Robust economic model predictive control based on a periodicity constraint

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Summary

This paper proposes robust economic model predictive control based on a periodicity constraint for linear systems subject to unknown-but-bounded additive disturbances. In this economic MPC design, a periodic steady-state trajectory is not required and thus assumed unknown, which precludes the use of enforcing terminal state constraints as in other standard economic formulations. Instead, based on the desired periodicity of system operation, we optimize the economic performance over a set of periodic trajectories that include the current state. To achieve robust constraint satisfaction, we use a tube-based technique in the economic MPC formulation. The mismatches between the nominal model and the closed-loop system with perturbations are limited using a local control law. With the proposed robust tube-based strategy, recursive feasibility is guaranteed. Moreover, under a convexity assumption, the closed-loop convergence of the closed-loop system is analyzed, and an optimality certificate is provided to check if the closed-loop trajectory reaches a neighborhood of the optimal nominal periodic steady trajectory using Karush-Kuhn-Tucker optimality conditions. Finally, through numerical examples, we show the effectiveness of the proposed approach.

KEYWORDS

convex optimization, economic MPC, linear uncertain systems, periodic operation

1 | INTRODUCTION

Economic model predictive control (MPC) has attracted a lot of attention in both the academia and engineering industry in the past decade.[1] Economic MPC is suitable for increasing the number of engineering applications, such as drinking/wastewater networks,[2-5] chemical processes,[1,6] and electrical grids.[7] The main objective is to optimize a general performance index (ie, economic cost) rather than to drive closed-loop trajectories to given reference steady states as standard MPC. In this case, the cost function of economic MPC controller is not necessarily defined to be a quadratic form that penalizes the difference between the predicted states and predefined references.[8,9]

In literature, a large amount of research in economic MPC have been reported. Economic model predictive controllers are designed for nonlinear systems, where the terminal constraint and cost are used; therefore, the closed-loop convergence can be proved under the dissipativity assumption.[10,11] Economic MPC based on extended horizon is proposed for nonlinear systems,[12] where an auxiliary local controller is used to guarantee an optimal asymptotic performance.

Naturally, in an economic framework, periodic operation arises when a potential periodic behavior is expected. Economic model predictive controllers for achieving a periodic operation are proposed for nonlinear systems by means of additional constraints to enforce the periodicity along the MPC prediction horizon.[13-15] Single-layer economic MPC for periodic operation is proposed for linear systems,[16] and an application to water distribution networks is presented. Cost-to-travel functions, as a generalization of cost-to-go functions, are also studied.[17] Based on the property of cost-to-travel functions, it is proved that every optimal periodic orbit is a steady-state trajectory for linear systems with strictly convex cost functions and constraints. Under this framework, economic MPC is formulated with a periodic return cost function, and as a result, local asymptotic orbital stability can be guaranteed. However, in existing research studies, economic model predictive controllers have been designed in nominal conditions without considering uncertainty.

From an application point of view, process systems may be affected by disturbances, which implies that a proper robust MPC strategy should be addressed for such systems, for instance.[18,19] Tube-based techniques have been proposed to guarantee robust constraint satisfaction in the presence of uncertainties for standard MPC and other applications, eg, in distributed approaches.[20] Robust MPC is proposed to track periodic trajectories online,[21] where a local control law is used to refine the constraints to guarantee the recursive feasibility in a closed loop. In recent years, several developments on adjusting the robustness of economic MPC have been studied,[22-24] where the strong terminal constraint and cost are also used to enforce the periodicity. In our work, we consider a pure economic cost function without using terminal constraint and terminal cost function. We consider the periodicity with nominal control system and build a tube between the nominal and perturbed states. The closed-loop properties will be also analyzed.

The main contribution of this paper is to propose robust economic MPC based on a periodicity constraint. The considered linear systems are affected by additive unknown-but-bounded disturbances. We use the nominal system as the prediction model in a novel economic MPC formulation based on a periodicity constraint to achieve an optimal periodic operation. In most existing studies, the optimal periodic trajectory is assumed known and is usually computed offline by a real-time optimizer. This trajectory is used to define a terminal constraint that forces the predicted trajectories to reach it at the end of the prediction horizon. This hierarchical approach is not robust to sudden changes in the economic cost function, which result in a change on the target periodic reference.[25,26] Following MPC for tracking approaches, several formulations for robust periodic economic MPC have been proposed.[21,23] However, these formulations need to duplicate the optimization variables resulting in more complex problems. In addition, economic performance is compromised because a tracking cost function is added to guarantee convergence. In the proposed formulation, the idea is based on designing an economic model predictive controller that minimizes the economic cost along a single period over all feasible periodic trajectories that start from the current state. To include robustness, we use the robust tube-based technique to guarantee recursive feasibility of the closed-loop systems. A local control gain and robust positively invariant (RPI) set are used to satisfy system constraints of the economic model predictive controller. The closed-loop system converges to a neighborhood of a periodic steady trajectory. When a defined optimality certificate is satisfied, this trajectory is equivalent to the optimal nominal periodic steady trajectory. The analysis of this certificate is based on the Karush-Kuhn-Tucker (KKT) optimality conditions under the convexity assumption.[27] In addition, the proposed controller is also robust to sudden changes in the economic cost function. Finally, two numerical examples are provided to demonstrate the proposed results.

This paper is organized as follows. The problem statement is addressed in Section 2. The robust economic MPC design is formulated in Section 3. The closed-loop properties of the system controlled by the proposed robust economic model predictive controller are discussed in Section 4. The simulation results of applying the proposed controller to the mass model and a counterexample are shown in Section 5. Finally, concluding remarks are drawn in Section 6.

2 | PROBLEM STATEMENT

Let us consider the following class of discrete-time linear time-invariant systems with additive disturbances:

$$x(k+1) = Ax(k) + Bu(k) + w(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and the input vectors, respectively; $w \in \mathbb{R}^n$ denotes the additive disturbance vector; and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are system matrices. The index k stands for the sampling time step.

Remark 1. From an application point of view, the additive disturbance $w(k)$ may include two parts as follows:

$$w(k) = B_d d(k) + \tilde{w}(k), \tag{2}$$

where $d \in \mathbb{R}^{n_d}$ denotes the vector of deterministic disturbances that is considered following a periodic behavior, that is, $d(k) = d(k+T)$ with a period $T > 0$ (see, for example, the work of Wang et al[3] and Broomhead et al[23]); $\tilde{w}(k)$ denotes the vector of unknown disturbances; and $B_d \in \mathbb{R}^{n \times n_d}$ is a distribution matrix.

