Technical Report

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Yet a Better Closed-Form Formula for the 3D Nearest Rotation Matrix Problem

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Abstract

This communication complements the results recently presented in [1] showing that they can be extended to define an efficient and robust method to determine the rotation matrix nearest to an arbitrary 3×3 matrix. This problem arises in different areas of robotics that range from the simple case in which we have to restore the orthogonality of a noisy rotation matrix to point-cloud registration or hand-eye calibration. We show that the new method compares favorably with the classical approaches to address this problem and also with more recent methods [2, 3].

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1 Introduction

Many fundamental problems in robotics require to solve the 3D nearest rotation problem. For instance, problems such as point cloud registration [4] or hand-eye calibration [5, 6] have to determine the rotation matrix $\hat{\boldsymbol{R}} \in SO(3)$ such that

$$\arg\min_{\hat{\boldsymbol{R}}} \operatorname{Trace}\left(\hat{\boldsymbol{R}}\boldsymbol{B}^\top\right), \tag{1}$$

from noisy estimations of $\hat{\boldsymbol{R}}$ and where $\boldsymbol{B} \in \mathbb{R}^{3\times 3}$ a kernel matrix specific for each problem.

The nearest rotation matrix is classically determined using the singular value decomposition (SVD). For the 3D case, this can also be achieved using eigenvalue decomposition of a 4×4 matrix. A recent paper presents two alternative methods, one exact and one approximate [3]. Concurrently with the reviewing process of this paper, another method for the same purpose was included in a paper on 3D point-cloud registration [1]. This latter method is exact, efficient and, with the modifications introduced herein, also robust. This communication, coauthored by some of the authors of both works, gives a concise presentation of this new method and compares it to the previous ones.

2 Exact-closed form formula

The problem addressed in this communication can be stated as follows: given the noisy rotation matrix $\boldsymbol{R} = (r_{ij})_{1\le i,j\le 3}$, find the rotation matrix $\hat{\boldsymbol{R}}$ that minimizes the Frobenius norm $\left\|\hat{\boldsymbol{R}}-\boldsymbol{R}\right\|_F^2$. Given $\boldsymbol{R}$, let us define the following symmetric matrix 4×4 associated to it

$$\boldsymbol{G} = \begin{pmatrix} r_{11}+r_{22}+r_{33} & r_{32}-r_{23} & r_{13}-r_{31} & r_{21}-r_{12} \\ r_{32}-r_{23} & r_{11}-r_{22}-r_{33} & r_{21}+r_{12} & r_{31}+r_{13} \\ r_{13}-r_{31} & r_{21}+r_{12} & r_{22}-r_{11}-r_{33} & r_{32}+r_{23} \\ r_{21}-r_{12} & r_{31}+r_{13} & r_{32}+r_{23} & r_{33}-r_{11}-r_{22} \end{pmatrix}. \tag{2}$$

It was proven in [7] that the dominant eigenvector of $\boldsymbol{G}$ is actually the vector of Euler parameters whose corresponding rotation matrix is $\hat{\boldsymbol{R}}$. The dominant eigenvector is the eigenvector associated with the eigenvalue whose absolute value is maximal. The determination of the maximal eigenvalue requires computing the roots of a quartic polynomial which can be performed using Ferrari's method [7]. Nevertheless, in [1], it was shown that it is not necessary to compute all roots to keep the largest one since a rather simple formula to compute it exists.

The characteristic polynomial of $\boldsymbol{G}$ can be expressed as

$$\lambda^4 + \tau_3\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0, \tag{3}$$

where

$$\begin{aligned} \tau_3 &= \text{Trace}(\boldsymbol{G}) = 0, \\ \tau_2 &= -2\sum_{i=1}^{3}\sum_{j=1}^{3} h_{i,j}^2 = -2\,\text{Trace}\left(\boldsymbol{R}^\top\boldsymbol{R}\right), \\ \tau_1 &= -8\det(\boldsymbol{R}), \\ \tau_0 &= \det(\boldsymbol{G}). \end{aligned}$$

