The Distance Geometry of the Generalized Lobster's Arm

Federico Thomas and Josep M. Porta

Abstract This paper proposes a distance-based formulation to solve the inverse kinematics of what is known as the *generalized Lobster's arm*: a 6R serial chain in which all consecutive revolute axes intersect. Since the solution of the inverse kinematics of a general 6R serial chain comes down to finding the roots of a 16th-degree polynomial, one might think that this polynomial also contains the solutions to the inverse kinematics of 6R serial chains with special geometric parameters as a mere particular case. Nevertheless, under certain geometric circumstances various problems can appear. Some are of numerical nature, but others are fundamental problems of the used method. For that reason, it is still useful to study 6R chains with special geometric parameters, especially when the new formulation leads to a simpler solution, gives new insights, and provides new connections with other problems, as is the case in this paper.

1 Introduction

In 1841, in a communication addressed to the Philosophical Society of Cambridge, Robert Willis (1800-1875) showed that the joints of a common crab's claw work in the same way as those of what we would today classify as a 5R kinematic chain [1]. Willis' description appeared later summarized at the end of his influential book "*Principles of Mechanism*" [2, pp. 461-463]. This description was accompanied by the drawing in Fig. 1. He observed that the crab's claw is composed of six rigid bodies (denoted by A, B, C, D, E, and F in the drawing) connected in series through five revolute joints (denoted by 1, 2, 3, 4 and 5 in the drawing). What makes the arrangement of these five joint axes remarkable is that any two consecutive rotation axes in the chain intersect.

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In 1979, J. Duffy and S. Derby, as a result of a suggestion by K. H. Hunt —who was aware of Willis' observations— studied the inverse kinematics (i.e., the determination of joint angles required to move the end-effector to a desired position and orientation) of what they called the *generalized lobster arm* [3]. This arm is a 6R kinematic chain where every two consecutive axes intersect (Fig. 2). The resolution of this problem was seen as an intermediate step worth to be solved before addressing the same problem for the general 6R arm which, few years earlier, was ranked as the "*Mount Everest of kinematic problems*" by F. Freudenstein [4]. Curiously enough, as we will see later, the arrangement of joints in the generalized lobster arm does not provide much simplifications with respect to the general 6R arm, at least in the number of solutions. J. Duffy and S. Derby showed, using a long and complicated process, how to reduced this inverse kinematics problem to the computation of the roots of a 24th-degree polynomial, which is now seen as an outdated result.

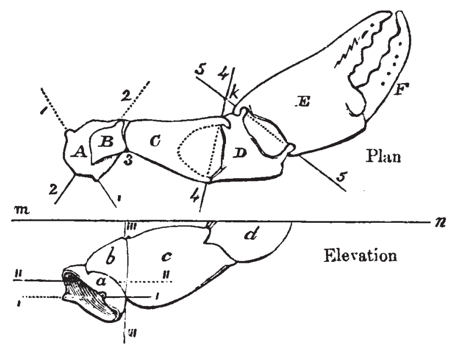


Fig. 1 Willis' drawing of the common crab's claw used to explain how its joint axes are arranged (adapted from [2, p. 462]).

In 1992, V. Murthy and K. J. Waldron revisited the problem in [5]. They solved it including a further generalization: the intersection between the second and the third axis and between the fourth and the fifth axis were no longer required. The standard

