LMI Conditions for Stability and State-Feedback $\mathcal{H}_\infty$ Control of Discrete-Time Multimode Multidimensional Systems

Ariel Medero, Olivier Sename, and Vicenç Puig

***Abstract*—This letter deals with stability and state feedback control of discrete-time Multi-Mode Multi-Dimensional (M^3D) linear systems. The M^3D switch dynamics are modeled through a state mapping describing the mode transitions. This M^3D model framework then allows to consider poly-quadratic Lyapunov functions to obtain Linear Matrix Inequalities conditions for stability proof and for the synthesis of state-feedback controllers under $\mathcal{H}_\infty$ performance. A numerical example illustrates the improvement of the controller synthesis conditions here introduced for discrete-time M^3D systems over independent pointwise mode solutions.**

***Index Terms*—LMIs, robust control, switched systems.**

I. Introduction

Most works dealing with switching systems consider multi-mode systems for which all modes share the same number of states and model structure. The study in [1] by Erik I. Verriest was an innovative work that presents for the first time tools allowing the description of Multi-Mode Multi-Dimensional switching system, categorized there as M^3D systems. Since then, the generalization of switching systems to the multi-dimensional mode case have gained in popularity thanks to the problems this framework allows to tackle. In [2], the authors provide a framework for the optimal control of M^3D switching systems. As a motivating example, they derived a model of an ice-skater with four distinct modes for which the optimal control algorithm provides the optimal switching instant and the forces to be applied. An interesting application of M^3D systems was presented in [3], where it is modeled a spacecraft group formation as a state-varying switched system in which new spacecrafts can join or leave the formation. Moreover, the authors provided results to analyze the stability and fault tolerance of the formation. In [4], conditions are given for checking the stability of Multi-Dimensional switching systems with additional state jump, based on parametric Lyapunov functions, given an application to the problem of consensus in open multi-agent systems. Meanwhile, in [5], the LQ control approach is studied for multi-agent dynamic systems with increasing state dimensions, and is applied to a tracking problem in leader-following dynamics.

As emphasized, the application space of M^3D systems covers many different fields and opens up the possibility of tackling new problems with a straightforward framework. However, to the best of the Authors knowledge no work has presented yet tools for the general stabilization and feedback control of M^3D systems. This letter concerns the domain of linear systems, for which many problems in control theory can be formulated using Linear Matrix Inequalities (LMI) [6]. The main contributions of this letter are the extension of well-known LMI conditions for discrete-time LTI systems, e.g., [7]–[10], to the stabilization and feedback control of discrete-time M^3D linear systems including:

- the proof of asymptotic stability,
- the computation of the $\mathcal{H}_\infty$ norm,
- the synthesis of $\mathcal{H}_\infty$ state-feedback controllers.

This letter is organized as follows: Section II presents the dynamical equations of discrete-time M^3D LTI systems. In Section III, the stability of discrete-time M^3D systems is studied. Meanwhile, in Section IV, conditions for computing the $\mathcal{H}_\infty$ norm of discrete-time M^3D systems are given, which in Section V are extended to the synthesis of state-feedback control for discrete-time M^3D systems. A numerical example is used in Section VI to explore the benefits of the proposed method. Finally, some conclusions about the present study and possible applications are discussed in Section VII.

This letter notation is the following. $\|\cdot\|_2$ represents the L_2-norm and $\|\cdot\|_\infty$ represents the $\mathcal{H}_\infty$ norm. x^T represents the transpose of x. X^{-1} represents the inverse of X, matrix $X > 0$ represents that X is positive-definite and $*$ in an LMI represents a symmetric element transposed. The term *mode*

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Ariel Medero is with the Institut de Robòtica i Informàtica Industrial (CSIC-UPC), 08028 Barcelona, Spain, and also with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, 38000 Grenoble, France (e-mail: ariel.medero@grenoble-inp.fr).

Olivier Sename is with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, 38000 Grenoble, France (e-mail: olivier.sename@grenoble-inp.fr).

Vicenç Puig is with the Institut de Robòtica i Informàtica Industrial (CSIC-UPC), 08028 Barcelona, Spain (e-mail: vicenc.puig@upc.edu).

indicates a point of operation of the M^3D system with individual system matrices, $x^{(i)}$ represents that x belongs to the mode i of the M^3D system, x_{ji} represents that x is an element involved in the transition from mode i to mode j.