The roots of (3) are real because $\boldsymbol{G}$ is symmetric. The application of Ferrari's method to obtain these roots is simplified because τ_3 is identically zero. In [1], it is shown that the largest real root of (3) can be expressed as

$$\lambda_{\text{max}} = \begin{cases} \sqrt{-\frac{\tau_1}{2}}, \text{ if } |\tau_1| < \zeta \text{ and } |\,k_1| < \zeta, \\ \\ \frac{1}{\sqrt{6}}\left(k_1+\sqrt{-k_1^2-12\tau_2-\frac{12\sqrt{6}\tau_1}{k_1}}\right), \text{ otherwise}, \end{cases}$$

where

$$k_0 = 2\tau_2^3 + 27\tau_1^2 - 72\tau_2\tau_0,$$
$$\theta = \operatorname{atan2}\left(\sqrt{4(\tau_2^2 + 12\tau_0)^3 - k_0^2},\ k_0\right),$$
$$k_1 = 2\sqrt{\left(\sqrt{\tau_2^2 + 12\tau_0}\right)\cos\frac{\theta}{3} - \tau_2}.$$

The threshold ζ is typically taken to be a small positive number, such as 10^{-5} [1].

Moreover, it can be proven that all the rows of the cofactor matrix of $(\boldsymbol{G} - \lambda_{\text{max}}\boldsymbol{I})$ are proportional to the eigenvector corresponding to λ_{max} [7]. In [1], some computational time is saved by computing only the last row of this cofactor matrix. Unfortunately, all the elements of this row are identically zero for rotations whose rotation axis lies on the xy-plane. Although, at least in theory, rotations whose rotation axes lie on the xy-plane can be seen as a set of measure zero in the space of quaternions, in practice it is enough to be close to this situation to generate large errors. Similar situations arise if we take any other row. Thus, for the sake of robustness, we have to compute all rows and take, for example, the one with the largest norm, say $\boldsymbol{q} = (q_1, q_2, q_3, q_4)^\top$. Then, the sought after rotation matrix $\hat{\boldsymbol{R}}$ is

$$\hat{\boldsymbol{R}} = \frac{1}{\boldsymbol{q}^\top\boldsymbol{q}} \begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & q_1^2-q_2^2-q_3^2-q_4^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{bmatrix} \tag{4}$$

3 Numerical validation

This short note has supplementary downloadable material which includes a C++ implementation of:

- The exact method presented in [3].
- The approximated method also introduced in [3].
- The approximated Cayley's method introduced in [2].
- The orthonormalization method based on the Jacobi singular value decomposition (SVD).
- The method using the implicit QR SVD method described in [8].
- The exact method presented in [1] with the modification described in this note.

To assess the performance of these methods, we apply the same procedure as in [3]:

1. Generate 10^6 random quaternions using the algorithm detailed in [9], which permits to generate sets of points uniformly distributed on the 3-D sphere $\mathbb{S}^3$.
2. Convert these quaternions to rotation matrices whose elements are then contaminated with additive uncorrelated uniformly distributed noise in the interval $[-\delta, \delta]$.
3. Compute the nearest rotation matrices for these 10^6 noisy rotation matrices using each of the above methods.
4. Compute the average and the maximum Frobenius norm between the noisy matrices and the obtained matrices for each method.
5. Compute the average and the maximum orthogonality error of the obtained matrices, computed as the Frobenius norm of $\boldsymbol{R}\hat{\boldsymbol{R}}^\top - \boldsymbol{I}$.