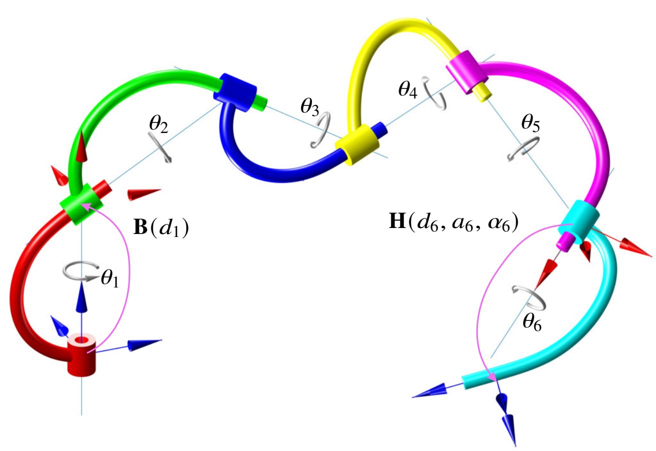


Fig. 2 A generalized lobster's arm, as defined by Duffy and Derby, is a 6R serial kinematic chain where all consecutive rotation axes intersect. In terms of standard DH-parameters, this means that $a_i = 0$ for $i = 1, \ldots, 5$.

DH-parameters of this kind of 6R serial chain appear in Table 1, where the parameters marked with an asterisk are actually irrelevant because they can be incorporated in the base and the hand transformations. In Fig. 2, these two transformations are represented by $\mathbf{B}(d_1)$ and $\mathbf{H}(d_6, a_6, \alpha_6)$, respectively. Murthy and Waldron reduced the resulting system of equations to a single univariate polynomial equation of degree 16. They also demonstrated that the used elimination process introduced no extraneous roots. This implied that the end-effector could reach a given position and orientation in at most sixteen different ways. This was an important improvement with respect to the 24th-degree polynomial solution of Duffy and Derby.

Table 1 Standard DH-parameters of the 6R serial chain analyzed by Murthy and Waldron [5]. The generalized lobster's arm can be seen as a particular case of this chain in which $a_2 = a_4 = 0$.

Link	θ_i	d_i	a_i	α_i
1	θ_1	*	0	α_1
2	θ_2	d_2	a_2	α_2
3	θ_3	d_3	0	α_3
4	θ_4	d_4	a_4	α_4
5	θ_5	d_5	0	α_5
6	θ_6	*	*	*

Six years earlier, in 1986, E. J. F. Primrose had already proved that the general 6R robot could have up to 16 inverse kinematic solutions. However, due to the complexity of the formulas, he could not come up with a way to remove the 16 redundant solutions of a polynomial equation of degree 32 [6]. This was considered as an important landmark in the history of kinematics. Nevertheless, in 1984, H.-Y. Lee already devised a method to explicitly obtain the 16th-degree polynomial in his Master Thesis written in Chinese [7]. This method became known to the western world when it appeared four years later in [8, 9]. Subsequently, Raghavan and Roth [10, 11] reformulated it in a cleared way using DH parameters. Since then, many other methods and variations have appeared leading to an extensive literature on the topic that we will not review here. However, it is worth observing that most methods reduce the problem to a single univariate polynomial in the one of the joints half angle tangent, and all the remaining joint variables follow from linear equations once the roots of the univariate polynomials are found.

Certain values of the kinematic parameters may reduce the degree of the resultant polynomial. Nevertheless, the intersections between the revolute axes in the 6R serial chain studied by Murthy and Waldron also lead to a 16th-degree polynomial. Unfortunately, Murthy and Waldron did not discuss the case in which all consecutive axes intersect. Thus, to the best of our knowledge, the question concerning the number of possible solutions for the inverse kinematics of the generalized lobster's arm remained open. We clarify this point in Section 2.

Since the roots of a 16th-degree polynomial equation gives the solution of the general case, one might think that it also contains the solutions to 6R serial chains with special geometric parameters as a mere particular case. Nevertheless, as it was pointed out in [13], under certain geometric circumstances, various problems appear. Some are of numerical nature, but others are fundamental problems of the used method. For that reason it is still useful to study 6R chains with special geometric parameters, such as the one discussed in this paper.