For notation simplicity, in this paper, we consider that, in general, $w(k)$ is unknown and the following assumption is made.

Assumption 1. The additive disturbance vector $w(k)$ is assumed unknown but bounded by a convex set $\mathcal{W}$, that is,

$$w(k) \in \mathcal{W}, \quad \forall k \in \mathbb{N}. \tag{3}$$

The state and control input vectors $x(k)$ and $u(k)$ are required to satisfy the following constraints:

$$x(k) \in \mathcal{X}, \quad u(k) \in \mathcal{U}, \ \forall k \in \mathbb{N}, \tag{4}$$

where $\mathcal{X}$ and $\mathcal{U}$ are convex and compact sets.

Denote the nominal state and input vectors as $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^m$, which follow

$$\bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k). \tag{5}$$

In principle, the nominal system (5) could be used as the prediction model in the MPC design. However, due to the existence of $w(k) \in \mathcal{W}, \forall k \in \mathbb{N}$, the predicted states have a mismatch with the real states of the system (1). Hence, a robust economic model predictive controller is required to guarantee recursive feasibility and robust constraint satisfaction in a closed loop.

Let us denote a periodic signal sequence along the period T as

$$p = \{p_1, \ldots, p_T\},$$

with $p_i \in \mathbb{R}^{m_p}$ for $i = 1, \ldots, T$ and the economic cost for the system (1) measured by

$$\ell\left(x(k), u(k), p_i\right), \quad i = \mathrm{mod}(k, T). \tag{6}$$

Assumption 2. In practice, the periodic signal sequence $p = \{p_1, \ldots, p_T\}$ is usually related to a price pattern, such as in water distribution networks and power systems, where the electricity price follows a periodic pattern. The economic cost function (6) including the periodic signal p_i is assumed to be strictly convex.

Following a receding horizon control strategy, we would like to design a robust economic model predictive controller for system (1) in the presence of disturbances to optimize the economic cost (6) in a finite-horizon optimization problem. In this case, the mismatch between the closed-loop perturbed states, and the open-loop nominal predicted states is corrected using a local control law.

3 | ROBUST ECONOMIC MPC BASED ON A PERIODICITY CONSTRAINT

In this section, we propose a robust economic model predictive controller based on a periodicity constraint. To guarantee the robustness, a local control law and RPI sets are used to refine constraints on system states and control inputs. Based on the robust tube-based technique, these tightened constraints will be used in the design of the robust economic model predictive controller. At the end, we present the so-called robust economic planner. The planner provides the best possible periodic trajectory with respect to the economic cost taking into account the set of tightened constraints. The resulting trajectory provides the optimal nominal periodic steady trajectory, which will be used for comparison purposes.

3.1 | Refining constraints on states and inputs

We consider that state x of system (1) is fully measured and the pair (A, B) is controllable. Following the so-called tube based approach,[20] to guarantee recursive feasibility, we will use a robustly stabilizing local control gain $K \in \mathbb{R}^{n \times m}$ to tighten the sets $\mathcal{X}$ and $\mathcal{U}$. First, we introduce the RPI set in the following definition

Definition 1 (RPI set).
Given the system $x(k+1) = (A+BK)x(k) + w(k)$, where $(A+BK)$ is Schur and $w(k) \in \mathcal{W}$, $\forall k \in \mathbb{N}$. A set $\mathcal{Z}$ is said to be an RPI set of this closed-loop system if

$$(A+BK)\mathcal{Z} \oplus \mathcal{W} \subseteq \mathcal{Z} \tag{7}$$

By Definition 1, the RPI set can be explicitly formulated as $\mathcal{Z} = \oplus_{i=0}^{\infty}(A+BK)^i\mathcal{W}$, where $\oplus$ denotes the Minkowski sum. Considering the set $\mathcal{W}$ defined by

$$\mathcal{W} := \left\{ w \in \mathbb{R}^{m_w} \mid |w| \leq \bar{w} \right\}, \tag{8}$$

an approximation of the RPI set $\mathcal{Z}$ can be obtained using the following lemma.

Lemma 1 (Approximation of RPI set[28,29]).
Given system (1) and the Schur matrix $A \in \mathbb{R}^{n\times n}$, the Jordan decomposition form of $A = V\Lambda V^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, and the compact disturbance set in (8), the set

$$\mathcal{Z} = \left\{ x \in \mathbb{R}^n : \left|V^{-1}x\right| \leq v + \varepsilon \right\} \tag{9}$$

is an RPI and attractive for all the trajectories of this system, where $v = (I - |\Lambda|)^{-1}\left|V^{-1}\right|\bar{w}$ and $\varepsilon \in \mathbb{R}^n$ is an arbitrary small vector with components $\varepsilon_1 > 0, \dots, \varepsilon_n > 0$.

Let the RPI set $\mathcal{Z}$ be a polytopic form as

$$\mathcal{Z} := \left\{ x \in \mathbb{R}^n : H^z x \leq b^z, H^z \in \mathbb{R}^{l_{in}\times n}, b^z \in \mathbb{R}^{l_{in}} \right\}.$$

Then, we refine the sets $\mathcal{X}$ and $\mathcal{U}$ to be $\mathcal{X}^r$ and $\mathcal{U}^r$, where

$$\mathcal{X}^r = \mathcal{X} \ominus \mathcal{Z}, \tag{10a}$$

$$\mathcal{U}^r = \mathcal{U} \ominus K\mathcal{Z}, \tag{10b}$$

and $\ominus$ denotes the Pontrygin difference.

Assumption 3. Without loss of generality, the refined sets $\mathcal{X}^r$ and $\mathcal{U}^r$ are not empty.

3.2 | Robust economic model predictive controller

With the tightened constraints, we now present a novel formulation of the robust economic model predictive controller. For standard MPC optimization problem, the current state $x(k)$ is usually set as the first prediction state. Instead of optimizing the future trajectories starting from the current state $x(k)$, the proposed robust economic MPC optimizes a single periodic trajectory that starts at time step zero and is close to $x(k)$ at the appropriate time step defined by $\mathrm{mod}(k, T)$. This formulation has the advantage that only this constraint is modified at each sampling time k in which a new measurement is available. Standard economic MPC approaches yield at each sampling time an equivalent optimization problem in which the optimization variables are rotated, which complicates the analysis of the controller properties. Using the proposed formulation, what is rotated at each sampling time is the constraint that includes the current state, instead of the definition of the cost function, being more appropriate for periodic operation controllers.

