When is a complete ideal in a rational surface singularity a multiplier ideal?

Maria Alberich-Carramiñana, Josep Àlvarez Montaner, and Víctor González-Alonso

Departament de Matemàtiques, Universitat Politècnica de Catalunya, Av. Diagonal 647, Barcelona 08028, Spain
maria.alberich@upc.edu, josep.alvarez@upc.edu

Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
gonzalez@math.uni-hannover.de

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Abstract: This is an extended abstract with some of the results that will appear in the forthcoming paper [1] in which we characterize when a given complete ideal in a two-dimensional local ring with a rational singularity can be realized as a multiplier ideal.

1 Introduction

Let X be a complex variety of dimension d which is $\mathbb{Q}$-Gorenstein and $\mathcal{O}_{X,O}$ its corresponding local ring at a point $O \in X$, with $\mathfrak{m} = \mathfrak{m}_{X,O}$ being the maximal ideal. Given an ideal $\mathfrak{b} \subseteq \mathcal{O}_{X,O}$ and a parameter $\lambda \in \mathbb{R}$ we may consider its corresponding *multiplier ideal* $\mathcal{J}(\mathfrak{b}^{\lambda}) \subseteq \mathcal{O}_{X,O}$. It follows from its construction that multiplier ideals are complete so it is natural to wonder how special are multiplier ideals among all complete ideals.

When X is smooth and $d = 2$, it was proved independently by Favre and Jonsson [2] and Lipman and Watanabe [7], that every complete ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ can be realized as a multiplier ideal; that is, we may find an ideal $\mathfrak{b}$ and a parameter λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda})$. This result is no longer true for $d \geq 3$ as it has been proved by Lazarsfeld and Lee in [4]. Indeed, they show some delicate properties regarding the vanishing of the syzygies of multiplier ideals which lead to the existence of complete ideals in higher dimension that cannot be realized as multiplier ideals.

Lazarsfeld, Lee and Smith [5] partially extended the results in [4] to the non-smooth case by giving some vanishing result on the first syzygy of multiplier ideals. This condition is still enough to cook up examples of complete ideals that cannot be realized as multiplier ideals when $d \geq 3$. They also quoted in [5, Question 3.12] the following question regarding the remaining case that is left open: *Is every complete ideal in a complex algebraic surface having a rational singularity a multiplier ideal?* A partial answer to this question was provided by Tucker in [8] by showing that this is indeed the case when X has a log-terminal singularity. In a forthcoming paper [1] we will give a characterization of complete ideals that can be realized as multiplier ideals by means of a new invariant that we introduce, the *limiting boundary* Δ_D^*, and we give examples where a complete ideal cannot be realized.

2 A reformulation of the problem via antinef closures

Let (Y, O) be a germ of complex surface with at worst a rational singularity. Let $\mathcal{O}_{Y,O}$ denote the local ring at O and let $\mathfrak{m} = \mathfrak{m}_{Y,O} \subseteq \mathcal{O}_{Y,O}$ be the maximal ideal. Let $\pi : X \to Y$ be a log-resolution of a $\mathfrak{m}$-primary complete ideal $\mathfrak{a} \subseteq \mathcal{O}_{Y,O}$. We say that $\mathfrak{a}$ is *realized as a multiplier*

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ideal in X if there exists another $\mathfrak{m}$-primary ideal $\mathfrak{b}$ such that π is also a log-resolution for $\mathfrak{b}$ and there is a rational number λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda})$. More precisely, let F and G be integral exceptional divisors such that $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-F)$ and $\mathfrak{b} \cdot \mathcal{O}_X = \mathcal{O}_X(-G)$. Let K_π be the relative canonical divisor which is a $\mathbb{Q}$-divisor with exceptional support. Then we want to find λ such that $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^{\lambda}) := \pi_* \mathcal{O}_X(\lceil K_\pi - \lambda G \rceil)$, where $\lceil \cdot \rceil$ denotes the round up of any $\mathbb{Q}$-divisor and is nothing but rounding up its coefficients..

