n_x\times n_x}$ such that $\forall\theta \in \Theta$ (Nguang and Shi, 2006):

$$\begin{aligned} &\alpha\otimes P + \beta \otimes PA(\theta) + \beta^T \otimes A(\theta)^T P \\ &= \left[\alpha_{kl}P + \beta_{kl}PA(\theta) + \beta_{lk}A(\theta)^T P\right]_{k,l\in\{1,\ldots,m} \prec 0 \end{aligned} \tag{5}$$

It can be observed that the time derivative of (4) when the LPV system (3) is taken into account is given by:

$$\begin{aligned} \dot{V}\left(x(t),\theta(t)\right) &= x(t)^T PA\left(\theta(t)\right)x(t) + x(t)^T A\left(\theta(t)\right)^T Px(t) \\ &\triangleq \Phi\left(x(t),\theta(t)\right) + \Phi\left(x(t),\theta(t)\right)^T \end{aligned} \tag{6}$$

and the quadratic $\mathcal{D}$-stability condition (5) can be interpreted as the following constraint on the Lyapunov function $V\left(x(t)\right)$ and its derivative $\dot{V}\left(x(t),\theta(t)\right)$:

$$\alpha V\left(x(t)\right) + \beta\Phi\left(x(t),\theta(t)\right) + \beta^T\Phi^T\left(x(t),\theta(t)\right) \prec 0 \tag{7}$$

In fact, (7) can be rewritten as:

$$\begin{aligned} &\alpha x(t)^T Px(t) + \beta x(t)^T PA\left(\theta(t)\right)x(t) \\ &\quad + \beta^T x(t)^T A\left(\theta(t)\right)^T Px(t) \prec 0 \end{aligned} \tag{8}$$

and, by defining:

$$X(t) = \operatorname{diag}\left(x(t), x(t), \ldots, x(t)\right) \tag{9}$$

one obtains that (8) is equivalent to:

$$X(t)^T\left(\alpha\otimes P + \beta\otimes PA(\theta) + \beta^T\otimes A(\theta)^T P\right)X(t) \prec 0 \tag{10}$$

from which (5) is obtained.

Taking into account the above interpretation of the quadratic $\mathcal{D}$-stability constraint as (7), the next section extends this approach to the class of nonlinear parameter varying (NLPV) systems with parameter-varying Lipschitz nonlinearity.

3. $\mathcal{D}$-STABILITY OF LIPSCHITZ NLPV SYSTEMS

A Lipschitz LPV system is defined as a finite-dimensional time-varying system whose state equation is composed of two terms: a linear part described by the state matrix and a nonlinear Lipschitz term, which are both functions of some varying parameters $\theta(t) \in \Theta \subset \mathbb{R}^{n_\theta}$ (with Θ known closed and bounded set), which are assumed to be unknown a priori, but that can be measured or estimated in real-time:

$$\dot{x}(t) = A(\theta(t))x(t) + f(x(t),\theta(t)) \tag{11}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $A\left(\theta(t)\right)$ is a matrix function of appropriate dimensions, and $f(x(t),\theta(t))$ is the parameter-varying Lipschitz function, which satisfies $f(0,\theta(t)) = 0$ and:

$$f(x(t),\theta(t))^T f(x(t),\theta(t)) \leq x(t)^T\lambda(\theta(t))^T\lambda(\theta(t))x(t) \tag{12}$$

for some known matrix function $\lambda\left(\theta(t)\right)$.

Inspired by the linear matrix inequality (LMI)-based results obtained by Mukherjee and Sengupta (2014) for the controller design for Lipschitz nonlinear systems, and taking into account the discussion in the previous section, we propose the following theorem, which proposes LMI-based analysis conditions for assessing the quadratic $\mathcal{D}$-stability of a Lipschitz NLPV system in the form (11).