In addition, the controller formulation is based on a tube-based approach. This implies two facts: (1) the constraints are tightened using a RPI set, and (2) the periodic trajectory will not meet through the current state k at prediction time $j = \mathrm{mod}(k, T)$, but instead, the difference has to be included in the aforementioned RPI set. These ingredients, together with the controller equations aiming to reduce this difference between the real state and the predicted state, will provide recursive robust constraint satisfaction. The closed-loop properties of the proposed robust economic model predictive controller will be demonstrated in the following section.

The proposed robust economic MPC is formulated by the following optimization problem:

$$V(k,x,p) = \underset{\substack{\bar{x}(0),\dots,\bar{x}(T),\\ \bar{u}(0),\dots,\bar{u}(T-1)}}{\text{minimize}} \; J_T(\bar{x},\bar{u},p) = \sum_{i=0}^{T-1} \ell\left(\bar{x}(i),\bar{u}(i),p_i\right), \tag{11a}$$

subject to

$$\bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i), \tag{11b}$$

$$\bar{x}(i) \in \mathcal{X} \ominus \mathcal{Z}, \tag{11c}$$

$$\bar{u}(i) \in \mathcal{U} \ominus K\mathcal{Z}, \tag{11d}$$

$$\bar{x}(0) = \bar{x}(T), \tag{11e}$$

$$x(k) - \bar{x}(j) \in \mathcal{Z}, \quad j = \mathrm{mod}(k, T). \tag{11f}$$

From the optimal solutions of (11a), with the robustly stabilizing local control gain $K \in \mathbb{R}^{n\times m}$, the control action at time step k is chosen as

$$u(k) = \bar{u}(j) + K\left(x(k) - \bar{x}(j)\right), \; j = \mathrm{mod}(k, T). \tag{12}$$

Using the formulation in (12), the mismatch between the predicted state $\bar{x}(j)$ for $j = \mathrm{mod}(k, T)$ and the closed-loop state $x(k)$ is attenuated by the local control gain K. In this case, due to constraint (11f), the closed-loop state trajectory $x(k)$ can always stay in a neighborhood of $\bar{x}(j)$, which is the tube defined by the RPI set $\mathcal{Z}$. Moreover, a periodic operation with the proposed robust economic MPC can be achieved using the periodicity constraint defined in 11e.

3.3 | Robust economic MPC planner

The control objective of the proposed robust model predictive controller is to drive the closed-loop system to a neighborhood of the optimal nominal periodic steady trajectory while robustly satisfying all the constraints. The economic performance of the closed-loop nominal system (5) is measured by

$$J_\infty\left(\bar{x}, \bar{u}, p\right) = \lim_{M\to\infty} \frac{1}{M} \sum_{i=0}^{\infty} \ell\left(\bar{x}(i), \bar{u}(i), p_j\right), \quad j = \mathrm{mod}(i, T).$$

In general, it is not possible to directly solve an infinite-horizon optimization problem. However, given the periodic nature of the dynamics, the considered constraints and economic cost function under Assumption 2, the uniqueness of the solution holds. Thus, the periodic nominal steady trajectory can be obtained by solving a finite-horizon open-loop problem (called the *planner*) that optimizes the economic cost over a period T (see equation (36) in the work of Angeli et al[10]).

$$\underset{\substack{\bar{x}(0),\,\dots\,,\bar{x}(T),\\ \bar{u}(0),\,\dots\,,\bar{u}(T-1)}}{\text{minimize}} \; J_T\left(\bar{x}, \bar{u}, p\right), \tag{13a}$$

subject to

$$\bar{x}(i+1) = A\bar{x}(i) + B\bar{u}(i), \tag{13b}$$

$$\bar{x}(i) \in \mathcal{X} \ominus \mathcal{Z}, \tag{13c}$$

$$\bar{u}(i) \in \mathcal{U} \ominus K\mathcal{Z}, \tag{13d}$$

$$\bar{x}(0) = \bar{x}(T). \tag{13e}$$

By solving the optimization problem (13a) offline, we can obtain the optimal solution denoted as $\bar{x}^*(0), \dots, \bar{x}^*(T)$ and $\bar{u}^*(0), \dots, \bar{u}^*(T-1)$. This optimal solution is considered as the optimal nominal periodic steady trajectory that will be used in the analysis of the closed-loop properties of the control system.

Remark 2. In the optimization problems (13a) and (11a), the index $i = 0, \dots, T-1$ is a prediction step along the MPC prediction horizon while the index $k \in \mathbb{N}$ is a time step for the simulation loop. This innovative optimization formulation will facilitate the analysis of the closed-loop properties presented in the next section.

4 | THE CLOSED-LOOP PROPERTIES OF THE CONTROL SYSTEM

In this section, we study the properties of the system (1) in a closed loop with the robust economic model predictive controller implemented by (11a), which are summarized in the following theorem.

Theorem 1. *Considering the system (1) with the robust economic model predictive controller implemented by (11a), the following closed-loop properties hold:*

a. *If the optimization problem (11a) is feasible from an initial state $x(0)$, then the closed-loop system satisfies all the constraints for all possible disturbances satisfying Assumption 1 and the optimal MPC cost $V(k, x, p)$, $\forall k \in \mathbb{N}$ is a nonincreasing sequence.*
b. ***[R1-1]** If there exists a time step $M > 0$ such that, for any $k \geq M$, $V(k+1, x, p) = V(k, x, p)$ and all the variables in the dual vector corresponding to the constraint (11f) are zero in the KKT optimality conditions, then the closed-loop system has reached a neighborhood (enclosed by the RPI set $\mathcal{Z}$ satisfying (7) of the optimal nominal periodic steady trajectory $\bar{x}^*(j)$ with $j = mod(k, T)$ obtained from the planner (13a).*

Proof. We first prove the closed-loop property expressed in the statement (a). In the following, we discuss recursive feasibility and robust constraint satisfaction of the closed-loop system. From these result, the closed-loop convergence is provided.

