Distributed Data-Driven UAV Formation Control Via Evolutionary Games: Experimental Results ⋆

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Abstract

This work proposes a novel data-driven distributed formation-control approach based on multi-population evolutionary games, which is structured in a leader-follower scheme. The methodology considers a time-varying communication graph that describes how the multiple agents share information to each other. We present stability guarantees for configurations given by time-varying interaction networks, making the proposed method suitable for real-world problems where communication constraints change along the time. Additionally, the proposed formation controller allows for an agent to leave or enter the group without the need to modify the behaviors of other agents in the group. This game-theoretical approach is evaluated through numerical simulations and real outdoors experimental results using a fleet of aerial autonomous vehicles, showing the control performance.

Key words: Data-driven control, evolutionary dynamics, game theory, mobile robots, formation control, UAVs, real implementation

1 Introduction

The formation control scheme of autonomous vehicles, and in particular of aerial autonomous vehicles (UAVs) is one of the key problems in the field of cooperative multi-agent navigation. This is due to the complex nature of the system and its significant potential impact in applications such as search and rescue, environmental monitoring and mapping, among many others [2], [3], [4]. Coordinated control involving formation usually implies relative spacial constraints on the vehicles. Furthermore, hardware limitations on on-board sensors and payload equipment define additional constraints such as communications range, restricting the information that a UAV can rely on for coordination purposes, introducing additional challenges to the overall formation control scheme.

⋆ Preliminary results reported in this paper have been presented without experiments in Proceedings of the 55th IEEE Conference on Decision and Control (CDC 2016) in [1].

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Table 1
Summary of literaure review for model-based/data-driven, and game-theory/non-game-theory techniques. [R1-1]

UAVs Control				
Model-based			**Data-driven**	
Game Theory	**Lagrangian Systems**	**Other Techniques**	**Game Theory**	**Other Techniques**
[13–19]	[9–12]	[20–26]	This paper	[27,28]

A wide variety of formation control approaches have been reported in the literature, many of them addressing the specific challenges of aerial vehicles [5,6] and considering distributed control architectures [7]. Some of the works found in the literature consider communication restrictions by modeling them with communication graphs, but the usual approach considers a constant graph during operation. This assumption may not address realistic needs found in real-world scenarios. For instance, in the coordinated navigation problem, if the formation shape varies dynamically over time, it may be necessary to rearrange the communication network to meet such new scenario. For instance, in [8], the communication is modeled by using hybrid graphs to capture information flow among agents. Another powerful approach to control multi-agent systems, as the problem discussed in this paper, is by using Lagrangian systems. In [9], an optimization-based distributed controller is designed for a group of autonomous Lagrangian systems while considering local cost functions. Either the consensus or the tracking problem can be addressed by using the Lagrangian systems as reported in [10], where distributed average tracking problem is solved. Related to the problem settings that we present in this manuscript, the Lagrangian systems allow the study of communication graphs and also leader-follower configurations. For example, in [11], Lagrangian systems play the role of followers tracking a virtual leader under asymmetric time-varying communication delays; and in [12], cooperative tracking is studied for a group of Lagrangian vehicle systems using a directed communication graph.

There exist two main approaches to control the formation of multi-agent systems as the UAVs. On one hand, there are control techniques requiring a dynamical model. On the other hand, there are some formation control mechanisms which are model-free or data oriented. Next, we discuss about the model-based controllers reported in the literature, and then how learning algorithms are used to solve the model-free formation problem, highlighting the relevance and advantages of the game theory approach addressed in this paper.

Regarding the model-based control strategies different from game theoretical techniques, there are several reported solutions in the literature, e.g., robust control, geometric control, flocking control, sliding mode, model predictive control, among others. In [24], a non-linear and robust tracking control is designed for UAVs, and [29] presents an attitude control for UAVs, illustrating the performance of the methodology by means of experimentation results/validation as we do in this paper. In [25], swarm systems with switching interaction topologies are used to solve time-varying formation control problems for UAVs. In [26], a formation control is designed by using negative imaginary systems theory. Trajectory-tracking control based on geometric control is presented in [20]. Other works propose distributed model-based control techniques as in [21] where flocking control with virtual leaders is designed. Discussing about the data-driven techniques different from game theory, in [27], cooperating-vehicle problems are studied with sensor-equipped UAVs for information gathering, and in [28], a data-driven modeling is presented by means of Gaussian process for the operation of multiple UAVs.

Game theory has been presented as a plausible alternative in the control of UAVs. Likewise, this approach could be designed either based on a model or model-free/data-driven. Some model-based game-theory controllers are presented next. In [13,14,18], game theory is used in the solution of problems involving UAVs in the context of communication networks. In [13], UAVs are considered as Internet-of-Things vehicles; in [14], also reinforcement learning is used considering the UAVs as quasi-base-stations in a communication network; and in [18], UAVs are used to assist wireless networks. Other type of problems with UAVs using game theory comprises the following. In [15], multi-UAVs concensus is solved by means of differential games. In [16], the formation problem is solved in the context of zero-sum games. In [17], as in [14], game theory is combined with reinforcement learning, and the heterogeneous formation control is addressed with a virtual leader. In [19], optimal coverage problem is solved by using UAVs and game theory in the context of potential games. Hence, to the best of our knowledge, data-driven game-theory-based control with real implementation for UAVs has not been reported in literature. This is one of the objective of this work. A summary of literature review is presented in Table 1.

Game theory is an approach that over recent years has proven to be very powerful for a myriad of applications requiring the design of control systems [30]. A key component that enables this approach is the concept of modeling interactions among alternative decision-makers. This is particularly useful when working with distributed systems [31]. When focusing on cooperative control and multiagent systems, some theoretical approaches have been proposed.

For instance, the concept of differential games was proposed to control formation of robots under fixed communication constraints [32]. In [33], differential-games were proposed to control a system that intercepts moving targets at an angle previously specified. Furthermore, evolutionary game theory, and in particular population games, have been shown to be suitable for distributed control of large-scale systems [30]. Besides, other works reported in the literature have shown several properties of the evolutionary dynamics. For instance, in [34], the outcome of a game with four strategies is studied under the framework of evolutionary dynamics including competition among cooperatives. In [35], the interaction among two populations evolving under the replicator dynamics is analyzed, i.e., evolutionary dynamics under co-evolving environment. In [36], a game with interactions between two communities, and two communities with the environment, is studied in the context of evolutionary dynamics. In [37], one of the fundamental evolutionary dynamics, i.e., replicator dynamics, are studied in a repeated setup. Indeed, the relationship and mixture of imitation dynamics has been studied in [38] in order to generate dynamics able to consider multiple individual constraints. Beyond the works mentioned above, games theory applied to multi-robot systems focused mainly on path planning in adversarial situations [39] and on the optimization of task allocation for cooperative missions, where a finite number of possible roles are assigned to members of a robot soccer team [40], or where agents need to coordinate actions such as landing, charging batteries, or picking up objects [41].