II. M^3D System Dynamics

This letter is concerned with the study of discrete-time LTI systems under Multi-Mode Multi-Dimensional (M^3D) switching conditions. In the absence of M^3D switching, the dynamics of the active mode i (given m modes) are given as:

$$M^{(i)} = \begin{cases} x_{k+1}^{(i)} = \mathcal{A}^{(i)} x_k^{(i)} + \mathcal{B}^{(i)} w_k \\ z_k = \mathcal{C}^{(i)} x_k^{(i)} + \mathcal{D}^{(i)} w_k \end{cases} \tag{1}$$

where $x_k^{(i)} \in \mathbb{R}^{n_i}$ is the state vector, $w_k \in \mathbb{R}^{n_w}$ is the vector of exogenous inputs with bounded energy such that $w_k \in L_2$ and $z_k \in \mathbb{R}^{n_z}$ is the vector of control performances. All along this letter, it is assumed that the switching signal is available in real-time, therefore, the active mode i is always known.

To account for the M^3D switching, we consider the framework introduced in [1], based on the notion of energy limited transitions. Such representations are of high interest when the structure and size of the system model can change accordingly to operating conditions. Let assume two modes with states $x^{(i)} \in \mathbb{R}^{n_i}$ and $x^{(j)} \in \mathbb{R}^{n_j}$ respectively. The M^3D system mode transition from mode i to mode j is defined by introducing the state mapping T_{ji} as:

$$x^{(j)} = T_{ji} x^{(i)}, \quad T_{ji} \in \mathbb{R}^{n_j \times n_i} \tag{2}$$

Now, the M^3D transition is assumed to occur during the switching from sampling instance k to sampling instance $k+1$.

With this assumption and from (1)-(2), the system dynamics during the M^3D transition from mode i to j are then given as:

$$M^{(ji)} = \begin{cases} x_{k+1}^{(j)} = T_{ji} x_{k+1}^{(i)} = T_{ji}\mathcal{A}^{(i)} x_k^{(i)} + T_{ji}\mathcal{B}^{(i)} w_k \\ z_k = \mathcal{C}^{(i)} x_k^{(i)} + \mathcal{D}^{(i)} w_k \end{cases} \tag{3}$$

III. Stability of M^3D Systems

As stated in [1] for energy limited transitions, a energy function $V^{(i)}(x_k^{(i)}) \in \mathbb{R}$ is associated with each mode i. By setting (2), the energy function at the switching instance must fulfill for energy dissipation:

$$V^{(j)}(x_{k+1}^{(j)}) = V^{(j)}(T_{ji} x_{k+1}^{(i)}) \leq V^{(i)}(x_k^{(i)}) \tag{4}$$

By considering in this letter a poly-quadratic energy function of the type

$$V^{(i)}(x_k^{(i)}) = x_k^{(i)^T} X^{(i)} x_k^{(i)}, \tag{5}$$

where $X^{(i)} \in \mathbb{R}^{n_i \times n_i}$ is a mode-dependent positive-definite symmetric matrix, as in [7], then the stability of a M^3D system can be proved if the following theorem holds true.

Theorem 1: A M^3D discrete-time autonomous system M is stable if, for each mode $i = 1, \ldots, m$ of M, there exist matrices $Q^{(i)} = Q^{(i)^T} > 0$, with $Q^{(i)} \in \mathbb{R}^{n_i \times n_i}$, and $G^{(i)} \in \mathbb{R}^{n_i \times n_i}$ such that the following conditions are satisfied:

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & G^{(i)^T} \mathcal{A}^{(i)^T} \\ * & Q^{(i)} \end{bmatrix} \geq 0$$

$$\forall i \text{ mode} \tag{6}$$

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & G^{(i)^T} \mathcal{A}^{(i)^T} T_{ij} \\ * & Q^{(j)} \end{bmatrix} \geq 0$$

$$\forall (i,j) \text{ connected pair of modes, } i \neq j. \tag{7}$$

Proof: Let us consider the active mode (1) restricted to the autonomous dynamics:

$$M^{(i)} = \left\{ x_{k+1}^{(i)} = \mathcal{A}^{(i)} x_k^{(i)}. \right. \tag{8}$$

By considering (2), the dynamics of the autonomous mode during the M^3D transition then are:

$$M^{(ji)} = \left\{ x_{k+1}^{(j)} = T_{ji}\mathcal{A}^{(i)} x_k^{(i)}. \right. \tag{9}$$

From (4), considering a energy function (5), the energy limited condition during the switching instance can be written as:

$$x_{k+1}^{(j)^T} X^{(j)} x_{k+1}^{(j)} - x_k^{(i)^T} X^{(i)} x_k^{(i)} \leq 0, \tag{10}$$

which according to (9) is equivalent to:

$$(T_{ji}\mathcal{A}^{(i)} x_k^{(i)})^T X^{(j)} (T_{ji}\mathcal{A}^{(i)} x_k^{(i)}) - x_k^{(i)^T} X^{(i)} x_k^{(i)} \leq 0 \tag{11}$$