(*Recursive feasibility*) Let $\bar{x}(j)$ and $\bar{u}(j)$ be feasible solutions of the optimization problem (11a) at time step k. We now prove that the optimization problem (11a) is also feasible at time $k+1$. From the robust economic MPC formulation in (11a), the constraints (11b)-(11e) do not depend on the time step k, so $\bar{x}(j)$ and $\bar{u}(j)$ satisfy them by definition. The only constraint that depends on the time step k is (11f). From (11b), we have that

$$\bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j), \quad j = \mathrm{mod}(k, T).$$

Taking into account (1) and the control action $u(k)$ chosen in (12), we can derive

$$\begin{aligned} x(k+1) &= Ax(k) + B\left(\bar{u}(j) + K\left(x(k) - \bar{x}(j)\right)\right) + w(k) \\ &= A\left(x(k) - \bar{x}(j) + \bar{x}(j)\right) + B\left(\bar{u}(j) + K\left(x(k) - \bar{x}(j)\right)\right) + w(k) \\ &= A\bar{x}(j) + B\bar{u}(j) + A\left(x(k) - \bar{x}(j)\right) + BK\left(x(k) - \bar{x}(j)\right) + w(k). \end{aligned}$$

Therefore, by subtracting above two equations, we have

$$x(k+1) - \bar{x}(j+1) = (A + BK)\left(x(k) - \bar{x}(j)\right) + w(k).$$

Considering the constraint (11f), we obtain

$$x(k+1) - \bar{x}(j+1) \in (A + BK)\mathcal{Z} \oplus \mathcal{W} \subseteq \mathcal{Z},$$

for any $w(k) \in \mathcal{W}$. Hence, the constraint (11f) holds at time $k+1$ and the optimization problem (11a) is also feasible at time $k+1$.

(*Robust constraint satisfaction*) With the feasible solution $\bar{x}(j)$ and $\bar{u}(j)$ at time step time k, we know $\bar{x}(j) \in \mathcal{X} \ominus \mathcal{Z}$ and $\bar{u}(j) \in \mathcal{U} \ominus K\mathcal{Z}$ for $j = \mathrm{mod}(k, T)$. Taking into account that constraint (11f) holds, the control action $u(k)$ at time step k is chosen in (12), which implies that

$$u(k) \in \mathcal{U} \ominus K\mathcal{Z} \oplus K\mathcal{Z} \subseteq \mathcal{U}.$$

From constraint (11f), we also have

$$x(k) \in \bar{x}(j) \oplus \mathcal{Z} \in \mathcal{X} \ominus \mathcal{Z} \oplus \mathcal{Z} \subseteq \mathcal{X}.$$

We have proven that the optimization problem (11a) is recursively feasible with an initial condition $x(0)$ and the constraints in (4) are satisfied. Since the optimal solution of the previous time step is always feasible, by optimality, we can know that the optimal MPC cost $V(k, x, p)$ is a nonincreasing sequence along the time step k, that is

$$V(k+1, x, p) \leq V(k, x, p), \quad \forall k \in \mathbb{N}. \tag{14}$$

We next prove the closed-loop property in the statement (b). For notation simplicity, we denote a periodic trajectory including states and control inputs over the MPC prediction horizon as the vector $z \in \mathbb{R}^{n+m}$, where

$$z = \left[\bar{x}(0)^\top \ \ldots \ \bar{x}(T)^\top \ \bar{u}(0)^\top \ \cdots \ \bar{u}(T-1)^\top\right]^\top; \tag{15}$$

therefore, the economic cost function $J_T(\bar{x}, \bar{u}, p)$ becomes $J_T(z, p)$. For the planner (13a), the optimal cost can be denoted as $J_T(z^*, p)$. Under Assumption 2 and convex constraints, the optimization problems (13a) and (11a) are strictly convex. In the following, we reformulate them in the convex forms. The optimization problem (13a) is equivalent to

$$\underset{z}{\text{minimize}}\, J_T(z, p), \tag{16a}$$

subject to

$$h_r(z) \leq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{16b}$$

$$g_i(z) = 0, \quad i = 1, \ldots, l_{\text{eq}}, \tag{16c}$$

where (16b)-(16c) are linear constraints. Specifically, (16b) corresponds to the refined constraints on states and inputs in (10a), and (16c) corresponds to the nominal prediction model.

Similarly, the optimization problem (11a) is equivalent to

$$V(k, x, p) = \underset{z}{\text{minimize}}\, J_T(z, p), \tag{17a}$$

subject to

$$h_r(z) \leq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{17b}$$

$$g_i(z) = 0, \quad i = 1, \ldots, l_{\text{eq}}, \tag{17c}$$

$$H_j^z x(k) - H_j^z Q^z(\sigma) z - b_j^z \leq 0, \quad j = 1, \ldots, l_{\text{in}}, \ \sigma = \text{mod}(k, T), \tag{17d}$$

where H_j^z and b_j^z denote the jth row of H^z and b^z, and $Q^z(\sigma)z = \bar{x}(\sigma)$ with $\sigma = \text{mod}(k, T)$.

For the optimization problem (17a) at time step k, we denote

$$z(k) = \arg\min_{z} J_T(z, p). \tag{18}$$

According to chapter 5.5.3 in the work of Boyd and Vandenberghe,[27] we can obtain the KKT optimality conditions of (17a) as follows:

$$\nabla J_T(z(k), p) + \sum_{r=1}^{l_{\text{ine}}} \lambda_r(k) \nabla h_r(z(k)) + \sum_{i=1}^{l_{\text{eq}}} \mu_i(k) \nabla g_i(z(k)) + \sum_{j=1}^{l_{\text{in}}} \nu_j(k) H_j^z Q^z(\sigma) = 0, \tag{19a}$$

$$h_r(z(k)) \leq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{19b}$$

$$g_i(z(k)) = 0, \quad i = 1, \ldots, l_{\text{eq}}, \tag{19c}$$

$$H_j^z x(k) - H_j^z Q^z(\sigma) z(k) - b_j^z \leq 0, \quad j = 1, \ldots, l_{\text{in}}, \ \sigma = \text{mod}(k, T), \tag{19d}$$

$$\lambda_r(k) \geq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{19e}$$

$$\lambda_r(k) h_r(z(k)) = 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{19f}$$

$$\nu_j(k) \geq 0, \quad j = 1, \ldots, l_{\text{in}}, \tag{19g}$$

$$\nu_j(k) \left(H_j^z x(k) - H_j^z Q^z(\sigma) z(k) - b_j^z \right) = 0, \quad j = 1, \ldots, l_{\text{in}}, \ \sigma = \text{mod}(k, T), \tag{19h}$$

where $\lambda_r(k)$, $\mu_i(k)$ and $\nu_j(k)$ are dual variables. Denote the following vectors

$$\lambda(k) = \begin{bmatrix} \lambda_1(k) \\ \vdots \\ \lambda_{l_{\text{ine}}}(k) \end{bmatrix}, \mu(k) = \begin{bmatrix} \mu_1(k) \\ \vdots \\ \mu_{l_{\text{eq}}}(k) \end{bmatrix}, \nu(k) = \begin{bmatrix} \nu_1(k) \\ \vdots \\ \nu_{l_{\text{in}}}(k) \end{bmatrix}. \tag{20}$$

In terms of the robust economic MPC planner in (13a), the equivalent convex form can be written in a similar form as (17a) excluding the constraint (17d).