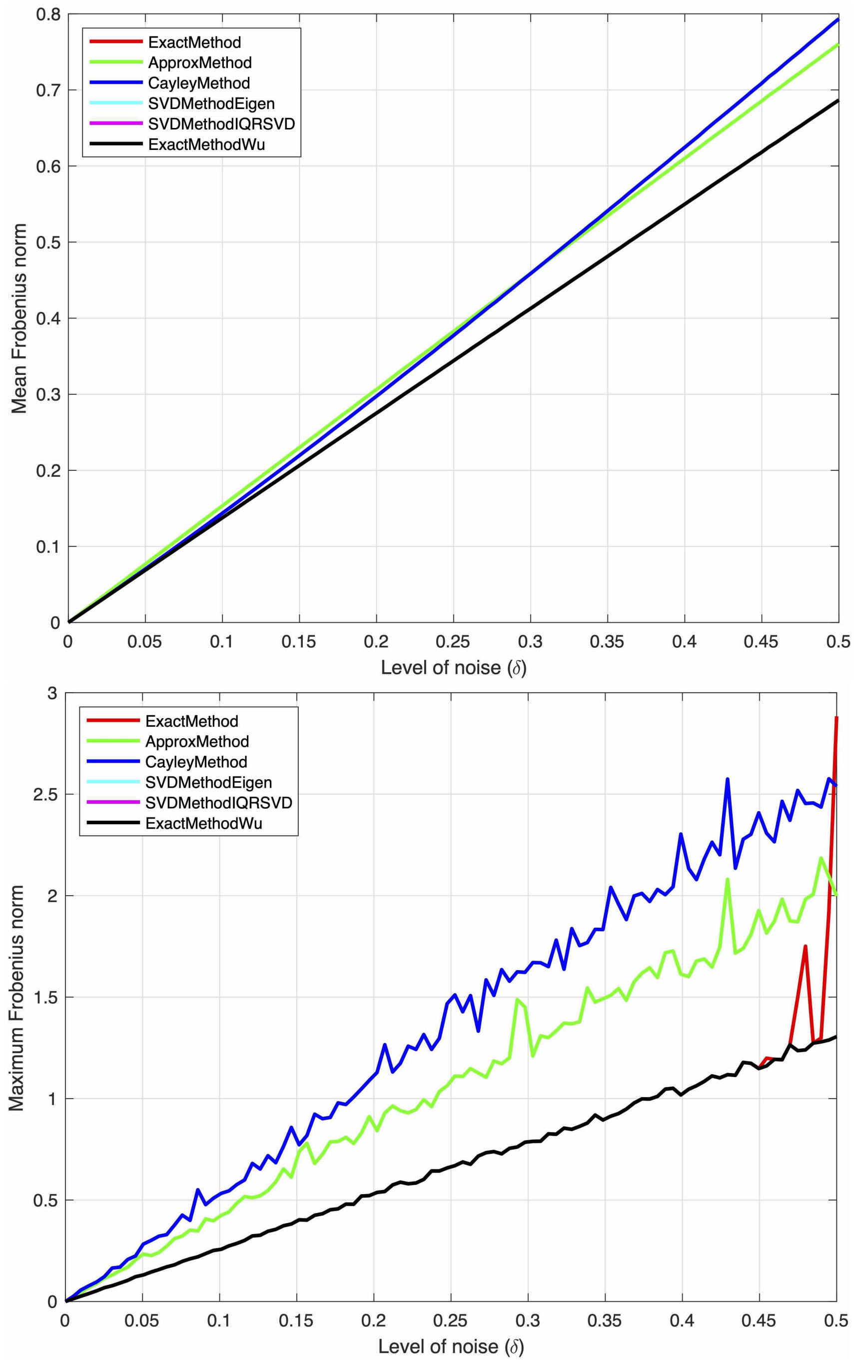

Figure 1: Mean (top) and maximum (bottom) Frobenius norm between randomly generated noisy rotation matrices and the corresponding nearest rotation matrices obtained with the six compared methods implemented in single precision arithmetics. 10^6 random matrices are generated for each value of δ.