This can then be rearranged as:

$$x_k^{(i)^T} \left[\mathcal{A}^{(i)^T} T_{ji}^T X^{(j)} T_{ji} \mathcal{A}^{(i)} - X^{(i)} \right] x_k^{(i)} \leq 0 \tag{12}$$

Using Schur complement, (12) is then equivalent to:

$$\begin{bmatrix} X^{(j)^{-1}} & T_{ji}\mathcal{A}^{(i)} \\ * & X^{(i)} \end{bmatrix} \geq 0 \tag{13}$$

Applying basic matrix row/column manipulation and a congruence transformation with $diag([G^{(i)}, I])$, where $G^{(i)} \in \mathbb{R}^{n_i \times n_i}$ is a general matrix, leads to:

$$\begin{bmatrix} G^{(i)^T} X^{(i)} G^{(i)} & G^{(i)^T} \mathcal{A}^{(i)^T} T_{ji}^T \\ * & X^{(j)^{-1}} \end{bmatrix} \geq 0. \tag{14}$$

Now, let us assume that condition (7) is true. By setting $X^{-1} \equiv Q$ in (7) the following inequality is obtained

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - X^{(i)^{-1}} & G^{(i)^T} \mathcal{A}^{(i)^T} T_{ji}^T \\ * & X^{(j)^{-1}} \end{bmatrix} \geq 0, \tag{15}$$

Using the simplified Young's relation [8], [11]:

$$G^{(i)^T} X^{(i)} G^{(i)} \geq G^{(i)^T} + G^{(i)} - X^{(i)^{-1}}$$

then (15) is a sufficient condition for (14), and thus for the stability of M during a M^3D transition. Notice that for the non-switching case, the same steps with $X^{(i)} = X^{(j)}$ and $T_{ji} = I \in \mathbb{R}^{n_i \times n_i}$ proves the sufficiency of condition (6), with (6) being a well known result for checking the asymptotic stability of discrete-time linear systems through the use of LMI [8]. This concludes the proof. ∎

Remark 1: Theorem 1 can be seen as an extension to discrete-time M^3D systems of well-known results for the stability of discrete-time switched linear systems, e.g., [9]. Note that Theorem 1 assumes a dwell-time equal to one sampling period T_s, which presupposes a completely random switching sequence and may be too strict. Instead, (7) can be modified to obtain the minimum dwell-time Δ_* for the M^3D system M, by discretizing M with a sampling time $\Delta_* T_s$ as explained in [9].

This remark about the switching dwell-time also applies to the following results presented in this letter.

In the next section, the stability condition for M^3D systems is extended with conditions for $\mathcal{H}_\infty$ performance.

IV. $\mathcal{H}_\infty$ Norm for Discrete M^3D Systems

Closed-loop systems need not only to be stable with respect uncertainties and disturbances but also being able to fulfill some performance requirements. To achieve this, one of the most well-known and powerful techniques in the control literature for LTI systems is the $\mathcal{H}_\infty$ robust control theory. The key concept being the $\mathcal{H}_\infty$ norm of systems, which is associated with the maximum effect γ_∞ the exogenous inputs w_k have over the control performance outputs z_k:

$$\frac{\|z\|_2}{\|w\|_2} \leq \gamma_\infty \tag{16}$$

The most common way of determining γ_∞ is making use of the Bounded Real Lemma [6]. The next theorem extends the Bounded Real Lemma to the case of discrete M^3D systems to determine an upper bound of its $\mathcal{H}_\infty$ norm.

Theorem 2: Consider a discrete M^3D system M and scalar $\gamma_\infty > 0$. If, for each mode $i = 1, \ldots, m$ of M, there exist matrices $Q^{(i)} = Q^{(i)^T} > 0$, with $Q^{(i)} \in \mathbb{R}^{n_i \times n_i}$, and $G^{(i)} \in \mathbb{R}^{n_i \times n_i}$ such that the following LMI problem is feasible:

$$Q^{(i)} > 0 \tag{17}$$

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & G^{(i)^T}\mathcal{A}^{(i)^T} & G^{(i)^T}\mathcal{C}^{(i)^T} & 0 \\ * & Q^{(i)} & 0 & \mathcal{B} \\ * & * & \gamma_\infty I & \mathcal{D} \\ * & * & * & \gamma_\infty I \end{bmatrix} \geq 0$$

$$\forall i \text{ mode} \tag{18}$$

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & G^{(i)^T}\mathcal{A}^{(i)^T}T_{ji}^T & G^{(i)^T}\mathcal{C}^{(i)^T} & 0 \\ * & Q^{(j)} & 0 & T_{ji}\mathcal{B} \\ * & * & \gamma_\infty I & \mathcal{D} \\ * & * & * & \gamma_\infty I \end{bmatrix} \geq 0$$

$$\forall (i,j) \text{ connected pair of modes, } i \neq j \tag{19}$$

Then, γ_∞ is an upper bound of the $\mathcal{H}_\infty$ norm of M, such that $\|M\|_\infty \leq \gamma_\infty$. If the optimal γ_∞ is required, the LMI minimization problem for γ_∞ is still an LMI problem with variables γ_∞, Q and G.