Different from the work presented in [1], this work presents outdoors-experimental results over real UAVs (see Section 5). Moreover, having contextualized the formation problem, and role of game theory and evolutionary games in engineering applications, we point out the specific contributions of our manuscript. The contributions of this paper are summarized as follows:

- We present a formation controller for UAVs with the following features: (i) Data-driven performance, (ii) Distributed communication scheme, and (iii) Time-varying communication scheme (see Section 3)

- We deduce the mean dynamics, which are mainly used in the generation of evolutionary dynamics, from the well known Kolmogorov-Forward equation, i.e., we present the equivalence and relationship between the master equation and the mean dynamics (see Section 3.1).

- This work builds on the results presented in [42]. We show that the recent results presented in [42] also hold for time-varying graphs, enlarging the number of applications in engineering including the formation control with information/communication constraints (see Corollary 1).

- Different from the work presented in [1], we present two different ways to design the control law incentives, i.e., with leader-dependent and leader-independent population mass (see Section 3.4).

- The proposed formation controller is designed by considering the two main game theoretical approaches. On one hand, the agents (UAVs) should make decisions to achieve the formation. This is an atomic problem since there is a finite number of players. On the other hand, we use evolutionary games where the decision-makers compose multiple populations. This is a non-atomic approach, (see Remarks 1 and 2).)

The article is organized as follows. Section 2 states the control problem corresponding to a formation of robots as a multi-agent system. Section 3 introduces the distributed formation control based on population-games theory under time-varying communication graphs. Section 4 develops a discussion about the obtained results. In Section 5 experimental results are both shown and discussed. Finally, the concluding remarks and future work are drawn.

2 Problem statement over information graphs

First, we present the preliminaries and introduce the formation problem statement by considering multiple communication constraints.

2.1 General preliminaries

The formation control strategy presented in this article is of data-driven nature. The distributed formation control based on evolutionary games does not require knowledge about the dynamical behavior of the UAVs but only information about their current position in order to compute the appropriate references that are assigned to model-based local controllers. The strategy relies on existing local controllers to be able to achieve the desired motions. The general scheme of the proposed approach is presented in Figure 1. Consequently, a simple model for the UAVs is considered while the implementation of such controllers at the robot level is beyond the reach of this work.

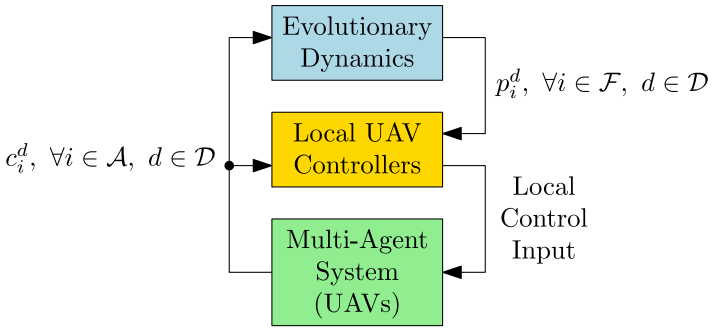


Fig. 1. General scheme for the data-driven distributed formation controller based on evolutionary dynamics.

The vehicles are considered to be stabilized multicopters. This assumption implies that each UAV carries internal equipment such as an autopilot that performs the necessary actions to keep the robot in the air in a hovering state in the absence of control commands, and that the commands that it accepts can produce motions in the four independent pose degrees of freedom, namely x, y, z, and yaw ϕ. Therefore, the position and orientation (pose) of the i^{th} unmmaned vehicle can be described as $\mathbf{c}_i = [c_i^x \quad c_i^y \quad c_i^z \quad c_i^\phi]^\top$. When commands are received, the internal autopilot implements a control law that guarantees tracking of the desired setpoints. The implementation of the models of these aerial unmanned vehicles for numerical simulation purposes is developed in the MATLAB/Simulink environment to present the results of Section 3.5. Experimental results of Section 5 are produced with commercial quadrotors equipped with Pixhawk autopilots. As the distributed control approach presented in this article is agnostic of the type of robot used to implement the results, in the rest of the article the aerial robots are simply going to be considered as agents being part of a multi-agent system.

2.2 Problem description

Let $\mathcal{A} = \{1, \ldots, n\}$ denote the set of $n \geq 2$ agents. One of them is defined as a leader $\ell \in \mathcal{A}$. The leader defines the behavior of the remaining $n-1$ followers, which belong to the set $\mathcal{F} = \mathcal{A} \backslash \{\ell\}$. The position of agent $i \in \mathcal{A}$ is assumed in the positive hyperoctant space, which is a technical but not restrictive assumption. The position coordinates are measurable and denoted by c_i^x, c_i^y, and $c_i^z \in \mathbb{R}_{\geq 0}$; and the orientation is described by the yaw angle $c_i^\phi \in \mathbb{R}_{\geq 0}$. All these measurements are grouped in $\mathbf{c}_i = [c_i^x \quad c_i^y \quad c_i^z \quad c_i^\phi]^\top$, for all $i \in \mathcal{A}$. The set of all agent pose parameters is denoted as $\mathcal{D} = \{x, y, z, \phi\}$.

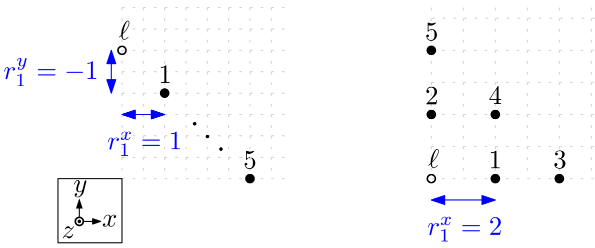
We consider that each agent has some restricted communication capacities determined by a communication range $\psi_i \in \mathbb{R}_{>0}$, for all agents $i \in \mathcal{A}$. The result of this is a communication network that changes over time as a function of the way the agents are spacially distributed as they move, i.e., $\mathcal{G}(t) = (\mathcal{A}, \mathcal{E}(t), \mathbf{A}(t))$, where $\mathcal{E}(t)$ is the set of edges representing the communication topology among nodes representing the agents, and $\mathbf{A}(t) = [a_{ij}(t)]$ is the associated adjacency matrix. The elements of the adjacency matrix are $a_{ij}(t) = 1$ if the i^{th} agent can establish a communication link with the j^{th} agent, and $a_{ij}(t) = 0$, if it does not. It is assumed that each communication channel is bidirectional, then the adjacency matrix is symmetric and the element $a_{ij}(t)$ depends on the communication range ψ_i, i.e., if $\xi_{ij} = \xi_{ij}(t) = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{1/2}$, then $a_{ij}(t) = 1$ if $\psi_i(t) \geq \xi_{ij}(t)$ and $a_{ij}(t) = 0$ otherwise. Finally, let $\mathcal{N}_i(t) = \{j : (i, j) \in \mathcal{E}(t)\}$ be the set of neighbor agents of $i \in \mathcal{A}$.