Lipman [6, §18] gave a correspondence between complete ideals and *antinef divisors* that will give us the right framework where we can address this question. Recall that an effective integral exceptional divisor $D \in \mathrm{EDiv}^{\geq 0}(X)$ is antinef if $D \cdot C_i < 0$ for all the irreducible components $C_1, \dots, C_r$ of the exceptional locus. Given any effective rational exceptional divisor $D \in \mathrm{EDiv}_{\mathbb{Q}}^{\geq 0}(X)$ we may either consider its:

· *Integral antinef closure* : $\widetilde{D} := \min \left\{ D' \in \mathrm{EDiv}^{\geq 0}(X) \mid D' \geq D, \quad D' \cdot C_i \leq 0 \quad \forall\, i \right\}$,

· *Rational antinef closure:* $\widetilde{D}^{\mathbb{Q}} = \min \left\{ D' \in \mathrm{EDiv}_{\mathbb{Q}}^{\geq 0}(X) \mid D' \geq D, \quad D' \cdot C_i \leq 0 \quad \forall\, i \right\}$.

The existence of the integral antinef closure can be found in [6, §18] and it can be computed using the *unloading procedure* described next: Set $D_0 = \lceil D \rceil$. For any $k \geq 0$, whenever there is an exceptional component C_i such that $D_k \cdot C_i > 0$, define $D_{k+1} = D_k + C_i$. If there is no such C_i, then $\widetilde{D} = D_k$.

The existence of the $\mathbb{Q}$-antinef closure follows from the cone structure of the set of antinef divisors. To describe it we use the $\mathbb{Q}$-*unloading procedure*, which can be deduced from [3], and is described next: Set $D_0 = D$. For any $k \geq 0$, whenever there is an exceptional component C_i such that $D_k \cdot C_i > 0$, define $D_{k+1} = D_k + \sum x_i C_i$, where the x_i are the solutions of the system of equations $\sum (C_i \cdot C_j)\, x_i = -D \cdot C_j, \quad \forall i, j$. If there are no such C_i, then $\widetilde{D}^{\mathbb{Q}} = D_k$.

The main result of this section is a reformulation of our initial problem in terms of the following boundary $\mathbb{Q}$-divisors that measure the difference between a divisor and its $\mathbb{Q}$-antinef closure. Namely, given any rational exceptional divisor D, we define

$$\Delta_D = \widetilde{(D + K_\pi)}^{\mathbb{Q}} - (D + K_\pi) \geq 0.$$

Now, given a convenient log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$, we want to check whether there exists an antinef divisor G and a rational number λ such that

$$\lfloor \widetilde{\lambda G - K_\pi} \rfloor = F \tag{1}$$

Notice that the rational divisor λG is antinef as well and, denoting $D = \lfloor \lambda G - K_\pi \rfloor$, we have that $D + K_\pi \leq \lambda G$. Therefore, the $\mathbb{Q}$-antinef closure of $D + K_\pi$ satisfies

$$D + K_\pi \leq \widetilde{(D + K_\pi)}^{\mathbb{Q}} \leq \lambda G$$

and thus

$$D = \lfloor D + K_\pi - K_\pi \rfloor \leq \left\lfloor \widetilde{(D + K_\pi)}^{\mathbb{Q}} - K_\pi \right\rfloor \leq \lfloor \lambda G - K_\pi \rfloor = D.$$

Under these premises, Equation 1 becomes

$$\left\lfloor \widetilde{\widetilde{(D + K_\pi)}^{\mathbb{Q}} - K_\pi} \right\rfloor = \lfloor \widetilde{D + \Delta_D} \rfloor = F. \tag{2}$$

Our approach to the problem is through the following

Proposition 1. *An $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ is realized as a multiplier ideal if and only if there is a log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ and an integral exceptional divisor D such that $D \geq \lfloor -K_\pi \rfloor$, $\widetilde{D} = F$, and $\lfloor \Delta_D \rfloor = 0$.*

3 Working in a fixed log-resolution

Let's start with a fixed log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$. It might well happen that we can not find an integral exceptional divisor D satisfying the conditions of Proposition 1. Indeed there are cases in which we may find such a divisor in a further log-resolution and cases where it will be impossible to find it, and thus giving examples of complete ideals that can not be realized as multiplier ideals (see Section 5). Even though working in a fixed log-resolution has a lot of shortcomings, the methods we present in this section will illustrate the main ideas behind our general method.