Proof: Let us consider a poly-quadratic energy function (5) such that during the M^3D mode transition the following condition holds true

$$V(x_{k+1}^{(j)}) - V(x_k^{(i)}) + \frac{1}{\gamma_\infty} z_k^T z_k - \gamma_\infty w_k^T w_k \leq 0. \tag{20}$$

Expanding the quadratic energy condition according to the M^3D switching dynamics (3), it can then be rearranged in matrix form as

$$\begin{bmatrix} x_k^{(i)} \\ w_k \end{bmatrix}^T \Phi \begin{bmatrix} x_k^{(i)} \\ w_k \end{bmatrix} \leq 0, \tag{21}$$

with Φ as in (22), shown at the bottom of the page. Applying a Schur Complement around $\frac{1}{\gamma_\infty}I$ followed by a Schur Complement around $X^{(j)}$, then (21) is equivalent to

$$\begin{bmatrix} X^{(j)^{-1}} & T_{ji}\mathcal{A}^{(i)} & T_{ji}\mathcal{B}^{(i)} & 0 \\ * & X^{(i)} & 0 & \mathcal{C}^{(i)^T} \\ * & * & \gamma_\infty I & \mathcal{D}^{(i)^T} \\ * & * & * & \gamma_\infty I \end{bmatrix} \geq 0 \tag{23}$$

Now, following the same way as in Theorem 1, from basic row/column manipulations followed by a congruence transformation by $diag([G^{(i)}, I, I, I])$ applied to (23), and making use of the simplified Young's relation, we can prove that the LMI (19) with $X^{-1} \equiv Q$ implies (23). As in the stability case, for the non switching condition, by setting $X^{(i)} = X^{(j)}$ and $T_{ji} = I \in \mathbb{R}^{n_i \times n_i}$ the same chain of steps proves sufficiency of condition (18), which is a well known result for the computation of the $\mathcal{H}_\infty$ performance of discrete-time systems through the use of LMI [10]. This concludes the proof. ■

V. $\mathcal{H}_\infty$ State-Feedback Control for Discrete M^3D Systems

The objective of this section is to introduce the $\mathcal{H}_\infty$ control of discrete-time M^3D systems. Let us consider the discrete-time M^3D system N, where the dynamics of the active mode i are:

$$N^{(i)} = \begin{cases} x_{k+1}^{(i)} = A^{(i)}x_k^{(i)} + B_u^{(i)}u_k + B_w^{(i)}w_k \\ z_k = C_z^{(i)}x_k^{(i)} + D_u^{(i)}u_k + D_w^{(i)}w_k \end{cases} \tag{24}$$

where $x_k^{(i)} \in \mathbb{R}^{n_i}$ is the state vector, $w_k \in \mathbb{R}^{n_w}$ is the vector of exogenous inputs with bounded energy such that $w_k \in L_2$, $z_k \in \mathbb{R}^{n_z}$ is the vector of control performances and $u_k \in \mathbb{R}^{n_u}$ is the vector of control inputs.

By introducing the discrete-time state-feedback control law

$$u_k = K^{(i)}x_k^{(i)}, \tag{25}$$

the $\mathcal{H}_\infty$ control problem is therefore to find suitable matrices $K^{(i)} \in \mathbb{R}^{n_u \times n_i}$ that render N closed-loop stable, and minimizes the influences of the exogenous inputs w_k on the control performances z_k, according to an $\mathcal{H}_\infty$ norm criterion. This is achieved if the following theorem holds true.

Theorem 3: Consider a discrete M^3D system N and scalar $\gamma_\infty > 0$. If, for each mode $i = 1, \ldots, m$ of N, there exist matrices $Q^{(i)} = Q^{(i)^T} > 0$, with $Q^{(i)} \in \mathbb{R}^{n_i \times n_i}$, $G^{(i)} \in \mathbb{R}^{n_i \times n_i}$ and $Y^{(i)} \in \mathbb{R}^{n_u \times n_i}$ such that the following LMI conditions are satisfied:

$$Q^{(i)} > 0 \tag{26}$$

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & \Psi_{1,2}^{(i)} & \Psi_{1,3}^{(i)} & 0 \\ * & Q^{(i)} & 0 & B_w^{(i)} \\ * & * & \gamma_\infty I & D_w^{(i)} \\ * & * & * & \gamma_\infty I \end{bmatrix} \geq 0 \tag{27}$$