Recall the optimal nominal periodic steady trajectory as z^*, where the variable assignment for z is defined in (15). Therefore, there exists a set of dual vectors λ^* and μ^* this optimal solution z^* also satisfies the KKT optimality conditions:

$$\nabla J_T\left(z^*, p\right) + \sum_{r=1}^{l_{\text{ine}}} \lambda_r^* \nabla h_r\left(z^*\right) + \sum_{i=1}^{l_{\text{eq}}} \mu_i^* \nabla g_i\left(z^*\right) = 0, \tag{21a}$$

$$h_r\left(z^*\right) \leq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{21b}$$

$$g_i\left(z^*\right) = 0, \quad i = 1, \ldots, l_{> \text{eq}}, \tag{21c}$$

$$\lambda_r^* \geq 0, \quad r = 1, \ldots, l_{\text{ine}}, \tag{21d}$$

$$\lambda_r^* h_r\left(z^*\right) = 0, \quad r = 1, \ldots, l_{\text{ine}}. \tag{21e}$$

(*Convergence*) We have proved that the optimal MPC cost $V(k, x, p)\ \forall k \in \mathbb{N}$ is a nonincreasing sequence. Without loss of generality, we also consider that this optimal MPC cost is lower bounded by the optimal MPC cost corresponding to the planner (13a). This implies that as the time step $k \to +\infty$, the optimal MPC cost can reach a constant value. **[R1-1]** Moreover, if there exists a time instant M such that for any $k \geq M$, $V(k+1, x, p) = V(k, x, p)$ holds, then the optimal MPC cost has reached a constant value. Because the economic cost function $J_T(z, p)$ is strictly convex, it follows $z(k+1) = z(k) = z^s$. It means that, after M time steps, we can obtain a steady periodic trajectory z^s.

(*Optimality Certificate*) Since z^s is a feasible solution of the optimization problem (17a), there exist dual vectors λ^s and μ^s and ν^s such that the KKT optimality conditions (19a) hold. Recall z^* and $J_T(z^*, p)$ as the optimal planner trajectories and the economic planner cost obtained by solving the optimization problem (17a). If the dual vector $\nu^s = 0$, then λ^s and μ^s satisfy the KKT conditions (21a) of the planner, which implies $z^s = z^*$ and $V(k, x, p) = J_T(z^*, p)$.

The condition $\nu_j^s = 0$ is called the *optimality certificate*. If this certificate is satisfied, then from the trajectory $z^s = z^*$, we denote $\bar{x}^*(\sigma) = Q^z(\sigma) z^*$, $\sigma = \text{mod}(k, T)$ corresponding to states. From constraint (11f), we obtain $x(k) - \bar{x}^*(\sigma) \in \mathcal{Z}$ for any $k \geq M$, which means the closed-loop system can reach a neighborhood (enclosed by the RPI set $\mathcal{Z}$) of the periodic nominal steady trajectory that is obtained by the planner.

Summing up, the proposed controller guarantees robust constraint satisfaction, recursive feasibility, and a nonincreasing optimal cost of the optimization problem (11a), which guarantees the convergence to a neighborhood of the optimal nominal periodic steady trajectory when the optimality certificate is satisfied. Based on the constraint (11f), as the time step $k \to +\infty$, the deviation of the closed-loop system trajectory from the nominal steady trajectory is bounded in the RPI set $\mathcal{Z}$. □

Remark 3. From Theorem 1, we have provided an optimality certificate, ie, all the variables in the dual vector corresponding to the inequality constraint (17d) are zero in the KKT optimality conditions, which can be verified online to know if the closed-loop convergence is optimal, ie, it reaches a neighborhood of the optimal periodic steady trajectory.

Remark 4. For the statement (b), a sufficient optimality condition with a finite-time step $M > 0$ is given, which can be verified at each iteration online. Based on the convergence analysis in the proof of Theorem 1, once this optimality condition is satisfied in a finite-time step by checking the value of corresponding dual variables, the closed-loop convergence to the planner can be anticipated.

Remark 5. For some certain systems, due to tight constraints leading to low degree of freedom, the optimality certificate may be not satisfied. Then, from the recursive feasibility, the corresponding KKT optimality conditions (19a) can still be satisfied. In this case, the optimization problem (17a) is also possible to reach a steady solution z^s with $|V(k+1, x, p) - V(k, x, p)| \leq \epsilon$, $\forall k \geq \bar{M}$ with an arbitrary small scalar ϵ. However, from the KKT optimality conditions (21a), z^s is a suboptimal solution. Thus, we can conclude that $z^s \neq z^*$ and $V_T(k, x, p) > J_T\left(z^*, p\right)$.

In this robust economic MPC design, the tube-based technique is used. As an example shown in Figure 1, the optimal nominal periodic steady trajectory obtained by the planner (13a) is plotted in red dashed line, the tubes defined by the RPI set $\mathcal{Z}$ are plotted in blue boundaries, and a closed-loop trajectory of system (1) with the proposed robust economic MPC (11a) is plotted in the blue line. Hence, we can conclude that once the closed-loop trajectory is close to the optimal nominal periodic steady trajectory, the optimal solution does not change because the state is in a tube and the input applied in (12) guarantees that it will not go outside the tube because it is defined as an RPI set.

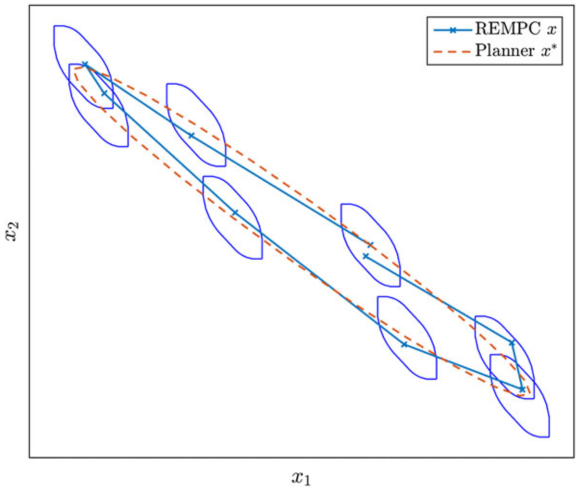


FIGURE 1 A closed-loop trajectory and optimal nominal steady periodic trajectory with tubes. REMPC, robust economic model predictive control [Colour figure can be viewed at wileyonlinelibrary.com]

5 | ILLUSTRATIVE EXAMPLES

In this section, we show two examples to illustrate the proposed theoretical results. Specifically, we first use the mass model taken from[23] to demonstrate the effectiveness of the proposed robust economic model predictive controller. Then, we also provide a counterexample to show the case when the optimality certificate is not satisfied.