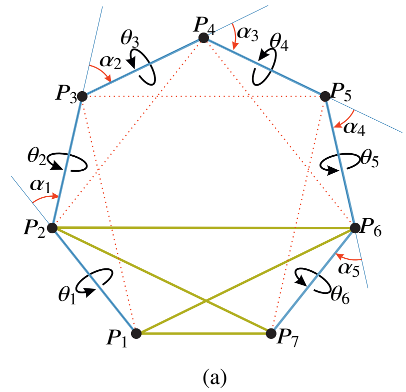


Fig. 3 (a) Bar-and-joint framework associated with the generalized lobster's arm. The lengths of the bars in blue and dotted red are determined by the geometric parameters of the arm; and those in green, by the closure condition. (b) 4-3 Gough-Stewart platform whose associated bar-and-joint framework has the same topology as that of the generalized lobster's arm.

The rest of this paper is organized as follows. Section 2 describes how the inverse kinematics problem of the generalized lobster's arm can be reformulated as the position analysis of seven points in $\mathbb{R}^3$ where some of their pairwise distances are known. This comes out to be equivalent to solving the forward kinematics of a 4-3 Stewart-Gough platform. Then, Section 3 shows how to solve this problem by computing a distance inversion in a strip of tetrahedra and Section 4 presents an example. Finally, the main contributions are summarized in Section 5.

2 A distance-based formulation

From a purely geometric point of view, a generalized lobster's arm can be described as the bar-and-joint framework depicted in Fig. 3(a). The points $P_2, \ldots, P_6$ correspond to the intersections between the six rotation axes and P_1 and P_7 can arbitrarily be taken on the first and the last rotation axes provided that they do not coincide with P_2 and P_6, respectively. The origin of the reference frames at the base and at the end-effector can be placed at P_1 and P_7, respectively. Then, we have that

$$|P_i P_{i+1}|^2 = d_i^2, \quad i = 1, \ldots, 6. \tag{1}$$

These distances are associated with the bars in solid blue in Fig. 3(a). Moreover, since the angle between consecutive joint axes is known and constant, we also have, using the cosine rule for supplementary angles, that

$$|P_i P_{i+2}|^2 = d_i^2 + d_{i+1}^2 + 2 d_i d_{i+1} \cos \alpha_i, \quad i = 1, \ldots, 5. \tag{2}$$

These distances are associated with the bars in dotted red in Fig. 3(a).

When the end-effector of the lobster's arm is fixed at a given location with respect to its base reference frame, the set of points $\{P_1, P_2, P_6, P_7\}$ defines a tetrahedron whose edge lengths are all known. The corresponding closing bars are depicted in light green in Fig. 3(a).

As a result of the above representation, the inverse kinematics problem of the generalized lobster's arm comes down to obtaining all the possible 3D embeddings of the bar-and-joint framework in Fig. 3(a). In general, one must be careful on how these embeddings are performed because they must respect the orientations of the involved tetrahedra [15] but, since in this case the framework contains just one tetrahedron, this can be ignored as it is done, for example, in [16].

Another interesting outcome of formulating the problem in terms of distances is that we can straightforwardly conclude that the inverse kinematics of the generalized lobster's arm and the forward kinematics of the 4-3 Gough-Stewart platform in Fig. 3(b) are equivalent position analysis problems. It is worth noting that this equivalency is not related to the series-parallel duality studied in [5]. Therefore, since the forward kinematics of this particular Gough-Stewart has up to 16 real solutions [15], so has the inverse kinematics of the generalized lobster's arm. In other words, the fact that any set of consecutive revolute axes of a 6R arm intersect at arbitrary angles does not reduce the number of solutions of its inverse kinematics. Some reductions are however obtained in the case that the axes intersect at right angles. Mavroidis and Roth gave a detailed investigation of these latter cases in [14].

3 The resolution process

Although it is not straightforward to see it at first glance, it is not difficult to express $d_{57}^2 = |P_5P_7|^2$ (a known distance) as a function of $d_{36}^2 = |P_3P_6|^2$ (an unknown distance). This is a one-to-many mapping whose inversion leads to a 16th-degree polynomial in the unknown distance. Each real root of this polynomial leads, using a sequence of trilaterations, to a valid configuration of the bar-and-joint framework. A detailed explanation of this method can be found in [17]. Next, we just give a brief summary.