$$\Phi = \begin{bmatrix} \mathcal{A}^{(i)^T}T_{ji}^T X^{(j)} T_{ji}\mathcal{A}^{(i)} - X^{(i)} + \frac{1}{\gamma_\infty}\mathcal{C}^{(i)^T}\mathcal{C}^{(i)} & \mathcal{A}^{(i)^T}T_{ji}^T X^{(j)} T_{ji}\mathcal{B}^{(i)} + \frac{1}{\gamma_\infty}\mathcal{C}^{(i)^T}\mathcal{D}^{(i)} \\ * & \mathcal{B}^{(i)^T}T_{ji}^T X^{(j)} T_{ji}\mathcal{B}^{(i)} + \frac{1}{\gamma_\infty}\mathcal{D}^{(i)^T}\mathcal{D}^{(i)} - \gamma_\infty I \end{bmatrix} \tag{22}$$

with

$$\Psi_{1,2}^{(i)} = G^{(i)^T} A^{(i)^T} + Y^{(i)^T} B_u^{(i)^T},$$
$$\Psi_{1,3}^{(i)} = G^{(i)^T} C_z^{(i)^T} + Y^{(i)^T} D_u^{(i)^T}$$
$\forall i$ mode

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & \Psi_{1,2}^{(ji)} & \Psi_{1,3}^{(ji)} & 0 \\ * & Q^{(j)} & 0 & T_{ji} B_w^{(i)} \\ * & * & \gamma_\infty I & D_w^{(i)} \\ * & * & * & \gamma_\infty I \end{bmatrix} \geq 0 \quad (28)$$

with

$$\Psi_{1,2}^{(ji)} = G^{(i)^T} A^{(i)^T} T_{ji}^T + Y^{(i)^T} B_u^{(i)^T} T_{ji}^T,$$
$$\Psi_{1,3}^{(ji)} = G^{(i)^T} C_z^{(i)^T} + Y^{(i)^T} D_u^{(i)^T}$$
$\forall (i, j)$ connected pair of modes, $i \neq j$

Then, there exists a state-feedback control law $u_k = K^{(i)} x_k^{(i)}$ such that $\frac{\|z\|_2}{\|w\|_2} \leq \gamma_\infty$. The state-feedback control matrices are recovered according to $K^{(i)} = Y^{(i)} G^{(i)^{-1}}$, for each mode $i = 1, \ldots, m$ of N. Now, if the optimal γ_∞ is required, the LMI minimization problem for γ_∞ is still an LMI problem with variables γ_∞, Q, G and Y.

Proof: Note that $N^{(i)}$ can be rewritten as $M^{(i)}$ in (1) considering:

$$\mathcal{A}^{(i)} = A^{(i)} + B_u^{(i)} K^{(i)} \;\; \mathcal{B}^{(i)} = B_w^{(i)}$$
$$\mathcal{C}^{(i)} = C_z^{(i)} + D_u^{(i)} K^{(i)} \;\; \mathcal{D}^{(i)} = D_w^{(i)} \quad (29)$$

Substitute the closed-loop system matrices $M^{(i)}$ from (18) and (19) with the system matrices of $N^{(i)}$, according to (29). Then, with the introduction of the linearizing change of variables $Y^{(i)} = K^{(i)} G^{(i)}$, the LMI conditions (27) and (28) are both recovered. With (27) recovering a well-known result for $\mathcal{H}_\infty$ state-feedback synthesis for discrete-time LTI systems [10]. This concludes the proof. ■

Remark 2: Note that, in some cases Theorem 3 may be too restrictive. Indeed, as formulated, the closed-loop $\mathcal{H}_\infty$ performance should be maintained even during a M^3D mode transition. If too conservative or unnecessary, a compromise may be to drop the strong requirement of $\mathcal{H}_\infty$ switching performance in favor of only requiring switching stability. This can be achieved by substituting the LMI condition (28) by

$$\begin{bmatrix} G^{(i)^T} + G^{(i)} - Q^{(i)} & \Psi_{1,2}^{(ji)} \\ * & Q^{(j)} \end{bmatrix} \geq 0, \quad (30)$$

which comes from applying the linearizing change of variables $Y^{(i)} = K^{(i)} G^{(i)}$ in (7).

Remark 3: Reduction of conservatism in Theorem 3 could also be achieved with the introduction of a new slack variable $Y^{(ji)} = K^{(ji)} G^{(ji)}$ in either LMI condition (28) or (30), such that the state-feedback controller $K^{(ji)} = Y^{(ji)} G^{(ji)^{-1}}$ is only active during the transition from mode i to mode j. This is similar as have been proposed for control of continuous-time switching systems in [12].