5.1 | Example 1: the mass model

The mass model with a spring and a damper is shown in Figure 2. Consider a discrete-time model of this mass model in the form as in (1) with system matrices

$$A = \begin{bmatrix} 0.9952 & 0.0950 \\ -0.0950 & 0.9002 \end{bmatrix}, \ B = \begin{bmatrix} 0.0048 \\ 0.0950 \end{bmatrix}, \ B_d = B,$$

and $w(k) := B_d d(k) + \tilde{w}(k)$, where the displacement and the velocity of the mass model are chosen as state variables in x and d is a periodic known signal with a period $T = 10$ that is given by a sequence $d(k) = d_i$ with $i = \mathrm{mod}(k, T)$. The disturbance $\tilde{w}(k) \in \mathcal{W}, \forall k \in \mathcal{W}$, where the set $\mathcal{W}$ is given in the form of (8) with $\bar{w} = [0.005,\ 0.01]^\top$. The constraints on states and inputs are given by the following sets:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^2 \,||x| \le [1.5 \ \ 0.75]^\top \right\},$$
$$\mathcal{U} = \{ u \in \mathbb{R} \,||u| \le 8 \}.$$

The local control law $K \in \mathbb{R}^{1\times 2}$ is computed using the LQR method with weighting matrices $Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ and $R = 0.01$ obtaining

$$K = [-1.8635 \ \ -2.5172].$$

The initial state is $x(0) = [-0.0890 \ \ 0.3570]^\top$. As defined by Broomhead et al,[23] the economic cost function is chosen to be $\ell(\bar{x}, \bar{u}, p) = 10(\bar{x}_2(i) - p_i)^2 + (\bar{u}(i))^2$ with a periodic signal p in order to test the proposed controller with sudden changes.

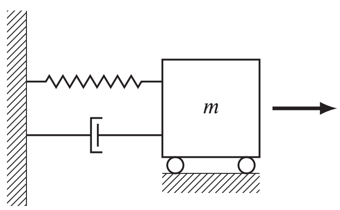


FIGURE 2 The mass model with a spring and a damper[23]

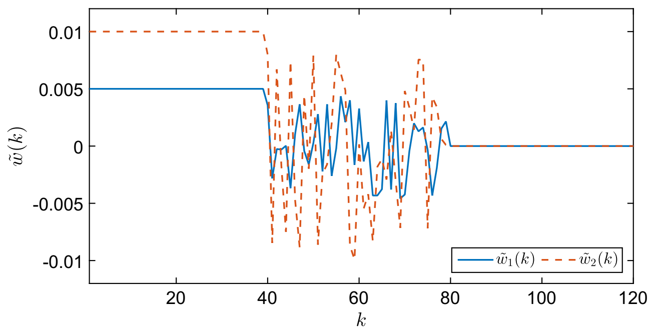


FIGURE 3 The sampled bounded disturbances [Colour figure can be viewed at wileyonlinelibrary.com]

These sudden changes are given by choosing different values of periodic signals d and p. In the simulation, the following two scenarios are considered:

- **Scenario 1**: For $k < 50$,

$$d_i = 5\cos\left(\frac{2\pi i}{T}\right), \quad i = 0, \ldots, T-1,$$
$$p_i = 0.1\sin\left(\frac{2\pi i}{T}\right), \quad i = 0, \ldots, T-1.$$

- **Scenario 2**: For $k \geq 50$,

$$d_i = 0.1\sin\left(\frac{2\pi i}{T}\right) + 0.5, \quad i = 0, \ldots, T-1,$$
$$p_i = 1.2\cos\left(\frac{2\pi i}{T}\right), \quad i = 0, \ldots, T-1.$$

The optimization problems (13a) and (11a) are solved using the Yalmip toolbox[30] and the MOSEK solver[31] in the MATLAB environment. For the previous scenarios, the planner has been applied. Then, two optimal nominal periodic steady trajectories and two different optimal MPC costs can be obtained.

The closed-loop simulation has been carried out for 120 sampling time steps with a sudden change defined in the previous two scenarios. As shown in Figure 3, the unknown disturbance $\tilde{w}(k)$ is defined as follows:

$$\tilde{w}(k) = \begin{cases} \bar{w}, & k < 40, \\ \tilde{w} \in \mathcal{W}, & 40 \leq k < 80, \\ 0, & k \geq 80. \end{cases}$$

The closed-loop results of state and control input trajectories are shown in Figures 4 and 5. For $k < 50$ (Scenario 1), starting from the feasible initial state $x(0)$, the closed-loop state and input trajectories converge to a neighborhood of the optimal nominal periodic trajectories obtained by the Planner 1. At the time step $k = 50$, there is a sudden change of the periodic signals d and p as defined in Scenario 2. For $k \geq 50$ (Scenario 2), the closed-loop system is also feasible, and the closed-loop state and input trajectories converge to a neighborhood of the optimal nominal periodic trajectories obtained by the Planner 2. From these results, it proves that the closed-loop system is always feasible from an initial state even with a sudden change.

Since we have discussed that the recursive feasibility mainly relies on the inequality constraint (11f), this constraint should be satisfied with the closed-loop state $x(k)$, $\forall k \in \mathbb{N}$. As shown in Figure 6, the mismatch between the closed-loop state and the optimal nominal state should be always inside the RPI set $\mathcal{Z}$. Hence, this result also proves that the closed-loop system can be always recursively feasible in the presence of unknown-but-bounded additive disturbances.