First of all observe that, if we remove the bar P_5P_7 and we add the bar P_3P_6 in the bar-and-joint framework in Fig. 3, the resulting framework can be seen as a strip of four tetrahedra: $P_7P_1P_2P_6$, $P_1P_2P_6P_3$, $P_2P_6P_3P_4$, and $P_6P_3P_4P_5$. Each tetrahedron shares a face with the following one in the strip. For example, the first and the second one share the face $P_1P_2P_6$. Now, let us suppose that the two neighboring tetrahedra in Fig. 4(left) belong to this strip. The squared distance between P_l and P_m can be expressed as (see [12] for details):

$$d_{l,m}^2 = \frac{2}{D(i,j,k)} \left(D(i,j,k,l;i,j,k,m)\Big|_{s_{l,m}=0} \pm \sqrt{D(i,j,k,l)\, D(i,j,k,m)} \right). \quad (3)$$

where the terms of the form $D(\cdot)$ stand for Cayley-Menger determinants or bideterminants, and the $\pm$ sign accounts for the two possible solutions depending on the relative orientation between the two tetrahedra.

If one of the points in the set $\{P_i, P_j, P_k\}$ does not belong to any other tetrahedron in the strip, it can be removed from the strip provided that a bar connecting P_l and P_m, with length given by Eq. (3), is added [see Fig. 4(right)]. This reduces the number of tetrahedra in the strip by one. Then, by repeating this operation three times, we end up with an expression for the known squared distance d^2_{57} as a function of the unknown squared distance d^2_{36}. This expression contains radicals, and singularity factors associated with the shared faces between consecutive tetrathedra in the strip, that can be easily cleared to finally obtain a closure polynomial of 16th-degree (see [17] for details).

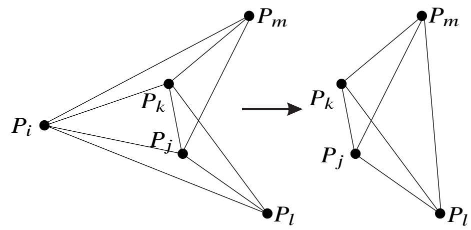


Fig. 4 Substitution rule in a strip of tetrahedra that permits reducing by one the number of tetrahedra in the strip.

4 Example

Let us consider the 6R kinematic chain with the standard DH-parameters given in Fig. 5(a). We can use the method described in the previous section to determine the joint angles required to move the end-effector to the location, with respect to the base reference frame, defined, for instance, by

$$\mathbf{E} = \mathbf{R}_x(-2.4019)\,\mathbf{R}_z(0.7047)\,\mathbf{T}(0,0,-2.2744). \tag{4}$$

For this particular example, we obtain 14 real solutions. The corresponding join angles appear numbered in Fig. 5(b) and the corresponding graphical representation of the arm configuration in Fig. 5(c).

Alternatively to the described method, we could straightforwardly use the implementation of the celebrated Manocha-Canny's method [18], available at [19]. Unfortunately, this implementation delivers 18 solutions, six of them being erroneous. These spurious solutions appear in Fig. 5(c) without number, and the missed solutions are marked with an asterisk.