Assumption 1 To accomplish the leader-tracking task, the graph $\mathcal{G}$ is assumed to be connected. ■

The leader $\ell \in \mathcal{A}$ tracks a predefined trajectory, and the agent $i \in \mathcal{F}$ that acts as a follower tracks the leader while satisfying a desired formation. Thus, the leader trajectory could be easily replaced by a moving target depending on the application.

The formation objective is determined by specifying a distance in euclidean space between each of the followers in the group and the designated leader, in each dimension x, y, and z, as well as a relative angle for the yaw ϕ. The specified reference distances are given by r_i^d, for all $i \in \mathcal{F}$, and $d \in \mathcal{D}$. Additionally, each agent has knowledge of such reference distance given by $\mathbf{r}_i = [r_i^x \quad r_i^y \quad r_i^z \quad r_i^\phi]^\top$, for all $i \in \mathcal{F}$.

As an example, the formation specified in Figure 2(a) results in a diagonal segment defined as $\mathbf{r}_i = [i \quad -i \quad 0 \quad 0]^\top$,



(a) (b)

Fig. 2. Alternative formation specifications in the xy plane for a given constant ϕ_i and z_i, for all $i \in \mathcal{F}$, and $\ell \in \mathcal{A}$. (a) agents in a diagonal segment configuration, and (b) agents in a triangular configuration.

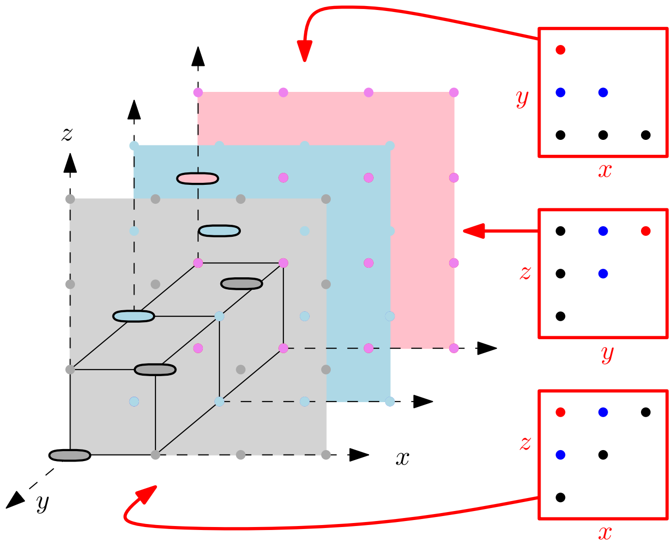


Fig. 3. Formation example involving the three directions x, y, and z.

for all $i \in \mathcal{F}$,. In Figure 2(b), a triangular definition is shown, where the formation specification is formulated as

$$\mathbf{r}_1 = [2 \quad 0 \quad 0 \quad 0]^\top, \tag{1a}$$
$$\mathbf{r}_2 = [0 \quad 2 \quad 0 \quad 0]^\top, \tag{1b}$$
$$\mathbf{r}_3 = [4 \quad 0 \quad 0 \quad 0]^\top, \tag{1c}$$
$$\mathbf{r}_4 = [2 \quad 2 \quad 0 \quad 0]^\top, \tag{1d}$$
$$\mathbf{r}_5 = [0 \quad 4 \quad 0 \quad 0]^\top. \tag{1e}$$

Figure 3 shows an example using three axis from $\mathcal{D}$, i.e., x, y, and z. The specification of the references is an important part of the definition of distributed evolutionary dynamics as discussed in Section 3.

Remark 1 (Atomicity/Non-negligible players) *Under the agents perspective, i.e., considering the set $\mathcal{A} = \{1, \ldots, n\}$, there is a finite number of interacting entities. Then, the proposed settings corresponds to an atomic game where the identity of the players matters.* ∎

Remark 1 shows that the agents are finite and that the proposed formation problem is of atomic class. Nevertheless, we propose to solve this finite-agent problem by means of a non-atomic game-theoretical strategy presented next.

3 Evolutionary Dynamics

This section introduces the evolutionary dynamics and their proposed design in order to solve the formation control introduced in Section 2.

3.1 Mean Dynamics

Consider a continuous-time Markov process involving n different states, i.e., $\mathcal{S}$. The Kolmogorov-Forward equation describes the evolution of the probability $[p_\ell]_{\ell\in\mathcal{S}}$ of a system to stay throughout each discrete set of n states along the continuous time, i.e.,

$$\dot{p}_k = \sum_{i\in\mathcal{S}} w_{ki} p_i, \tag{2}$$

where w_{ki} denotes the transition probability from state i to state k, and the sum of probabilities is $\sum_{i\in\mathcal{S}} p_i = 1$. Then, notice that, in order to guarantee the preservation of the sum probability under the Kolmogorov-Forward equation, it is necessary that

$$\sum_{k\in\mathcal{S}\setminus\{i\}} w_{ki} = -w_{ii}, \tag{3}$$

it follows that (2) is re-written as

$$\dot{p}_k = \sum_{i\in\mathcal{S}\setminus\{k\}} w_{ki} p_i - \sum_{i\in\mathcal{S}\setminus\{k\}} w_{ik} p_k, \tag{4}$$

which corresponds to the Master Equation that is directly associated to the Mean Dynamics [43], and from which the fundamental evolutionary dynamics are deduced [42].

3.2 Preliminaries

Consider a multi-population case composed of decision-makers involving four populations represented by the set $\mathcal{D} = \{x, y, z, \phi\}$. At each population, decision makers can choose from n diverse strategies, i.e., let $\mathcal{A} = \{1, \dots, n\}$ be the set of the available strategies for all $d \in \mathcal{D}$. As described in Section 2, let $\mathcal{F} = \mathcal{A}\backslash\{\ell\}$, being $\ell \in \mathcal{A}$ a fictitious strategy as in [44] associated to the leader. Strategy ℓ is said to be fictitious since the leader has a predefined trajectory.

Let $p_i^d \in \mathbb{R}_{\geq 0}$ denote a subset of decision makers that select a strategy $i \in \mathcal{F}$ in the population $d \in \mathcal{D}$. In addition, the value p_i^d represents the specified reference pose of the i^{th} agent in the coordinate $d \in \mathcal{D}$. Thus, the total desired pose of the $i \in \mathcal{F}$ agent including all coordinates is $\mathbf{p}_i = [p_i^x \quad p_i^y \quad p_i^x \quad p_i^\phi]^\top \in \mathbb{R}_{\geq 0}^4$. Additionally, it should be noted that there is a fictitious pose related to the leader denoted by $p_\ell^d \in \mathbb{R}_{\geq 0}$, where $\ell \in \mathcal{A}, \ell \notin \mathcal{F}$.