Taking into account three different selections of bounded additive disturbances, for $k < 40$, the closed-loop state and input trajectories are periodic based on the periodicity constraints and meanwhile approaching to the optimal nominal periodic steady trajectories obtained by Planner 1. For $40 \leq k < 80$, the closed-loop trajectories are close to the optimal nominal periodic steady trajectories and with the sudden change, the optimal nominal periodic steady trajectories are switched to the ones obtained by the Planner 2. Moreover, the closed-loop trajectories in Figure 4B and Figure 5 stay close to the optimal nominal periodic steady trajectories in the tube (defined by the RPI set $\mathcal{Z}$). For $k \geq 80$, since $w(k) = 0$, the

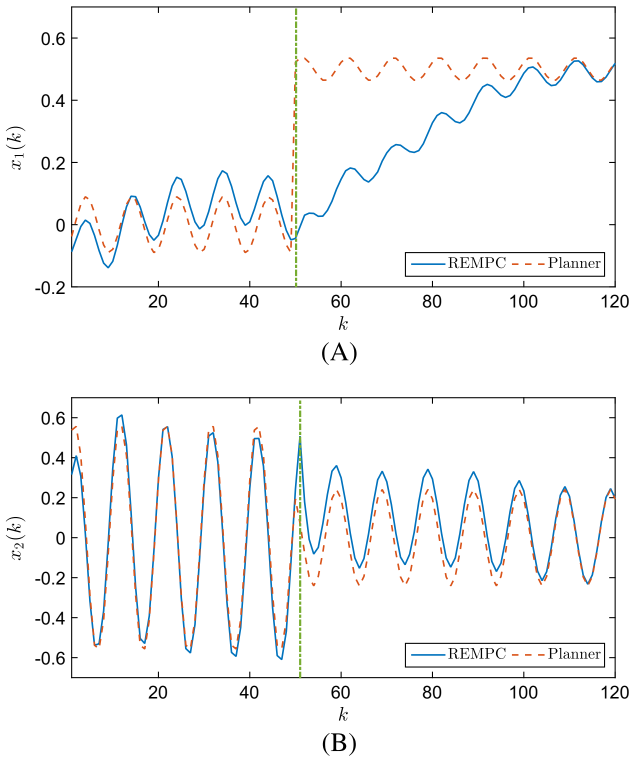


FIGURE 4 The closed-loop state trajectories of the mass model. A, x_1; B, x_2. REMPC, robust economic predictive model control [Colour figure can be viewed at wileyonlinelibrary.com]

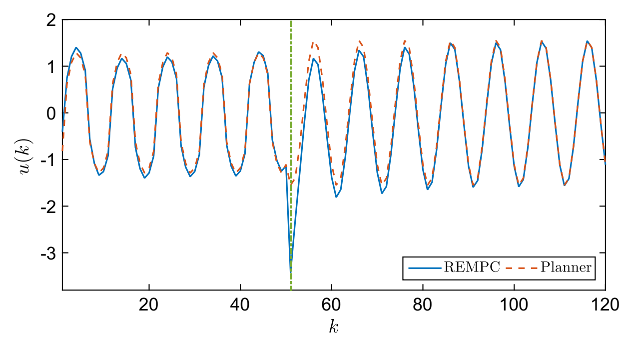


FIGURE 5 The closed-loop input trajectory of the mass model. REMPC, robust economic model predictive control [Colour figure can be viewed at wileyonlinelibrary.com]

closed-loop state and input trajectories are able to match the optimal nominal periodic steady trajectories of the Planner 2 after a transient time.

Moreover, from the offline computation results of the planners, two optimal MPC costs are also shown in Figure 7A. Since the optimality certificate is verified online, the closed-loop optimal MPC cost can converge to the optimal one for each scenario with a sudden change in the closed-loop cost between two scenarios.

As discussed in Theorem 1, the optimality certificate given by checking whether all the variables in the dual vector $\nu(k)$ corresponding to (11f) is zero. From the online closed-loop simulation, these dual variables can be extracted together with the optimal solution of (11a) at each time step. To verify the optimality certificate, the two-norm of $\nu(k)$ as $\|\nu(k)\|_2$ is shown in Figure 7B. For these scenarios considered, two steady situations are expected to be observed. Despite sudden changes in the controller design parameters, after a transient time, the two-norm of $\nu(k)$ converges to zero. Also, as shown in Figures 4 and 5, the closed-loop trajectories are able to reach a neighborhood of the optimal nominal periodic steady trajectories obtained by each planner.

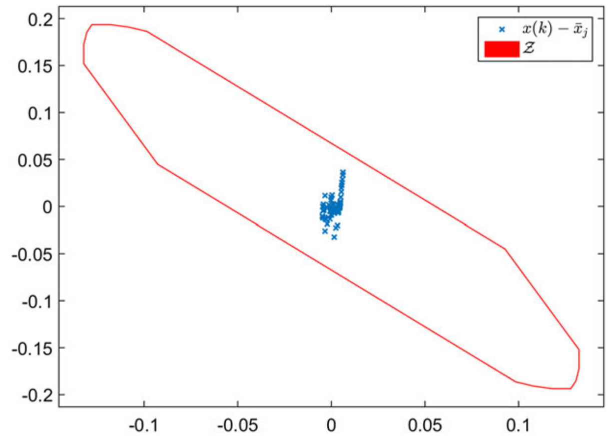


FIGURE 6 The validation of the recursive feasibility for the mass model [Colour figure can be viewed at wileyonlinelibrary.com]

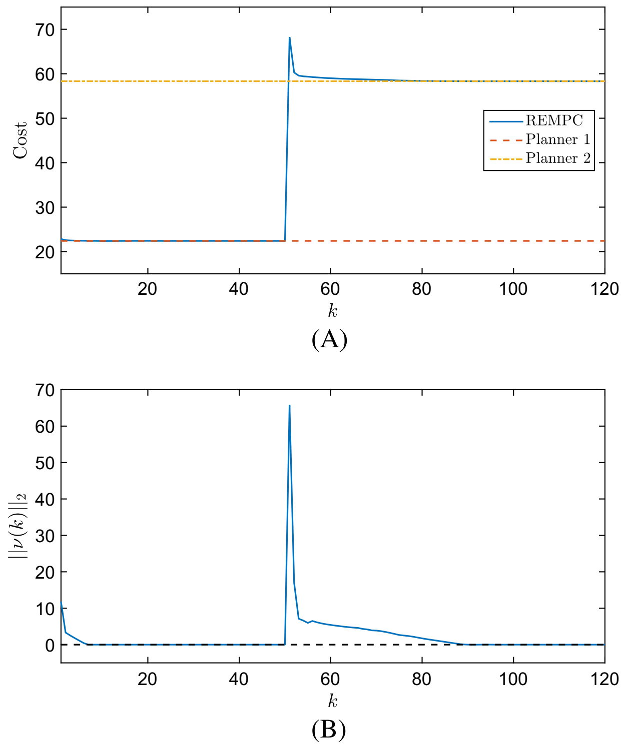


FIGURE 7 The closed-loop economic cost and the online verification of the optimality certificate for the mass model. A, The closed-loop economic cost; B, The optimality certificate. REMPC, robust economic predictive model control [Colour figure can be viewed at wileyonlinelibrary.com]

5.2 | Example 2: a counterexample

To illustrate the optimality certificate, we next show a counterexample to show the case that the optimality certificate is not satisfied. Consider the discrete-time system (1) with the disturbance vector in the form of (2), where

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.25 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the period is given by $T = 3$. The periodic known signal $d(k) = d_i$ for $i = \text{mod}(k, T)$ is given by $d_0 = -0.1$, $d_1 = -0.2$, and $d_2 = -0.1$. The disturbance $\tilde{w}(k)$ is bounded by the set $\mathcal{W}$ also in the form of (8) with $\bar{w} = 0.01$.