Theorem 1. (Quadratic $\mathcal{D}$-stability of a Lipschitz NLPV system). Given an LMI region defined as in (1)-(2), the autonomous Lipschitz NLPV system (11) is quadratically $\mathcal{D}$-stable if there exist a scalar $\gamma > 0$ and a matrix $Q \in \mathbb{S}_+^{n_x\times n_x}$ such that $\forall\theta \in \Theta$:

$$\begin{bmatrix}
\Pi_{11}(\theta) & \cdots & \Pi_{1m}(\theta) & \beta_{11}I & \cdots & \beta_{1m}I & Q\lambda(\theta)^T & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Pi_{m1}(\theta) & \cdots & \Pi_{mm}(\theta) & \beta_{m1}I & \cdots & \beta_{mm}I & 0 & \cdots & Q\lambda(\theta)^T \\
\beta_{11}I & \cdots & \beta_{m1}I & -\gamma I & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\beta_{1m}I & \cdots & \beta_{mm}I & 0 & \cdots & -\gamma I & 0 & \cdots & 0 \\
\lambda(\theta)Q & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{\gamma}I & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \lambda(\theta)Q & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\gamma}I
\end{bmatrix} < 0 \tag{13}$$

where:

$$\Pi_{ij}(\theta) = \alpha_{ij}Q + \beta_{ij}A(\theta)Q + \beta_{ji}QA(\theta)^T, \quad i,j = 1,\ldots,m \tag{14}$$

Proof: Consider the Lyapunov function $V(x(t))$ defined in (4), for which the $\mathcal{D}$-stability constraint (7) is required. Taking into account (11), we have:

$$\begin{aligned}&\alpha x(t)^T Px(t) + \beta x(t)^T P\left(A(\theta(t))x(t) + f(x(t),\theta(t))\right)\\&+\beta^T\left(x(t)^T A(\theta(t))^T + f(x(t),\theta(t))^T\right)Px(t) \prec 0\end{aligned} \tag{15}$$

which, by defining $x(t)$ as in (9) and:

$$F(t) = \mathrm{diag}\left(f(x(t),\theta(t)),\ldots,f(x(t),\theta(t))\right) \tag{16}$$

can be rewritten as:

$$\begin{aligned}&X(t)^T\left(\alpha\otimes P + \beta\otimes PA(\theta) + \beta^T\otimes A(\theta)^T P\right)X(t)\\&+X(t)^T\beta\otimes PF(t) + F(t)^T\beta^T\otimes PX(t) \prec 0\end{aligned} \tag{17}$$

According to the condition (12), we have that:

$$\gamma\left(X(t)^T\lambda(\theta(t))^T\lambda(\theta(t))X(t) - F(t)^T F(t)\right) \succeq 0 \tag{18}$$

From (17) and (18), we obtain:

$$\begin{aligned}&X(t)^T\left(\alpha\otimes P + \beta\otimes PA(\theta) + \beta^T\otimes A(\theta)^T P\right)X(t)\\&+X(t)^T\beta\otimes PF(t) + F(t)^T\beta^T\otimes PX(t)\\&+\gamma\left(X(t)^T\lambda(\theta(t))^T\lambda(\theta(t))X(t) - F(t)^T F(t)\right) \prec 0\end{aligned} \tag{19}$$

which can be rewritten in the compact form:

$$\begin{bmatrix}X(t)^T & F(t)^T\end{bmatrix}\Gamma(\theta(t))\begin{bmatrix}X(t)\\F(t)\end{bmatrix} \prec 0 \tag{20}$$

where:

$$\Gamma(\theta(t)) = \begin{bmatrix}\Gamma_{11}(\theta(t)) & \cdots & \Gamma_{1m}(\theta(t)) & \beta_{11}P & \cdots & \beta_{1m}P\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Gamma_{m1}(\theta(t)) & \cdots & \Gamma_{mm}(\theta(t)) & \beta_{m1}P & \cdots & \beta_{mm}P\\ \beta_{11}P & \cdots & \beta_{m1}P & -\gamma I & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \beta_{1m}P & \cdots & \beta_{mm}P & 0 & \cdots & -\gamma I\end{bmatrix} \tag{21}$$