VI. Numerical Example

In this section, a numerical example is given to illustrate the potential of the synthesis conditions provided in this letter for M^3D systems. First, the M^3D system is presented. Then, the use and interest of the provided theorems are illustrated. Finally, some simulation results are carried out together with some analysis of the obtained results.

A. System Description

It is considered a discrete-time M^3D system N such that the active mode i dynamics are given by:

$$N^{(i)} = \begin{cases} x_{k+1}^{(i)} = A^{(i)} x_k^{(i)} + B_u^{(i)} u_k + B_w^{(i)} w_k \\ z_k = C_z^{(i)} x_k^{(i)} \end{cases} \quad (31)$$

The system N has three modes with state dimensions $x_k^{(1)} \in \mathbb{R}^4$, $x_k^{(2)} \in \mathbb{R}^{10}$ and $x_k^{(3)} \in \mathbb{R}^6$. The system matrices are given by:

$$A^{(1)} = \begin{bmatrix} 0.25 & 0.16 & 0.44 & -0.08 \\ 0.2 & -0.07 & -0.28 & 0.06 \\ 0.48 & -0.29 & 0.18 & -0.02 \\ 0.00 & 0.06 & 0.00 & 0.60 \end{bmatrix}, \quad (32)$$

$$A^{(3)} = \begin{bmatrix} 1.5 & -0.15 & 0.06 & 0.18 & -0.10 & 0.09 \\ 0.14 & -0.02 & -0.29 & 0.42 & 0.35 & -0.13 \\ -0.08 & -0.27 & 0.21 & 0.06 & -0.27 & -0.06 \\ 0.24 & 0.34 & 0.09 & 0.15 & -0.16 & -0.16 \\ 0.10 & 0.32 & -0.28 & -0.11 & -0.12 & -0.07 \\ 0.07 & -0.07 & -0.09 & -0.15 & -0.03 & 0.46 \end{bmatrix} \quad (33)$$

and

$$A^{(2)} = \begin{bmatrix} A^{(1)} & A_{1,2}^{(2)} \\ A_{2,1}^{(2)} & A^{(3)} \end{bmatrix} \quad (34)$$

with

$$A_{1,2}^{(2)} = \begin{bmatrix} 0.26 & -0.28 & 0.16 & -0.27 & 0.15 & -0.13 \\ -0.08 & 0.07 & -0.21 & -0.34 & 0.30 & -0.11 \\ -0.08 & 0.34 & 0.07 & -0.20 & 0.00 & 0.42 \\ 0.14 & 0.26 & 0.46 & 0.04 & 0.07 & 0.04 \end{bmatrix}$$

$$A_{2,1}^{(2)} = \begin{bmatrix} 0.27 & -0.17 & -0.16 & -0.02 \\ -0.12 & 0.03 & 0.33 & 0.19 \\ 0.09 & -0.18 & 0.08 & 0.51 \\ -0.23 & -0.37 & -0.23 & -0.02 \\ 0.26 & 0.27 & -0.01 & 0.00 \\ -0.14 & -0.09 & 0.43 & 0.08 \end{bmatrix} \quad (35)$$

Notice from (34) that modes 1 and 3 are subsystems of mode 2. The chosen system aims to represent a multi-component system with coupled dynamics. However, note that having a mode encompassing all other modes is not a requirement of the methods presented in this letter. Also, note from the first diagonal element in (33), that mode 3 (thus, mode 2 too) has unstable open-loop dynamics, with $A^{(2)}$ having unstable poles at $p = 1.53$ and at the unit circle, and $A^{(3)}$ having a single unstable pole at $p = 1.49$.

On the other hand the system N has two control inputs, with the input matrix of each mode given by:

$$B_u^{(1)} = \begin{bmatrix} -1.53 & 0 & -1.96 & 0.73 \\ -1.01 & -0.52 & 1.96 & 0 \end{bmatrix}^T, \quad (36)$$

$$B_u^{(3)} = \begin{bmatrix} 0.83 & -0.10 & 0.43 & 0.30 & 0 & -0.68 \\ 0 & 0 & 0 & 0.89 & 0 & 0.06 \end{bmatrix}^T \quad (37)$$

and

$$B_u^{(2)} = \begin{bmatrix} B_u^{(1)^T} & B_u^{(3)^T} \end{bmatrix}^T \tag{38}$$

All modes are affected by disturbance inputs, such that the disturbance input matrix of each mode is given by

$$B_w^{(i)} = 0.1 \cdot B_u^{(i)}. \tag{39}$$

The performance output matrix for each of the three modes of N are chosen as:

$$C_z^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \tag{40}$$

$$C_z^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{41}$$

and

$$C_z^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{42}$$

It is worth noticing that the dimension of the control performance output vector $z(k)$ is different for all modes. This has been chosen to illustrate that the proposed method can handle cases where the dimensions of $u(k)$, $w(k)$ and $z(k)$ change during mode transitions. This can be accomplished without any modification on the results provided in previous sections. It is also important to note that the control performance output matrices $C_z^{(i)}$ indicate the signals to be minimized following the $\mathcal{H}_\infty$ criterion. Of course, for state feedback control, it is moreover assumed that all states are available.