Hence, vector $\mathbf{p}^d = [p_1^d \quad \dots \quad p_n^d]^\top \in \mathbb{R}_{\geq 0}^n$ provides information about the strategic distribution of all the decision makers in the population $d \in \mathcal{D}$. Denoting $m^d \in \mathbb{R}_{>0}$ as the size of the respective population, the set of all the possible strategic distributions corresponding to the population $d \in \mathcal{D}$ is

$$\Delta^d = \left\{ \mathbf{p}^d \in \mathbb{R}_{\geq 0}^n : \sum_{i\in\mathcal{A}} p_i^d = m^d \right\}. \tag{5}$$

Note that this mass m^d determines a space constraint in the respective coordinate, i.e., x, y, z or rotation ϕ. Therefore, the relative interior of Δ^d is defined as follows: $\text{int}\Delta^d = \left\{ \mathbf{p}^d \in \mathbb{R}_{>0}^n : \sum_{i\in\mathcal{A}} p_i^d = m^d \right\}$. The aforementioned simplex set in (5) implies that the decision makers are constrained as follows: $\sum_{i\in\mathcal{F}} p_i^d \leq m^d$, or equivalently, that the UAVs positions satisfy a constraint $\left[c_i^x \ c_i^y \ c_i^z \ c_i^\phi \right]^\top \leq \left[m_i^x \ m_i^y \ m_i^z \ m_i^\phi \right]^\top$, for all $i \in \mathcal{F}$.

Decision makers select among different strategies by pursuing an enhancement over their benefits. Such benefits have a mapping $f_i^d : \Delta^d \mapsto \mathbb{R}$, for all $i \in \mathcal{A}$. Thus, the vector $\mathbf{f}^d$ collects all the fitness functions, whose mapping is $\mathbf{f}^d : \Delta^d \mapsto \mathbb{R}^n$.

Remark 2 (Non-atomicity/Negligible players) *Under the populations perspective, there is a large number of decision-makers. Notice that a single decision-maker has no impact over the population distribution neither over the fitness functions since an individual decision-maker is negligible. Thus, the proposed approach belongs to a non-atomic game class.* ■

Remark 2 shows that the population dynamics belongs to a non-atomic class. Nevertheless, notice that the atomic settings presented in Section 2 are connected to the non-atomic evolutionary games by means of the direct relationship between the set of agents and the set of strategies in the population.

The objective within the multi-population is to reach a Nash equilibrium at each population. The following definitions have been adapted from [43] and will be used in the next sections.

Definition 1 (Nash Equilibria) The set of Nash equilibria is given by $\{\mathbf{p}^{d*} \in \Delta^d : p_i^{d*} > 0 \Rightarrow f_i^d(\mathbf{p}^{d*}) \geq f_j^d(\mathbf{p}^{d*})\}$, for all $i, j \in \mathcal{A}$. ■

Definition 2 (Monotone Fitness) The game $\mathbf{f}^d$ in the population $d \in \mathcal{D}$ is monotone decreasing if

$$(\mathbf{p}^d - \mathbf{q}^d)^\top \left(\mathbf{f}^d(\mathbf{p}^d) - \mathbf{f}^d(\mathbf{q}^d)\right) \leq 0, \tag{6}$$

for all $\mathbf{p}^d, \mathbf{q}^d \in \Delta^d$. ■

Definition 3 (Potential Games) The population game $\mathbf{f}^d$ is a full-potential if there exists a function $V(\mathbf{p}^d)$ such that $\mathbf{f}^d(\mathbf{p}^d) = \nabla V(\mathbf{p}^d)$, for all $\mathbf{p}^d \in \Delta^d$. ■

3.3 Distributed replicator dynamics

There are different distributed population dynamics, or population dynamics considering migration constraints [42]. These dynamics are computed from the Mean Dynamics or Master Equation as presented in Section 3.1. In particular, the distributed replicator dynamics are

$$\dot{p}_i^d = p_i^d \left(f_i^d(\mathbf{p}^d) \sum_{j \in \mathcal{N}_i} p_j^d - \sum_{j \in \mathcal{N}_i} p_j^d f_j^d(\mathbf{p}^d) \right), \tag{7}$$

for all $i \in \mathcal{A}$, and $d \in \mathcal{D}$. Alternatively, (7) is

$$\dot{\mathbf{p}}^d = \mathrm{diag}\left(\mathbf{p}^d\right)\left[\mathrm{diag}\left(\mathbf{f}^d(\mathbf{p}^d)\right)\mathbf{A}\mathbf{p}^d - \mathbf{A}\,\mathrm{diag}\left(\mathbf{f}^d(\mathbf{p}^d)\right)\mathbf{p}^d\right], \tag{8}$$

for all $d \in \mathcal{D}$, where $\mathrm{diag}\left(\mathbf{p}^d\right) \in \mathbb{R}_{\geq 0}^{n \times n}$.

The distributed replicator dynamics can be expressed in terms of Laplacian matrix corresponding to a weighted graph for the interactions (see [42, Proof of Thm 3]) depending on the population states, i.e.,

$$\dot{\mathbf{p}}^d = \mathbf{L}^{(\mathbf{p})}\,\mathbf{f}^d(\mathbf{p}^d), \;\; \forall d \in \mathcal{D}, \tag{9}$$

where $\mathbf{L}^{(\mathbf{p})}$ is symmetric positive semidefinite.

The rest points denoted by $\mathbf{p}^{d*} \in \Delta^d$ of (7) implies that either $p_i = 0$ for some $i \in \mathcal{A}$, or $f_i^d(\mathbf{p}^{d*}) = f_j^d(\mathbf{p}^{d*})$, for all $i, j \in \mathcal{A}, d \in \mathcal{D}$. Consequently, if $\mathbf{p}^{d*} \in \mathrm{int}\Delta^d$ then the equilibrium point means that $f_i^d(\mathbf{p}^{d*}) = f_j^d(\mathbf{p}^{d*})$, for all $i, j \in \mathcal{A}, d \in \mathcal{D}$. Theorem 1 below presents the stability analysis of the aforementioned equilibrium point under the distributed replicator dynamics.

Theorem 1 (Stability) (from [42, Thm 3]) Let $\mathbf{f}^d = \nabla V(\mathbf{p}^d)$, where $V(\mathbf{p}^d)$ is strictly concave, and let $\mathbf{p}^{d*} \in \mathrm{int}\Delta^d$ be a Nash equilibrium. If the graph $\mathcal{G} = (\mathcal{A}, \mathcal{E}, \mathbf{A})$ is connected, then under the distributed replicator dynamics (7), $\mathbf{p}^{d*}$ is asymptotically stable. ■

Corollary 1 below highlights the fact that Theorem 1 is also valid for time-varying graphs, which became a powerful feature for a wide variety of engineering applications.