with:

$$\begin{aligned}\Gamma_{ii}(\theta(t)) &= \alpha_{ii}P + \beta_{ii}\left(PA(\theta(t)) + A(\theta(t))^T P\right)\\&\quad+\gamma\lambda(\theta(t))^T\lambda(\theta(t))\\ \Gamma_{ij}(\theta(t)) &= \alpha_{ij}P + \beta_{ij}PA(\theta(t)) + \beta_{ji}A(\theta(t))^T P, i\neq j\end{aligned}$$

Hence, $\mathcal{D}$-stability of the autonomous Lipschitz NLPV system (11) can be assessed by checking that:

$$\Gamma(\theta) \prec 0 \qquad \forall\theta\in\Theta \tag{22}$$

Let us pre- and post-multiply (22) by:

$$\begin{bmatrix}Q & 0 & \cdots & 0 & 0\\ 0 & Q & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0\\ 0 & 0 & \cdots & 0 & I\end{bmatrix} \tag{23}$$

where $Q = P^{-1}$. Hence, we obtain:

$$\Pi_\star(\theta) \prec 0 \qquad \forall\theta\in\Theta \tag{24}$$

where:

$$\Pi_\star(\theta) = \begin{bmatrix}\Pi^\star_{11}(\theta) & \cdots & \Pi_{1m}(\theta) & \beta_{11}I & \cdots & \beta_{1m}I\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Pi_{m1}(\theta) & \cdots & \Pi^\star_{mm}(\theta) & \beta_{m1}I & \cdots & \beta_{mm}I\\ \beta_{11}I & \cdots & \beta_{m1}I & -\gamma I & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \beta_{1m}I & \cdots & \beta_{mm}I & 0 & \cdots & -\gamma I\end{bmatrix} \tag{25}$$

where:

$$\Pi^\star_{ii}(\theta) = \alpha_{ii}Q + \beta_{ii}\left(A(\theta)Q + QA(\theta)^T\right) + \gamma Q\lambda(\theta)^T\lambda(\theta)Q$$

and Π_{ij}, $i\neq j$, are defined as in (14).

Inequality (24) can be rewritten as (26) (see top of the next page), from which, by applying Schur's complements, (13) is obtained, which completes the proof. □

Remark 1. Note that (13) is not an LMI due to the appearance of both terms γ and $1/\gamma$. However, for predefined values of γ, (13) becomes an LMI parameterized by the parameter θ.

4. CONTROLLER DESIGN

In this section, the design process of a state feedback control for the Lipschitz NLPV system will be presented. Let us consider the following NLPV system, obtained from (11) by adding a control input $u\in\mathbb{R}^{n_u}$ which affects the system's dynamics through the parameter-varying input matrix $B(\theta(t))$:

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) + f(x(t),\theta(t)) \tag{27}$$

and let us use a gain-scheduled state-feedback control law given by:

$$u(t) = K(\theta(t))x(t) \tag{28}$$

where $K(\theta(t))$ is the controller gain to be designed. Then, the closed-loop system is given by:

$$\dot{x}(t) = \Big(A(\theta(t)) + B(\theta(t))K(\theta(t))\Big)x(t) + f(x(t),\theta(t)) \tag{29}$$

for which the following corollary can be obtained from Theorem 1.