Note also that, later on, the control performance output of the mode 1 is referred to as $z_a(k)$, while it is referred to as $z_b(k)$ for the mode 3, and therefore, for the mode 2 the considered performance output vector is denoted $z(k) = [z_a(k), z_b(k)]^T$.

The M^3D system N is considered to switch with arbitrary conditions and no restrictions, such that the active mode in the next sampling instance could potentially (but not necessarily) switch to any of the other two modes. As a result, it is considered all mode pairs (i, j), $i \neq j$, are connected. The state mappings T_{ji} for each Multi-Dimensional transition are:

$$T_{12} = \begin{bmatrix} I^{4\times4} & \mathbf{0}^{4\times6} \end{bmatrix}, \tag{43}$$

$$T_{32} = \begin{bmatrix} \mathbf{0}^{6\times4} & I^{6\times6} \end{bmatrix}, \tag{44}$$

$$T_{13} = \begin{bmatrix} \mathbf{0}^{4\times6} \end{bmatrix} \tag{45}$$

and

$$T_{21} = T_{12}^T, \quad T_{23} = T_{32}^T, \quad T_{31} = T_{13}^T. \tag{46}$$

Note that M^3D transitions involving T_{21}, T_{23} and T_{31} imply a dilation of the state vector. In these cases, it is assumed that new appearing state variables are initialized with zero initial condition.

B. Control Design

Two different control approaches are considered in this section. In the first baseline approach, independent state-feedback controllers are designed for each mode i of N. In order to check the stability of the global system N in closed-loop, Theorem 1 is then employed. The second approach follows our proposed method, so the design of the state-feedback control law is carried out applying Theorem 3 to the global system N.

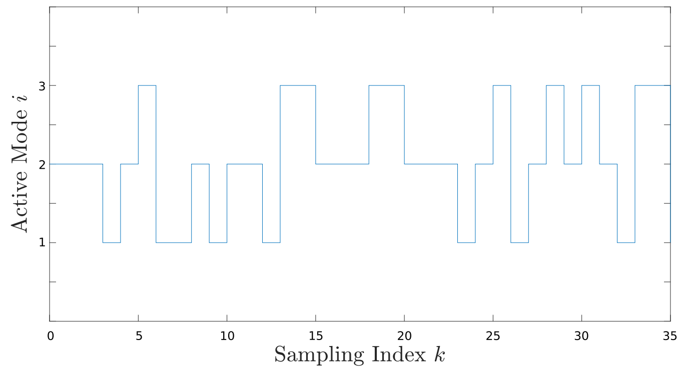

Fig. 1. Active mode i during arbitrary switching conditions.

As mentioned, the synthesis of controllers for the first approach is done as an independent discrete-time LTI control synthesis problem for each mode of N. The computation of all controllers $K^{(i)}$ is performed considering the LMI condition (27) from Theorem 3 only, without accounting for the transition's effect (so actually using the method in [10]). As each controller is computed independently of the others, this results in three different LMI optimization problems where the optimal $\mathcal{H}_\infty$ norms found in each case are $\gamma_\infty^{(1)} = 0.0732$, $\gamma_\infty^{(2)} = 0.1442$ and $\gamma_\infty^{(3)} = 0.0686$. However, it is well known that stable systems can be rendered unstable under arbitrary switching conditions [13]. With the independently computed state-feedback controllers $K^{(i)}$ and relation (29), the stability of the switched M^3D closed-loop system can be tested employing Theorem 1. In this scenario, the M^3D closed-loop system could not be proved to be stable as the conditions from Theorem 1 were not satisfied.

Concerning the second scenario (our approach), the controllers $K^{(i)}$ are computed considering the global M^3D system N by employing Theorem 3. To tackle the design problem it is required to solve a total of nine LMI conditions, with three LMI according to (27) for the $\mathcal{H}_\infty$ control of each mode plus six LMI conditions according to (28) to account for all the possible Multi-Dimensional mode transitions. The obtained optimal upper bound on the $\mathcal{H}_\infty$ norm of the closed-loop system is $\gamma_\infty = 0.19$.