Corollary 1 (Time-varying Graphs) The asymptotic stability of $\mathbf{p}^{d^*} \in \mathrm{int}\Delta^d$ stated in Theorem 1 is also valid for connected time-varying graphs $\mathcal{G}(t) = (\mathcal{A}, \mathcal{E}(t), \mathbf{A}(t))$. ■

Moreover, the simplex Δ^d is an invariant under the distributed replicator dynamics. This fact can be derived from the property $\mathbf{1}_n^\top \mathbf{L}^{(\mathbf{p})} = 0$, and by using the Laplacian representation of the dynamics, i.e., $\dot{\mathbf{p}}^d = \mathbf{L}^{(\mathbf{p})} \mathbf{f}^d(\mathbf{p}^d)$, for all $d \in \mathcal{D}$. Then, $\mathbf{1}_n^\top \dot{\mathbf{p}}^d = 0$, for all $d \in \mathcal{D}$.

3.4 Fitness functions design

Note that the distributed formation task is equivalent to maintaining some desired constant distances among the agents as previously described in Section 2. Therefore, if each agent is tasked with keeping a constant distance with respect to the leader by tracking it, then the specified formation is achieved. From the distributed replicator dynamics, and based on the asymptotic convergence to a Nash equilibrium $\mathbf{p}^{d^*} \in \text{int}\Delta^d$, an adequate fitness function can be defined. We discuss two different ways in which the fitness functions can be designed, i.e., with leader-dependent and leader-independent population mass.

3.4.1 Leader-Independent Population Mass

Let $f_\ell^d = -c_\ell^d$ be the fitness function associated with the subset of decision makers selecting the fictitious strategy corresponding to the group leader $\ell \in \mathcal{A}, \ell \notin \mathcal{F}$. Furthermore, let $f_i^d(p_i^d) = r_i^d - p_i^d$, for all $i \in \mathcal{F}$. First, it should be noted that these fitness functions collected in $\mathbf{f}^d(\mathbf{p}^d)$ satisfy the condition described in Definition 2) regarding a stable game. The non-positive diagonal matrix $D\mathbf{f}^d(\mathbf{p}^d)$ has only one null element over the diagonal corresponding to the leader agent $\ell \in \mathcal{A}$.

Assuming that the population of the portion of decision makers does not become extinct, the fact that $\mathbf{p}^{d^*} \in \text{int}\Delta^d$ implies that $f_i^d(\mathbf{p}^{d*}) = f_j^d(\mathbf{p}^{d*})$, for all $i, j \in \mathcal{A}$, and $d \in \mathcal{D}$. Then, $f_\ell^d(\mathbf{p}^{d^*}) = f_i^d(\mathbf{p}^{d^*})$, and $-c_\ell^d = r_i^d - p_i^{d^*}$, it follows that:

$$p_i^{d^*} = c_\ell^d + r_i^d, \tag{10}$$

for all $i \in \mathcal{F}$, and $d \in \mathcal{D}$. The equilibrium point condition in (10) shows that for all $i \in \mathcal{F}$, p_i^d converges to the specified formation references for all pose coordinates (position and orientation) $d \in \mathcal{D}$ (see Figure 2). Furthermore, the portion of decision makers p_ℓ^d corresponding to all $d \in \mathcal{D}$, as well as for the fictitious strategy, converges to values that satisfy the simplex Δ^d.

3.4.2 Leader-Dependent Population Mass

Let us considered a scenario where the mass is determined by the position of the leader, i.e., m^x, m^y, and m^z depends on the position of the leader c_ℓ^x, c_ℓ^y, and c_ℓ^z as follows:

$$m^d = |\mathcal{F}| c_\ell^d, \ \forall d \in \mathcal{D}. \tag{11}$$

Therefore, notice that the population mass changes as the leader agent moves along the space. Thus, a new leader-position-dependent simplex $\Delta^d(c_\ell^d)$ is introduced, i.e.,

$$\Delta^d(c_\ell^d) = \left\{ \mathbf{p}^d \in \mathbb{R}_{\geq 0}^{n-1} : \sum_{i \in \mathcal{F}} p_i^d = |\mathcal{F}| c_\ell^d \right\}. \tag{12}$$

This approach is suitable for symmetric formation and it consists on assigning the following fitness functions for the follower agents $\mathcal{F}$:

$$f_i^d = r_i^d - p_i^d, \ \forall i \in \mathcal{F}, \ d \in \mathcal{D}. \tag{13}$$

For simplicity, let us consider a three-agents case, i.e., two followers $\mathcal{F} = \{1, 2\}$ and the leader, with $-r_1^d = r_2^d = r^d$. In the interior equilibrium point we have