Corollary 1. (Quadratic $\mathcal{D}$-stabilization of a Lipschitz NLPV system). Given an LMI region defined as in (1)-(2), if there exist a scalar $\gamma > 0$, a matrix $Q\in\mathbb{S}_+^{n_x\times n_x}$ and a matrix function $Y(\theta)\in\mathbb{R}^{n_u\times n_x}$ such that $\forall\,\theta\in\Theta$:

$$\begin{bmatrix}\Psi_{11}(\theta) & \cdots & \Psi_{1m}(\theta) & \beta_{11}I & \cdots & \beta_{1m}I & Q\lambda(\theta)^T & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Psi_{m1}(\theta) & \cdots & \Psi_{mm}(\theta) & \beta_{m1}I & \cdots & \beta_{mm}I & 0 & \cdots & Q\lambda(\theta)^T\\ \beta_{11}I & \cdots & \beta_{m1}I & -\gamma I & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \beta_{1m}I & \cdots & \beta_{mm}I & 0 & \cdots & -\gamma I & 0 & \cdots & 0\\ \lambda(\theta)Q & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{\gamma}I & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda(\theta)Q & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\gamma}I\end{bmatrix} \prec 0 \tag{30}$$

where:

$$\begin{aligned}\Psi_{ij} &= Q\alpha_{ij} + \beta_{ij}A(\theta)Q + \beta_{ij}B(\theta)Y(\theta)\\&\quad+\beta_{ij}QA(\theta)^T + \beta_{ij}Y(\theta)^T B(\theta)^T, \quad i,j = 1,\ldots,m\end{aligned} \tag{31}$$

then, the controller gain $K(\theta) = Y(\theta)Q^{-1}$ ensures the Lipschitz NLPV system (29) to be quadratically $\mathcal{D}$-stable.

Proof: This corollary is obtained from Theorem 1 by replacing in (13) $A(\theta)$ with $A(\theta)+B(\theta)K(\theta)$ and using the change of

$$
\begin{bmatrix}
\Pi_{11}(\theta) & \cdots & \Pi_{1m}(\theta) & \beta_{11}I & \cdots & \beta_{1m}I \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Pi_{m1}(\theta) & \cdots & \Pi_{mm}(\theta) & \beta_{m1}I & \cdots & \beta_{mm}I \\
\beta_{11}I & \cdots & \beta_{m1}I & -\gamma I & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\beta_{1m}I & \cdots & \beta_{mm}I & 0 & \cdots & -\gamma I
\end{bmatrix}
+
\begin{bmatrix}
Q\lambda(\theta)^T & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & Q\lambda(\theta)^T \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\gamma I & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \gamma I
\end{bmatrix}
\begin{bmatrix}
\lambda(\theta)Q & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \lambda(\theta)Q & 0 & \cdots & 0
\end{bmatrix}
\prec 0 \quad (26)
$$

variables $K(\theta)Q = Y(\theta)$ to avoid the product of unknown variables $K(\theta)$ and Q in the term $B(\theta)K(\theta)Q$, which would generate a bilinear matrix inequality (BMI) instead of an LMI. $\square$

It is necessary to mention that the LMI (30) implies satisfying an infinite number of conditions, which leads to a computational issue. In order to reduce the number of conditions from infinite to finite, the most common way to solve this problem is to use the polytopic framework. The NLPV system (27) is said to be polytopic if it can be represented by state-space matrices $A(\theta(t))$ and $B(\theta(t))$ which range over a convex set:

$$
\dot{x}(t) = \sum_{k=1}^{N} \mu_k(\theta(t))\Big(A_k x(t) + B_k u(t)\Big) + f(x(t), \theta(t)) \quad (32)
$$

where μ_k are the non-negative coefficients of the polytopic decomposition such that:

$$
\sum_{k=1}^{N} \mu_k(\theta(t)) = 1, \quad \mu_k(\theta(t)) \geq 0 \quad \forall k = 1, ..., N, \quad \forall \theta \in \Theta \quad (33)
$$

and if the matrix function $\lambda(\theta(t))$ can be expressed as:

$$
\lambda(\theta(t)) = \sum_{k=1}^{N} \mu_k(\theta(t)\lambda_k \quad (34)
$$

It follows that:

$$
A(\theta(t)) = \sum_{k=1}^{N} \mu_k(\theta(t))A_k \quad B(\theta(t)) = \sum_{k=1}^{N} \mu_k(\theta(t))B_k \quad (35)
$$