The starting point of our method comes from the unloading procedure. We can reach every $D \geq \lfloor -K_\pi \rfloor$ with $\widetilde{D} = F$ by starting with $D = F$ and then go backwards replacing D by $D - C$ for any exceptional component with $(D - C) \cdot C > 0$, and contained in the support of $D - \lfloor -K_\pi \rfloor$. If this is the case we say that going from D to $D - C$ is an *admissible subtraction*. Moreover, without getting into technical details, the multiplicities of Δ_{D-C} are smaller than the multiplicities of Δ_D when $(D + K_\pi + \Delta_D) \cdot C < 0$. We will say in this case that we have a *strict subtraction*. If a subtraction is admissible and strict we say that it is a *good subtraction*.

Our goal would be to find a chain of admissible subtractions $F > D_1 > \cdots > D_n = D$ such that $\lfloor \Delta_D \rfloor = 0$. In the case that every subtraction in the chain is also strict, hence good, we will say that $D < F$ is a *good subdivisor* and it is characterized as follows:

Proposition 2. *$D < F$ is a good subdivisor if and only if $\operatorname{mult}_C(\Delta_D) < 1$ for every subtracted component $C \subset \operatorname{supp}(F - D)$.*

It leads to the following characterisation:

Proposition 3. *An $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ is realized as a multiplier ideal if and only if there is a log-resolution $\pi : X \to Y$ of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ and a good subdivisor $D < F$ such that $\lfloor \Delta_D \rfloor = 0$.*

This provides an efficient algorithm to decide whether a complete ideal can be realized as multiplier ideal in X. Obviously if $\lfloor \Delta_F \rfloor = 0$ then $\mathfrak{a}$ is a multiplier ideal. Otherwise, we can take F and consider recursively all the possible strict subtractions, until we either find some D with $\lfloor \Delta_D \rfloor = 0$ or we run out of divisors (in which case $\mathfrak{a}$ cannot be realized as multiplier ideal in X). We point out that we may find examples of surfaces with a log-terminal singularity and ideals that can not be realized in a given log-resolution. We already know, by Tucker's result [8], that they must be realized in a further log-resolution.

4 Comparing log-resolutions

In general, we have to study how the Δ_D behave in different log-resolutions, in order to obtain the best good chains possible. In order to get a minimal Δ_D we would consider only strict subtractions $D - C$ and, in the case that they are not admissible, it would require to blow-up $m = 1 - (D - C) \cdot C \geq 0$ smooth points of C to make them admissible, and thus good. This process can be quite involved but we can speed it up using what we call

Standard procedure with length N: Let $\pi : X \to Y$ be a log-resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$ and consider (X, F) as our starting pair. Given a positive integer N we will produce a sequence $X_n^{(N)} \to \cdots \to X_1^{(N)} \to X \to Y$, hence a sequence of pairs $(X_n^{(N)}, D_n^{(N)})$ as follows:

- If some initial irreducible component $C_i \subset X$ is good-subtractible from $D_n^{(N)}$, then take $X_{n+1}^{(N)} = X_n^{(N)}$ and $D_{n+1}^{(N)}$ as the result of subtracting C_i and all subsequent possible good subtractions of non-initial components.
- If some initial irreducible component $C_i \subset X$ is strict-subtractible but not admissible, set $m_{n,i} = 1 - (D_n^{(N)} - C_i) \cdot C_i$. Then blow up C_i at $m_{n,i}$ smooth points, further blow-up each of the resulting $m_{n,i}$ exceptional components at a smooth point, and

then blow-up each of the newest exceptional components, and so on until we have added $Nm_{n,i}$ exceptional components, forming $m_{n,i}$ *tails* of length N attached to the original exceptional divisor at C_i. Then subtract C_i and all subsequent possible good subtractions of non-initial components (including the newest ones).

- If no initial component is strict-subtratible, stop.

Remark 4. Each pair $(X_n^{(N)}, D_n^{(N)})$ is determined by data on the initial log-resolution $\pi : X \to Y$ if one also remembers how many tails have been created from each initial exceptional component. More precisely, each step can be codified by the pair $(\overline{D_n^{(N)}}, m_n)$, where $\overline{D_n^{(N)}}$ is the image of $D_n^{(N)}$ in X and $m_n = (m_{n,1}, \ldots, m_{n,r}) \in \mathbb{Z}^r_{\geq 0}$ is the vector such that at this step there are $m_{n,i}$ tails attached to the initial components $C_1, \ldots, C_r$.