$$f_1^d(p^{d\star}) = f_2^d(p^{d\star}), \tag{14a}$$

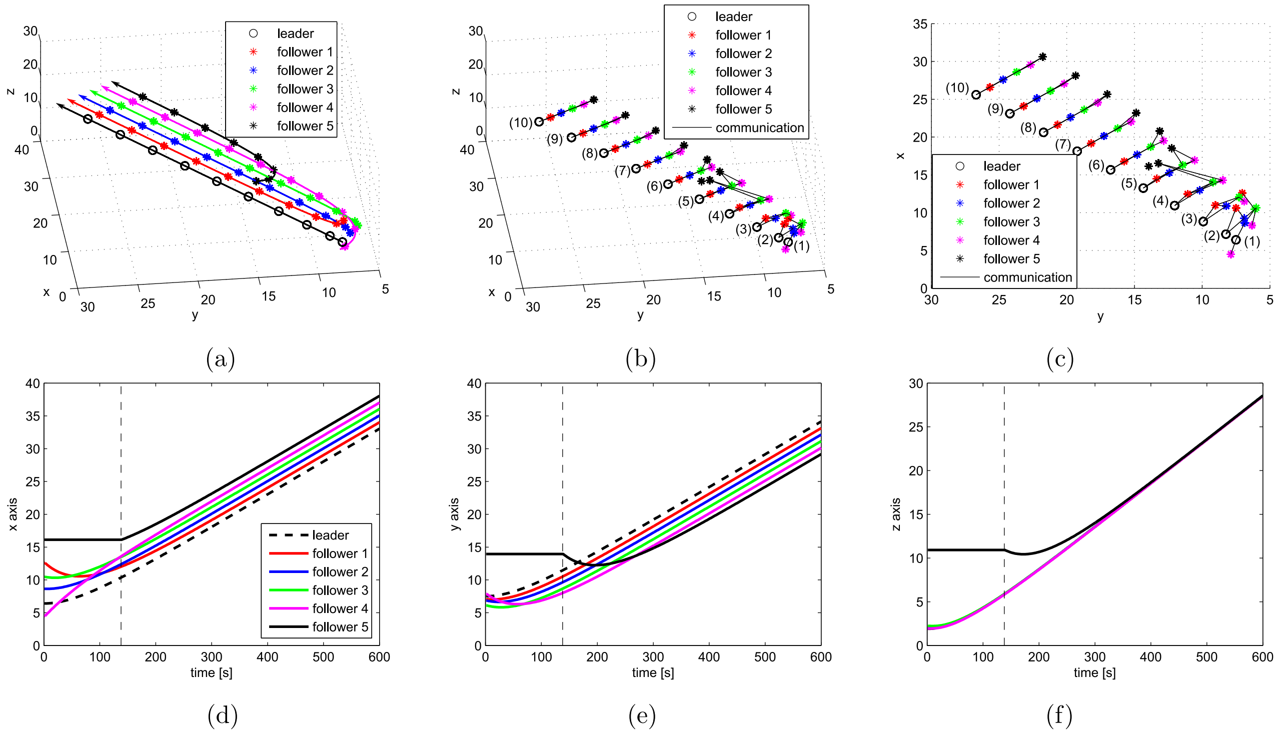


Fig. 4. Simulation results corresponding to Scenario 1 for a linear formation using population dynamics, corresponding to Figure 2(a)), with a maximum communication range of $\psi_i = 3$ m, for agents $i \in \mathcal{A}$. A new agent is incorporated into the formation when it establishes communication with other agents.

$$r_1^d - p_1^{d\star} = r_2^d - p_2^{d\star}, \tag{14b}$$

given that $p_1^{d\star} + p_2^{d\star} = |\mathcal{F}|c_\ell^d$ yields

$$r_1^d - 2c_\ell^d + p_2^{d\star} = r_2^d - p_2^{d\star}, \tag{15a}$$

$$p_2^{d\star} = \frac{r_2^d - r_1^d + 2c_\ell^d}{2}, \tag{15b}$$

$$p_2^{d\star} = r + c_\ell^d, \tag{15c}$$

and

$$p_1^{d\star} = 2c_\ell^d - p_2^{d\star}, \tag{16a}$$

$$= c_\ell^d - r, \tag{16b}$$

achieving the desired formation, i.e., one follower located at $c_\ell^d - r$ and another placed at $c_\ell^d + r$ keeping a distance given by r.

3.5 Population games-based simulation results

Two alternative scenarios are considered for $n = 6$ agents to exemplify the proposed population games-based formation control approach. The two scenarios are defined as follows:

(1) **Scenario 1:** Linear formation as represented in Figure 2(a). The leader is chosen as agent $1 \in \mathcal{F}$. Follower agent $5 \in \mathcal{F}$ initial position lays outside the communication range of the rest of the agents. The trajectory of the leader is specified so that at some point the isolated agent gets within the communication range, and therefore is automatically taken into the formation.

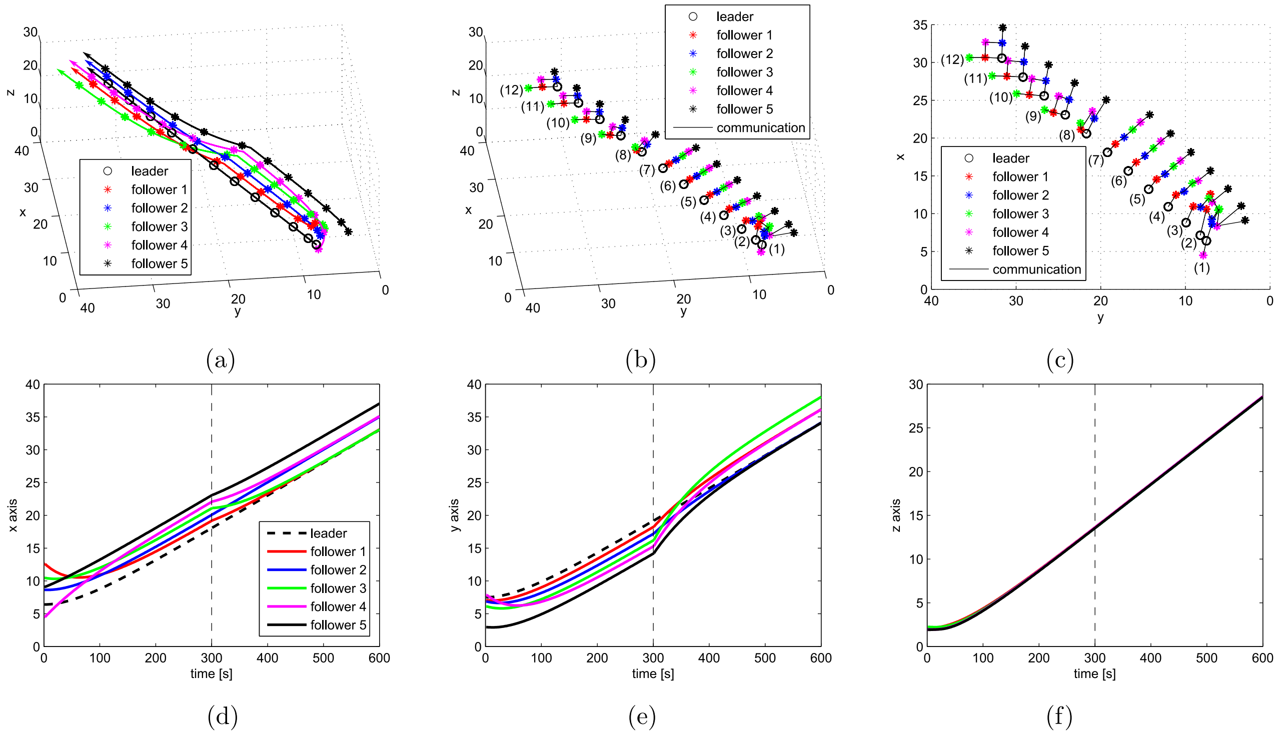


Fig. 5. Simulation results corresponding to Scenario 2 for a time-varying topology using population dynamics. An initial linear formation (Figure 2(a)) is transitioned to a triangular formation (Figure 2(b)) at $t = 300$ s, the maximum communication range is $\psi_i = 2.5$ m, for all $i \in \mathcal{A}$.

(2) **Scenario 2:** Linear formation as represented in Figure 2(a) that evolves to a triangular configuration as the one represented in Figure 2(b). This scenario intends to illustrate the behavior of the distributed evolutionary-dynamics approach for a case when the formation specification shows time-varying characteristics.

Simulation results of the first scenario are shown in Figure 4. In particular Figure 4(a) shows the trajectories of the agents over time. Figures 4(b)-4(c) show the communication links between agents at different times to highlight how the communication network evolves. It should be noted that initially, the follower agent $5 \in \mathcal{F}$ does not have a communication link with other agents. As a result the agent does not move. At time $t = 138$ s, the agent establishes communication with agents three and four, as they come within the communication range threshold. At this point, agent five is incorporated into the formation. Furthermore, Figures 4(d)-4(f) illustrates the position of the agents c_i^d, for all $i \in \mathcal{A}$ and $d \in \mathcal{D}\backslash\{\phi\}$ over time. The followers maintain the desired spacial configuration with respect to the leader in the coordinates x, and y, and they track the leader position in the z coordinate as they move.