Then, the controller gain can be chosen to be polytopic as well, which leads to:

$$
K(\theta(t)) = \sum_{k=1}^{N} \mu_k(\theta(t))K_k \Rightarrow Y(\theta) = \sum_{k=1}^{N} \mu_k(\theta)Y_k \quad (36)
$$

However, the reduction of (30) to a finite number of conditions is not trivial, due to the fact that:

$$
B(\theta)Y(\theta) = \sum_{k=1}^{N} \mu_k(\theta(t)) \sum_{l=1}^{N} \mu_l(\theta(t))B_k Y_l \quad (37)
$$

which leads to the problem of verifying the negativity of a double polytopic sum.

A possible approach to address this issue is the application of Polya's theorem on positive forms in the standard simplex. This procedure, proposed by Sala and Arino (2007) leads to a set of sufficient conditions to assess the positiveness of double sums, which are progressively less conservative when a complexity parameter increases. However, in order to keep the mathematical complexity simpler, we provide the following corollary for the special case in which the input matrix is constant, such that (32) becomes:

$$
\dot{x}(t) = \sum_{k=1}^{N} \mu_k(\theta(t))A_k x(t) + Bu(t) + f(x(t), \theta(t)) \quad (38)
$$

Corollary 2. (Polytopic conditions for controller design) Given an LMI region defined as in (1)-(2), if there exist a scalar $\gamma > 0$, a matrix $Q \in \mathbb{S}_+^{n_x \times n_x}$ and matrices $Y_k \in \mathbb{R}^{n_u \times n_x}$ such that $\forall\, k = 1, \dots, N$:

$$
\begin{bmatrix}
\Psi_{11,k} & \cdots & \Psi_{1m,k} & \beta_{11}I & \cdots & \beta_{1m}I & Q\lambda_k^T & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Psi_{m1,k} & \cdots & \Psi_{mm,k} & \beta_{m1}I & \cdots & \beta_{mm}I & 0 & \cdots & Q\lambda_k^T \\
\beta_{11}I & \cdots & \beta_{m1}I & -\gamma I & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\beta_{1m}I & \cdots & \beta_{mm}I & 0 & \cdots & -\gamma I & 0 & \cdots & 0 \\
\lambda_k Q & \cdots & 0 & 0 & \cdots & 0 & -\frac{1}{\gamma}I & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \lambda_k Q & 0 & \cdots & 0 & 0 & \cdots & -\frac{1}{\gamma}I
\end{bmatrix} \prec 0 \quad (39)
$$

where:

$$
\begin{aligned}
\Psi_{ij,k} &= Q\alpha_{ij} + \beta_{ij}A_k Q + \beta_{ij}BY_k \\
&\quad + \beta_{ij}QA_k^T + \beta_{ij}Y_k^T B^T, \quad i, j = 1, \dots, m
\end{aligned} \quad (40)
$$

then, the controller gain (36), with $K_k = Y(\theta)Q^{-1}$ ensures the Lipschitz NLPV system (38) to be quadratically $\mathcal{D}$-stable.

Proof: It follows from the basic property of matrices that any linear combination of negative definite matrices with non-negative coefficients, of which at least one different from zero, is negative definite. Hence, using the linear combination brought by (33), (39) leads to (30). $\square$

5. SIMULATION RESULTS

Let us consider a Lipschitz polytopic NLPV system as in (38) with matrices given by:

$$
A_1 = \begin{bmatrix} -53.7037 & 16.6296 \\ -185.9630 & 55.7037 \end{bmatrix} \quad A_2 = \begin{bmatrix} -0.6667 & 1.0000 \\ -102.7778 & 2.6667 \end{bmatrix}
$$

$$
A_3 = \begin{bmatrix} -0.4000 & 0.2000 \\ -509.8000 & 2.4000 \end{bmatrix} \quad A_4 = \begin{bmatrix} -36.4286 & 16.5714 \\ -90.5714 & 38.4286 \end{bmatrix}
$$