5 Conclusion

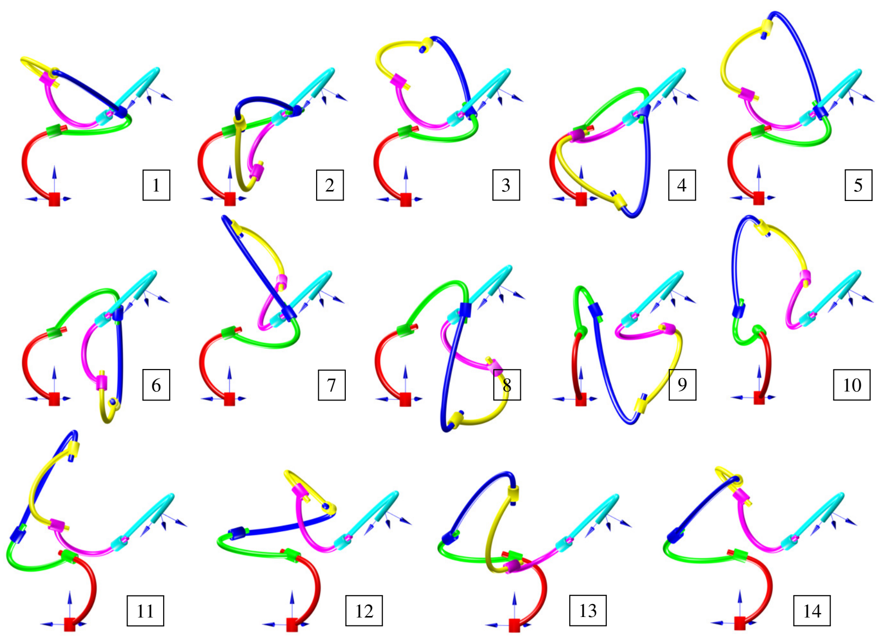
We have derived a 16th-degree polynomial whose roots determine the inverse kinematics solutions of the generalized lobster's arm. Contrarily to other methods that also derive closing polynomials, the one obtained here needs no variable elimina-

(a)

Link	θ_i	d_i	a_i	α_i
1	θ_1	1.0000	0	1.2661
2	θ_2	1.0000	0	1.9058
3	θ_3	1.5969	0	2.3505
4	θ_4	1.2247	0	1.6116
5	θ_5	1.0000	0	1.3181
6	θ_6	1.0000	0	2.7411

(b)

sol.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
1	0.3529	-2.1181	-0.7380	0.1745	-0.7304	1.9820
2	0.3321	-1.5990	0.9899	-0.1750	0.8327	0.1452
3	0.3293	2.4415	0.7239	-0.1752	1.7406	1.9539
4	0.2668	-0.6467	-0.6026	0.1863	-1.4280	0.9062
5	0.2633	2.8899	0.5955	-0.1873	1.4031	1.6535
6	0.4520	0.5264	-1.2307	0.1923	-2.3244	-0.2700
7	0.1679	3.0171	-1.4765	0.2258	-1.3087	-2.9700
8*	0.0836	-0.1470	1.6851	-0.2728	1.7280	-1.2528
9*	1.8664	-0.4997	-1.7695	1.1613	2.3125	-2.0415
10	2.4236	-3.1147	1.4208	-1.5051	-1.2302	-3.0822
11	3.0179	-3.0882	-0.5596	1.7591	1.2549	1.5303
12	3.0816	-1.9218	-0.9766	1.7763	2.1124	2.4124
13	-2.4803	2.1982	-0.6337	1.7982	0.7401	0.8245
14	-2.5240	2.3406	0.8787	-1.8062	-0.7739	2.2421
	1.8664	2.7047	-2.0793	-0.1201	-3.1357	2.5302
	0.0836	2.2749	-0.4562	-1.3347	1.5828	-2.8315
	2.9407	-2.6665	0.8593	-1.9783	2.2324	2.6025
	2.6512	-3.1377	-0.2420	1.5604	-3.0005	0.7550
	154.9700	-119.1354	-6.8803	-179.6604	144.9683	1.5549
	130.3419	-26.1367	-76.4750	90.6894	-162.2357	1.5229



(c)

Fig. 5 Using the method described in this paper, 14 real solutions are obtained for the inverse kinematics of the arm with the DH-parameters given in (a) and the end-effector located at Eq. (4). These solutions are numbered from 1 to 14 in (b), and the corresponding configurations are depicted to visually verify their correctness in (c).

tions. Moreover, the polynomial variable is a distance, and all its coefficients also come from operations with distances. No angles are involved at any point. We have also shown how Manocha and Canny's method fails to deliver the correct solutions for the analyzed 6R serial arm. As Dietmaiyer already observed, this is a common problem for general methods when some parameters are set to zero.