Simulation results of Scenario 2 are shown in Figure 5(a) where the agents follow a time-varying configuration. Initially, the agents converge to a linear formation. Then at time $t = 300$ s, the formation switches to a triangular configuration. Figures 5(b)-5(c) show how the communication link between agents evolve over time. Furthermore, Figures 5(d)-5(f) depicts the position of the agents c_i^d for all $i \in \mathcal{A}$ and $d \in \mathcal{D}\backslash\{\phi\}$ as a function of time.

4 Discussion

It has been shown that the distributed population games-based control achieves the formation objective by using partial information about the position of other agents. First, notice that the population-games approach generates as output the desired positions (or trajectories) for each agent. This feature calls for the need of a local controller for each agent that guarantees the achievement of the established desired positions. Such controllers need to account for any dynamics the agents may have.

On the other hand, the relative positioning of the agents is achieved differently for the two mentioned strategies. Notice that, with the population-dynamics approach, the task for an agent to keep distance from another agent in

Fig. 6. Testbed: unmanned aerial vehicle used in the experimental tests.

a positive or negative offset for any axis, i.e., the formation is determined with respect to the global coordinates in space. Furthermore, it is possible to modify the formation shape dynamically for both cases, and with the population-dynamics approach, it is also possible to integrate or remove agents dynamically to/from the formation task seamlessly. Finally, it should be noted that the distributed replicator dynamics have been used as the population-dynamics approach. However, any distributed population dynamics may be implemented for the formation control, i.e., distributed Smith dynamics, distributed projection dynamics, among others [42].

5 Experimental Results

To experimentally validate the proposed formation control, test cases of static formations moving in a 3D space and formation tasks changing over time. To this end, three UAVs are considered as follows: one virtual agent, and two real UAVs as followers. Notice that, in order to evaluate the distributed characteristics of the proposed approach, it is quite important to incorporate more than two agents in the strategic interaction. Otherwise, the unique possible communication graph is given by a complete graph. Next, we describe the equipment of the follower agents (UAVs).

The testbed consists of two multi-rotor vehicles based on a DJI F450 frame with 920KV motors, 9545 propellers, and 4s LiPo batteries (see Figure 6). A Pixhawk autopilot (with PX4 firmware version 1.6.5) is used as flight computer. In order to validate the algorithm in an outdoor environment, a standard L1 GPS and a 3-axis magnetometer are connected to the flight computer, where an Extended Kalman Filter merges GPS data with inertial measurements and a barometric altimeter to provide navigation information (position, velocity and attitude). The navigation data is transmitted to a ground station computer through a UHF transceiver, and velocity (position) control commands are sent back to the vehicles. The communication is based on the MAVLink (Micro Air Vehicle Communication) protocol. The ground station is a PC running Linux with ROS (Robot Operating System) installed. MAVROS package is used to handle communications with the vehicles, since it easily allows to interpret the MAVLink protocol. Once the ground station receives the navigation parameters of each vehicle, the control algorithm computes the input commands for the vehicles.

Two experimental test cases are presented as explained next. Figure 7 presents the testbed and the performance of the real implementation at ITBA. It can be seen that only two physical/real UAVs have been used for the experimentation with a third virtual UAV determining the desired trajectories for the formation. In the first experimental test case, the virtual leader follows an elliptical trajectory with semi-axes of 6 m in the x dimension and 10 m in the y dimension while staying at a constant altitude of 7 m. The two follower UAVs maintain their position 2 m away from the leader in the y dimension. During the experiment, the followers keep a constant communication link with the virtual leader. Figure 8 shows the results for the population-games formation.

In the second experimental test case, the virtual leader varies its altitude from 8 m to 18 m and the followers rotate around it (effectively changing the formation spatial configuration) at a distance of 4 m, producing an ascending spiral motion. Communication is again maintained throughout the experiment. Figure 9 shows the results for the population games formation for this case.

Again, results show an acceptable performance validating their functionality in an experimental setup as discussed next. In addition, rhere are two facts to take into consideration when observing the experimental results. First, the proposed distributed game-theory-based control-formation strategy is data-driven. Second, there are many disturbances related to both the position measurements computed by the GPS sensors and by the wind pushing randomly the UAVs in the real experimentation. Because of the aforementioned reasons it is expected to observed a noisy

Fig. 7. Testbed: outdoors experiment performed at ITBA.

behavior. Hence, according to the magnitude of the space we are considering in the testbed, the obtained errors correspond to around a 10%. For instance, Figure 8 shows that the average desired position in x-axis is 10m and the followers 1 and 2 exhibit a behavior with a standard error deviation around 1m. Likewise, we can observe a noisy behavior because of the disturbances emerging in a real experimentation for the second example in the Figure 9. Recalling Figure 1, the distributed evolutionary dynamics provide the appropriate axis set-points to the local UAV controller. Therefore, the performance illustrated in Figures 8 and 9 can be potentially improved, specially in the z-axis where big errors due to perturbations are observed, by enhancing the local UAV controllers.