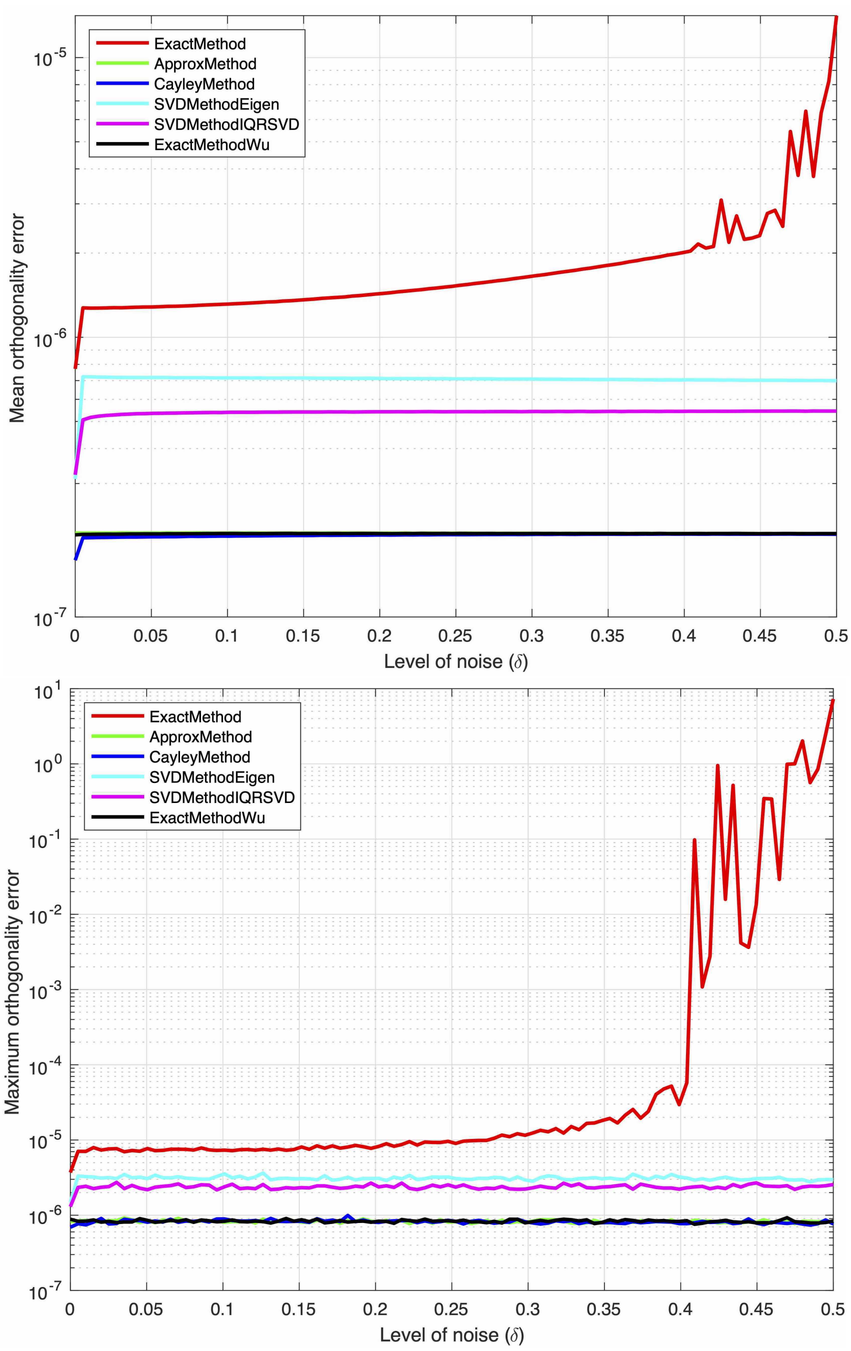

Figure 2: Mean (top) and maximum (bottom) orthogonality error of the rotation matrices obtained with the six compared methods implemented in single precision arithmetics. 10^6 random matrices are generated for each value of δ.

If this procedure is repeated for values of δ ranging from 0 to 0.5, the plots in Figs. 1 and 2 are obtained. The curves for the mean Frobenius norm error obtained using the exact method summarized in this note overlap with that of all other exact methods. The curves for the maximum Frobenius norm error also overlap, except for the exact formula presented in [3] which is numerically unstable for high levels of noise. The mean and maximum orthogonality error for the new exact formula is negligible since the last step of this method consists in applying equation (4) which returns an exact orthogonal matrix for any $\boldsymbol{q}$, provided that $\|\boldsymbol{q}\| > 0$. Using single-precision arithmetic on a PC with a 4,2 GHz Intel Core i7 processor, the average execution time of the six methods compared here is 0.12μs, 0.04μs, 0.03μs, 1.28μs, 0.22μs, and 0.12μs, respectively.

Summing up, the method introduced in [1] is as accurate as the exact method presented in [3], but without its numerical instabilities for large levels of noise. Both of them are clearly more efficient than the classical orthonormalization methods based on the SVD. Thus, according to the presented comparisons, the method introduced in [1], together with the modification included here for the sake of robustness, should be the method of choice for the exact orthonormalization of noisy rotation matrices.

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