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References

1. Willis, R.: On the arrangement of the joints of crustaceous animals. Trans. of the Cambridge Philosophical Society, May 10th (1841)
2. Willis, R.: Principles of Mechanism. Longmans, Green, and Co., London (1870)
3. Duffy, J., Derby, S.: Displacement analysis of spatial 7R mechanism: A generalized lobster's arm. ASME Journal of Mechanical Design, **101**(2), 224–231 (1979)
4. Freudenstein, F.: Kinematics: past, present and future, Mechanism and Machine Theory, **8**(2), 151–160 (1973)
5. Murthy, V., Waldron, K.J.: Position kinematics of the generalized lobster arm and its series-parallel dual. ASME J. of Mechanical Design, **114**(3), 406–413 (1992)
6. Primrose, E.J.F.: On the input–output equation of the general 7R mechanism, Mechanism and Machine Theory, **21**(2), 509–510 (1986)
7. Lee, H.-Y.: Analysis of a spatial mechanism (In Chinese), Master Thesis, Beijing Institute of Posts and Telecommunications (1984)
8. Lee, H.-Y., Liang, C.G.: A new vector theory for the analysis of spatial mechanisms, Mechanism and Machine Theory, **23**(3), 209–217 (1988)
9. Lee, H.-Y., Liang, C.G.: Displacement analysis of the general spatial 7-link 7R mechanism, Mechanism and Machine Theory, **23**(3), 219–226 (1988)
10. Raghavan, M., Roth, B.: Kinematic analysis of the 6R manipulator of general geometry, In: Miura, H., Arimoto, S. (eds.), Proc. of the 5th International Symposium on Robotics Research, MIT Press, Cambridge, MA (1990)
11. Raghavan, M., Roth, B.: Inverse kinematics of the general 6R manipulator and related linkages, ASME Journal of Mechanical Design, **115**(3), 502–508 (1993)
12. Rojas, N., Thomas, F.: The closure condition of the double banana and its application to robot position analysis, Proc. of the IEEE Int. Conf. on Robotics and Automation, 4641–4646 (2013)
13. Dietmaier, P.: Displacement analysis of the homogeneous orthogonal 7-link 7R space mechanism. In: Lenarčič, J., Stifter, S. (eds.), Advances in Robot Kinematics, pp. 302–309, Springer-Verlag, Wien (1991)
14. Mavroidis, C., Roth, B.: Structural parameters which reduce the number of manipulator configurations. ASME Journal of Mechanical Design, **116**(1), 3–10 (1994)
15. Porta, J.M., Thomas, F.: Yet another approach to the Gough-Stewart platform forward kinematics, Proc. of the IEEE International Conference in Robotics and Automation, Brisbane, Australia (2018)
16. Thomas, F.: A distance geometry approach to the singularity analysis of 3R robots. ASME Journal of Mechanisms and Robotics, **8**(1), 011001 (2015)
17. Thomas, F., Porta, J.M.: Closure polynomials for strips of tetrahedra, In: Lenarčič, J., Merlet, J.-P. (eds.), Advances in Robot Kinematics, pp. 303-312 (2016)
18. Manocha, D., Canny, J.: Efficient inverse kinematics for general 6R manipulators. IEEE Transactions on Robotics and Automation, **10**(1), 648–657 (1994)
19. Manocha, D.: Inverse kinematics code for serial manipulators. GAMMA Research Group, http://gamma.cs.unc.edu/software/downloads/IK/ik-1.0.tar.gz (Last accessed: January 2022)