$$
B = [1\ \ 0]^T
$$

which depends on two varying parameters $\theta_1(t), \theta_2(t) \in [0, 1]$, such that:

$$
\begin{aligned}
\mu_1(\theta(t)) &= (1 - \theta_1(t))(1 - \theta_2(t)) \\
\mu_2(\theta(t)) &= (1 - \theta_1(t))\,\theta_2(t) \\
\mu_3(\theta(t)) &= \theta_1(t)\,(1 - \theta_2(t)) \\
\mu_4(\theta(t)) &= \theta_1(t)\theta_2(t)
\end{aligned}
$$

and for which the Lipschitz nonlinearity is given by:

$$f(x(t), \theta(t)) = \begin{bmatrix} \theta_1(t) \sin x_1(t) \\ 0 \end{bmatrix}$$

which satisfies:

$$\theta_1^2(t) sin^2 x_1(t) \leq \theta_1^2(t) x_1^2(t) \tag{41}$$

such that:

$$\lambda(\theta(t)) = \begin{bmatrix} \theta_1(t) & 0 \\ 0 & 0 \end{bmatrix} \tag{42}$$

Let us note that it is straightforward to check that the open-loop NLPV system is unstable, since it is composed by the interpolation of four unstable time-invariant systems with state matrix's poles located at $[1 + 10j, 1 - 10j]$. Indeed, if an LMI region $\mathcal{D}$ is chosen to be completely contained in the left-hand part of the complex plane, the application of Theorem 1 does not provide a feasible solution. This fact can be further exemplified by plotting the free response starting from a non-zero initial condition, as shown in Fig. 1 for the particular varying parameter trajectory given by:

$$\theta_1(t) = \frac{1}{2} + \frac{1}{2} \sin(1.7t) \tag{43}$$

$$\theta_2(t) = \frac{1}{2} + \frac{1}{2} \sin(2.3t) \tag{44}$$

At this point, let us apply Corollary 2 to solve the design problem for an LMI region chosen as the conic sector with angle ϕ, hence described by matrices:

$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{bmatrix}$$

For such a choice of the region $\mathcal{D}$, the LMIs (39) become:

$$\begin{bmatrix} \Psi_{11,k} & \Psi_{12,k} & \sin\phi I & \cos\phi I & -Q\lambda_k^T & 0 \\ \Psi_{21,k} & \Psi_{22,k} & -\cos\phi I & \sin\phi I & 0 & -Q\lambda_k^T \\ \sin\phi I & -\cos\phi I & -\gamma I & 0 & 0 & 0 \\ \cos\phi I & \sin\phi I & 0 & -\gamma I & 0 & 0 \\ -\lambda_k Q & 0 & 0 & 0 & -\frac{1}{\gamma} I & 0 \\ 0 & -\lambda_k Q & 0 & 0 & 0 & -\frac{1}{\gamma} I \end{bmatrix} < 0 \tag{45}$$

where

$$\Psi_{11,k} = A_k Q \sin\phi + BY_k \sin\phi + QA_k^T \sin\phi + Y_k^T B^T \sin\phi$$
$$\Psi_{12,k} = A_k Q \cos\phi + BY_k \cos\phi - QA_k^T \cos\phi - Y_k^T B^T \cos\phi$$
$$\Psi_{21,k} = QA_k^T \cos\phi + Y_k^T B^T \cos\phi - A_k Q \cos\phi - BY_k \cos\phi$$
$$\Psi_{22,k} = A_k Q \sin\phi + BY_k \sin\phi + QA_k^T \sin\phi + Y_k^T B^T \sin\phi$$

Figs. 2-3 show the closed-loop state trajectories and the control input for simulations starting from the initial condition $x(0) = [15 \ -5]^T$, obtained with the controllers designed using angles $\phi = \pi/2$ (blue), $\phi = \pi/2.1$ (red) and $\phi = \pi/3$ (green), and a prefixed value of $\gamma = 1$. In Chilali and Gahinet (1996), the conic sectors of angle ϕ are used to control the minimum damping ratio and the maximum undamped natural frequency. This in turn bounds the maximum overshoot, the frequency of oscillatory modes, the delay time, the rise time, and the settling time. As we can see that in Fig 2, by decreasing the angle ϕ, the overshoot oscillations can be reduced significantly.