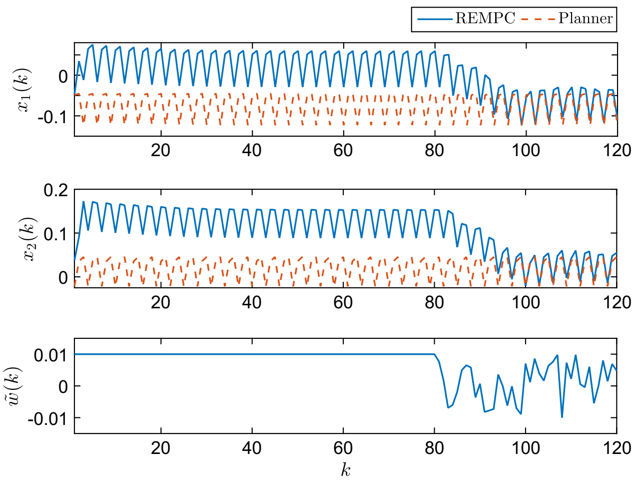
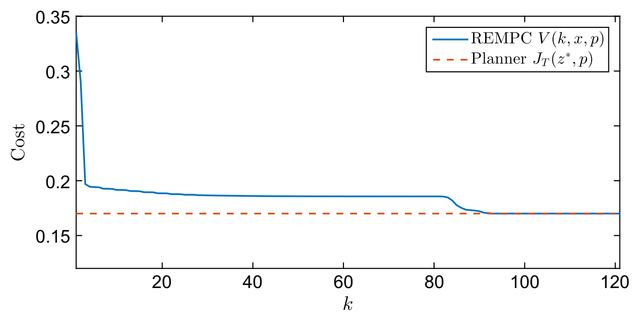


FIGURE 8 The closed-loop state trajectories of the counterexample. REMPC, robust economic model predictive control [Colour figure can be viewed at wileyonlinelibrary.com]

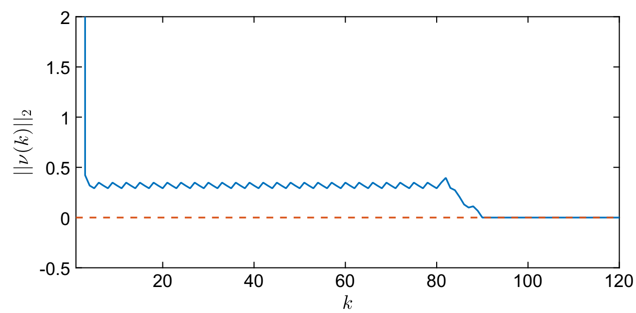
The constraints on state and input are set as $\mathcal{X} = \{x \in \mathbb{R}^2 | |x| \leq [0.15 \;\; 0.2]^\top\}$ and $\mathcal{U} = \{u \in \mathbb{R} | |u| \leq 0.1\}$. Again, by means of the LQR method with weighting matrices $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 100$, we obtain a local control gain $K = [-0.1543, \; -0.0909]^\top$. Given a convex cost function in a standard quadratic form $J_T(z) = \frac{1}{2} z^\top H z + f^\top z$, where

$$H = \mathrm{diag}\left([1 \;\; 1 \;\; 10 \;\; 1 \;\; 1 \;\; 20 \;\; 1 \;\; 1 \;\; 10]\right),$$
$$f^\top = [0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1].$$

The closed-loop state trajectories and the planner trajectories are plotted in Figure 8 with $\tilde{w}(k) \in \mathcal{W}$ in two scenarios: (1) $w(k) = \bar{w}$ for $k < 80$, and (2) $\tilde{w}(k) \in \mathcal{W}$ with random values for $k \geq 80$. In Scenario (1), it is shown that the closed-loop state trajectories are converging to periodic trajectories but different from those of the planner. Note that in Figure 9, the measure of the dual variable $\nu(k)$ is not always equal to zero. Hence, the optimality certificate is not satisfied. Moreover,



(A)



(B)

FIGURE 9 The closed-loop economic cost and the optimality certificate of the counterexample. A, The closed-loop economic cost; B, The optimality certificate. REMPC, robust economic model predictive control [Colour figure can be viewed at wileyonlinelibrary.com] the cost $V(z(k), p)$ is more expensive than the planner one, which is the optimal cost $J_T(z^*, p)$. In this case, the optimality certificate is not satisfied because constraints of this optimization problem are too tight. Consequently, the closed-loop trajectory may be trapped into a periodic trajectory that is not optimal.

On the other hand, in Scenario (2) with random $\tilde{w}(k)$, the closed-loop trajectories are able to reach a neighborhood of the optimal nominal periodic steady trajectory obtained by the planner. In this case, since the corresponding dual variables are zero, ie, the optimality certificate is satisfied, the closed-loop cost is reducing to the optimal MPC cost of the planner.

6 | CONCLUSION

In this paper, we have proposed a robust economic MPC based on a periodicity constraint for linear systems subject to unknown but bounded disturbances. We have proven that, from a feasible initial condition, the closed-loop system is always feasible even with sudden changes and all the system constraints can be satisfied in the presence of disturbances. We have also proven the closed-loop convergence with the proposed robust economic model predictive controller. Moreover, an optimality certificate of the proposed robust MPC has been given. If this certificate is satisfied, the closed-loop trajectory has reached a neighborhood of the optimal nominal periodic steady trajectory that can be obtained by the planner, where the region is defined by the considered RPI set. Finally, we have tested the proposed robust strategy with two examples. Inside the results, a counterexample has been also provided, where the optimality certificate is not always satisfied. Due to constraints of the corresponding optimization problem being set too tight, the optimality certificate cannot be satisfied, and consequently, the closed-loop trajectory may be trapped in another periodic trajectory. As future work, an extension of the proposed robust economic MPC framework to nonlinear systems for periodic operation deserves to be investigated, where a suitable robust strategy for nonlinear systems should be addressed. Moreover, finding an additional condition in the MPC design to enforce the satisfaction of the optimality certificate could be interesting.

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