C. Simulation Analysis

Following the discussed control design approaches, two simulation scenarios are proposed. The first scenario considers the case where independent controllers have been designed for each mode, while the second scenario presents the results of the global design approach proposed in Theorem 3. In both scenarios, the system N evolves under arbitrary switching conditions, where the switching sequence is the same for both cases, as shown in Fig. 1. Moreover the disturbance signals were generated such that they values change every five sampling instances. The randomized value for each of the two disturbance inputs was chosen to be with zero mean and variance equal to 1 and 2, respectively.

The control output performance $z(k)$ for each scenario is shown in Fig. 2. On the top figure, it is shown the control performances obtained during the first scenario with independent controllers for each mode. On the bottom, it is given

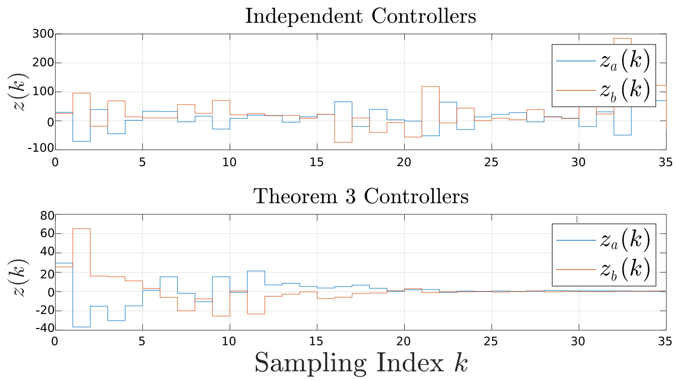

Fig. 2. Control performance output $z(k)$.

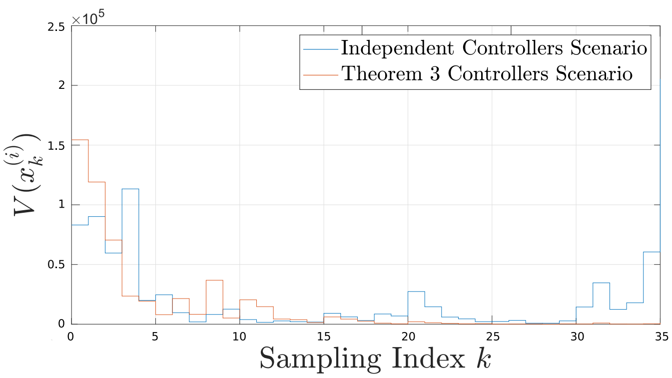

Fig. 3. Energy Storage Function $V^{(i)}(x_k^{(i)})$ of system N under arbitrary switching.

the control performances output in the second scenario with state-feedback controllers computed for the system N globally according to Theorem 3.

From the simulation results given in Fig. 2, it can be seen that during the first scenario, the control performances do not converge and, in fact, they increase in magnitude with time as the system N in this case is unstable. For the second scenario however, the closed-loop system is stable despite the presence of the arbitrary switch conditions and state dimension and system structure changes during mode transitions.

The stability of the closed-loop system can also be assessed considering a poly-quadratic storage function as in (5), with $X \equiv Q^{-1}$, where Q is the symmetric positive-definite matrix found from applying Theorem 3. Fig. 3 shows the evolution of the storage function during both simulation scenarios.

In the first scenario (in blue), the energy in the system initially converges towards zero. However, the energy in the last sampling instances starts to increase dramatically indicating closed-loop instability affected by the arbitrary Multi-Dimensional transitions. In the second scenario (in red), the energy can be seen to increase for a sampling period after some transitions, which is to be expected when using discontinuous Lyapunov functions in switching systems [9], [13]. However, as seen by Fig. 3, the synthesis method based on Theorem 3 ensures the energy of the M^3D system N is globally decreasing, which shows that the system has been stabilized despite arbitrary switching conditions.

VII. Conclusion

In this letter, new conditions using LMI formulation have been provided in order to test the stability of discrete-time M^3D LTI systems, to compute the $\mathcal{H}_\infty$ norm of such systems, as well as to design state-feedback controllers. The synthesis conditions were tested on a numerical M^3D system composed of three modes, two of which are open-loop unstable, all of different dimensions and under arbitrary switching conditions.

Therefore, the results here presented allow to study control problems associated with complex Multi-Dimensional systems while retaining strong stability guarantees. Some application examples could be: the control of nonlinear systems modeled as piecewise reduced linearizations of different dimensions (as commented in [14]) or the control of MIMO systems with subsystems that could be discarded or not. Furthermore, the results presented in this letter may be extended with the use of structured Lyapunov Functions in order to reduce the conservatism on the synthesis conditions [15]. Another interesting extension concerns the adaption of the results to the Linear Parameter Varying framework, allowing to study problems beyond the scope of LTI systems.

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