At each step we may consider the corresponding $\Delta_{D_n^{(N)}}$ and its images $\overline{\Delta_{D_n^{(N)}}} \subset X$ decrease and have a limit Δ_n^* when $N \to \infty$ that can be computed as follows

Proposition 5. *Let $\left(X^{(N)}, D^{(N)}\right)$ be a pair computed using the standard procedure of length N, and for each $i = 1, \ldots, r$ let m_i be the number of tails attached to the initial exceptional component $C_i \subset X$. Then there exists $\Delta_D^* = \lim_{N\to\infty} \overline{\Delta_{D^{(N)}}}$, which can be computed as the smallest solution of the system of inequalities*

$$\left(\overline{D^{(N)}} + K_0 + \Delta_D^*\right) \cdot C_i < -m_i \quad i = 1, \ldots, r.$$

The fact that the limiting boundary Δ_D^* can be computed on the initial log-resolution by taking into account the tail-counting vector m motivates the following definitions.

Definition 6. A divisor $D \subset X$ is an *asymptotically good subdivisor* of F if for big enough $N \in \mathbb{N}$ there is a pair $\left(X^{(N)}, D^{(N)}\right)$ obtained by the standard procedure of length N such that the image of $D^{(N)}$ in X is D.

Let $D \leq F$ be an asymptotically good subdivisor and $C \subset X$ an (initial) exceptional component. We say that the subtraction $D > D - C$ is *asymptotically good* if for big enough $N \in \mathbb{N}$ there is a pair $\left(X^{(N)}, D^{(N)}\right)$ obtained by the standard procedure of length N such that the image of $D^{(N)}$ in X is D and $D^{(N)} > D^{(N)} - C$ is a good subtraction (where we identify $C \subset X_0$ with its strict transform in $X^{(N)}$).

Asymptotically good subtractions can be numerically characterized in the original log-resolution with the help of the tail-counting vector $m \in \mathbb{N}^r$.

Lemma 7. *Let (D, m) be a pair given by $D \subset X$ and $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$. The subtraction $D > D - C_i$ of the exceptional component C_i is asymptotically good with m_i tails constructed from each exceptional component C_i if and only if*

$$(D + K_\pi + \Delta_D^*) \cdot C_i < -m_i$$

It follows from the definition that a subdivisor $D \leq F \subset X$ is asymptotically good if it can be reached from F by a chain of asymptotically good subtractions

$$(F, 0) > (D_1, m_1) > \cdots > (D_n, m_n) = (D, m),$$

where the convention that we follow is that an *asymptotic subtraction* is $(D, m) > (D', m')$ where $D' = D - C_i$ for some exceptional component C_i, $m_i \leq m'_i$ and $m_j = m'_j$ for all $j \neq i$. The main result of this work is

Theorem 8. *Let $\pi : X \to Y$ be a log-resolution of an $\mathfrak{m}$-primary complete ideal $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-F)$. The ideal $\mathfrak{a}$ is realized as a multiplier ideal in a further log-resolution if and only if there is an asymptotically good chain from $(F, 0)$ to a pair (D, m) such that $\lfloor \Delta_D^* \rfloor$=0.*

Example 9. Consider the rational singularity given by the intersection matrix

$$M = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

with relative canonical divisor $K_\pi = \left(-1, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, 0\right)$. In particular it is a log-canonical singularity. Consider the antinef divisor $F = (2, 1, 1, 1, 1, 4)$ and let's look for asymptotically good chains. We first compute $\Delta_F = \Delta_F^* = -K_\pi = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)$, with

$$(F + K_\pi + \Delta_F^*) \cdot M = F \cdot M = (0, 0, 0, 0, 0, -2) \leq (0, 0, 0, 0, 0, 0) = -m_0.$$

The only asymptotically strict subtraction is that of C_6, but since $F \cdot C_6 = -2 \leq -1 = C_6^2$, two tails need to be added. This means we have to take $D_1 = F - C_6 = (2, 1, 1, 1, 1, 3)$ and $m_1 = (0, 0, 0, 0, 0, 2)$. Then we have $\Delta_{D_1}^* = K_\pi + C_6 = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$, with

$$(D_1 + K_\pi + \Delta_1^*) \cdot M = F \cdot M = (0, 0, 0, 0, 0, -2) = -m_1.$$

No further asymptotically strict subtraction is thus possible. Since both $\lfloor \Delta_F^* \rfloor, \lfloor \Delta_{D_1}^* \rfloor \neq 0$, the ideal defined by F is not a multiplier ideal.

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