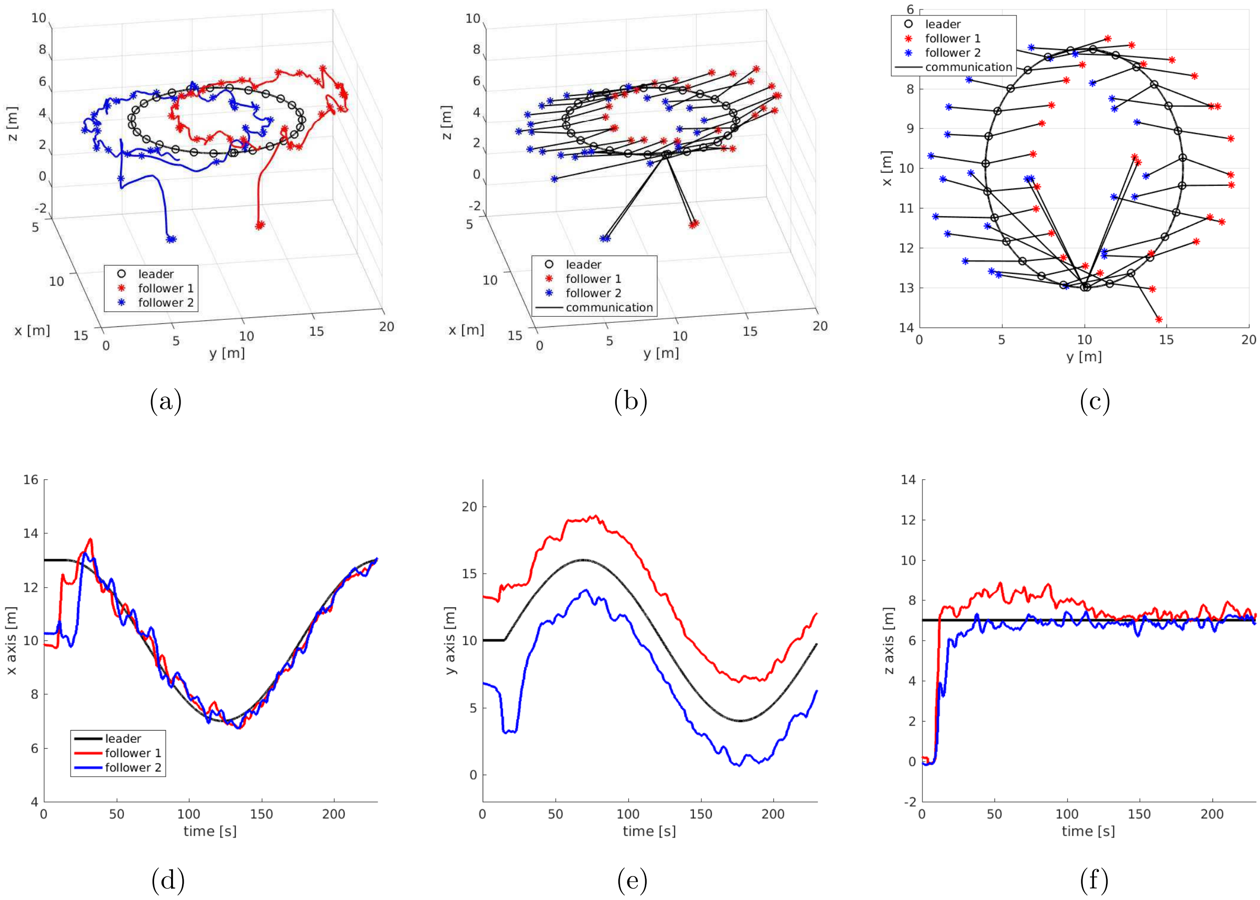


Fig. 8. Experimental result of the population dynamics method following an elliptical trajectory.

6 Conclusion and Future Directions

This paper presented a distributed formation control based on population games and evolutionary dynamics, which has been applied to unmanned aerial vehicles. It has been shown that this method is stable and has several advantages with respect to previous reported methods. First, it is of data-driven nature, i.e., it does not rely on a physical model of the system. The proposed approach requires the use of local controllers to guarantee that the UAVs achieve the imposed position computed by the population-games controller. Therefore, different complex models could be considered without having to modify the population-games-based control design. Secondly, it has been formally shown that, the approach proposed in this article can work with time-varying graphs with stability guarantees. Finally, the proposed controller has some plug-and-play features, i.e., it is possible to integrate and/or remove agents to/from the the specified formation in a gracefully dynamical manner, i.e., without redesigning the closed-loop control scheme. As future investigation, we propose the consideration of obstacles and space constraints. On the other hand, the consideration of uncertainties in the fitness functions measurements or in the model of the unmanned vehicles is also proposed as further work.

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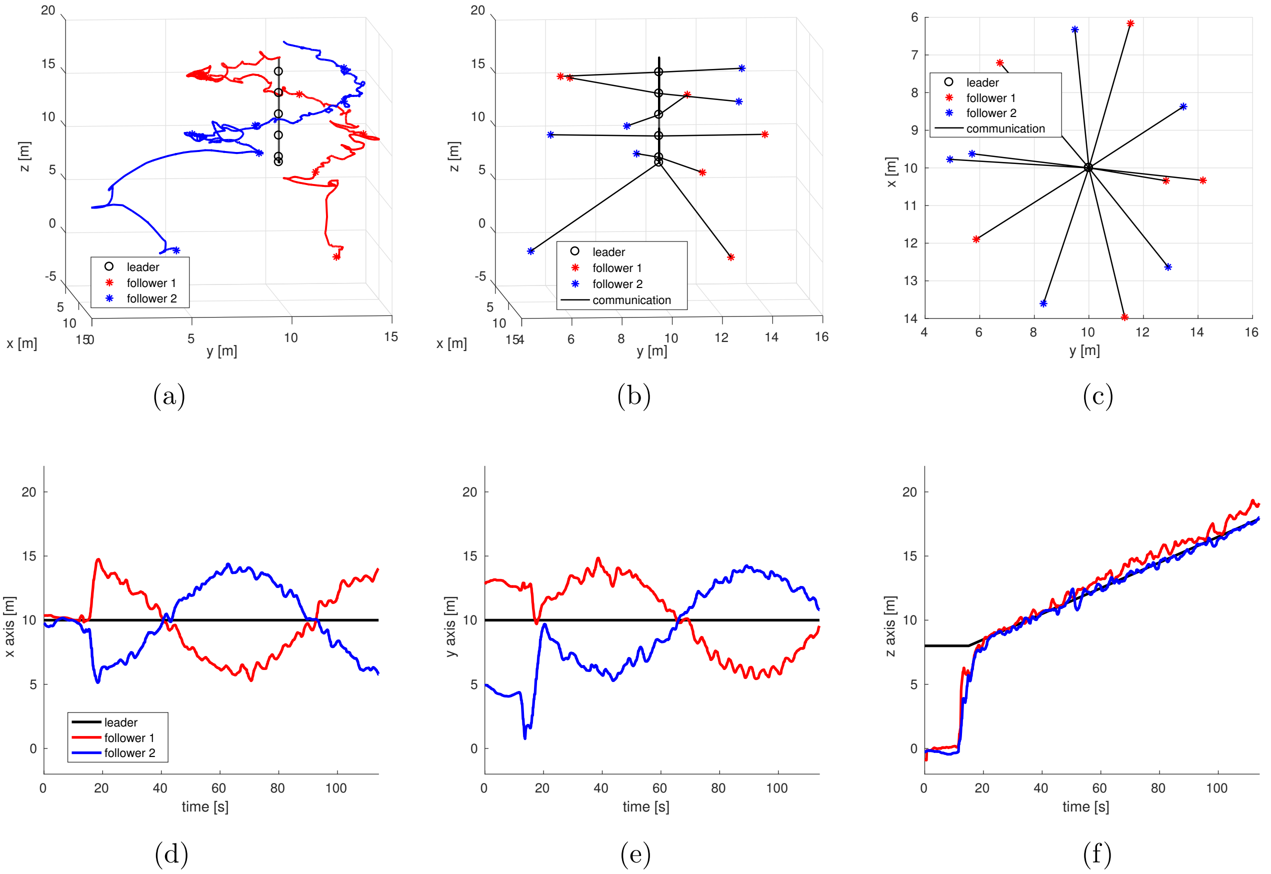

Fig. 9. Experimental result of the population dynamics method performing an ascending spiral motion.

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