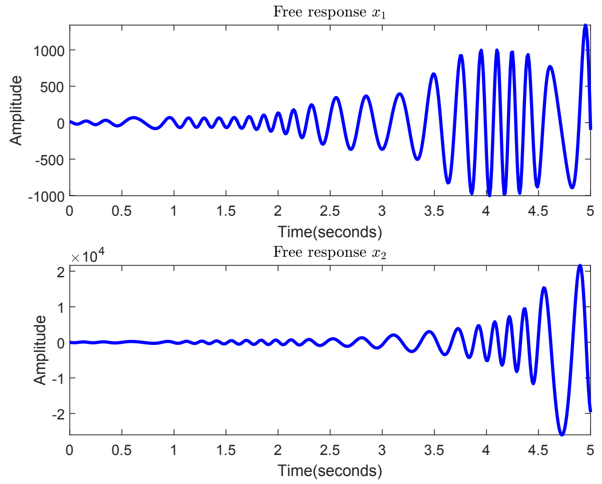

Fig. 1. Free response of the open-loop system.

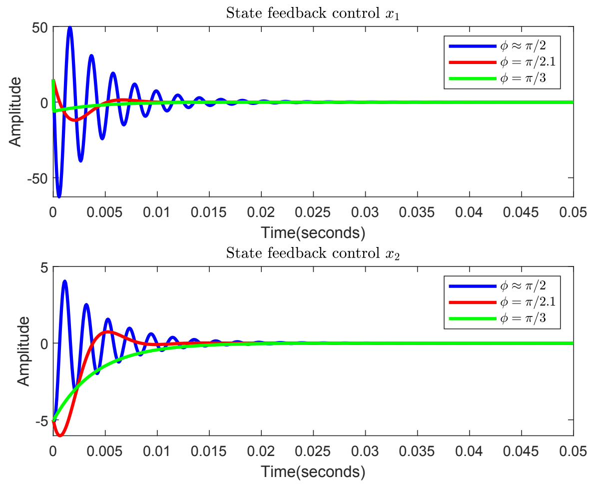

Fig. 2. Closed-loop state trajectories.

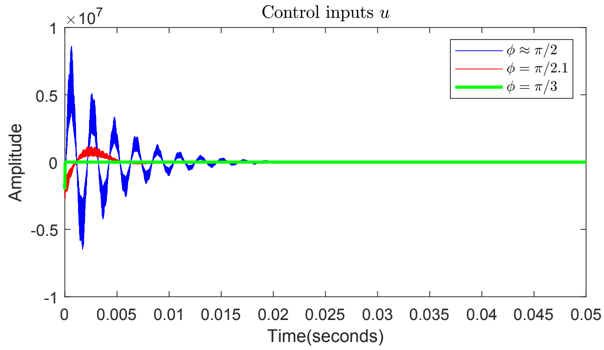

Fig. 3. Control input to the system

6. CONCLUSIONS

This paper has proposed a state-feedback controller design approach for a class of nonlinear parameter varying (NLPV) systems in which the nonlinearity can be expressed as a parameter-varying Lipschitz term. The controller has been designed to satisfy a $\mathcal{D}$-stability specification that imposes constraints on the closed-loop pole location. The design conditions have been obtained using a quadratic Lyapunov function that allow obtaining the linear matrix inequalities (LMIs), which can be solved efficiently using available solvers. The effectiveness of the proposed method has been demonstrated by means of a numerical example.

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