On Distributed Nash Equilibrium Seeking in a Class of Contractive Population Games

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***Abstract*— In this paper, we consider the framework of population games and evolutionary dynamics. Based on such a framework, we formulate a novel approach for distributed Nash equilibrium seeking under partial-decision information for a class of evolutionary dynamics and a family of contractive population games. As the main contribution, we provide sufficient conditions to guarantee the asymptotic stability of the set of Nash equilibria of the underlying game. To the best of our knowledge, this is the first paper to address the problem of distributed Nash equilibrium seeking under partial-decision information in the aforementioned context of population games and evolutionary dynamics.**

I. Introduction

Consider a set of $N \in \mathbb{Z}_{\geq 2}$ populations, each comprised of a large and constant number of strategic decision-making agents. Throughout, the set of populations is indexed by $\mathcal{P} = \{1, 2, \ldots, N\}$, the set of strategies available to the agents of population $k \in \mathcal{P}$ is indexed by $\mathcal{S}^k = \{1, 2, \ldots, n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$, and the totality of agents of each population $k \in \mathcal{P}$ is modeled as a continuum of mass $m^k \in \mathbb{R}_{>0}$. At any time, the mass of agents choosing strategy $i \in \mathcal{S}^k$ at population $k \in \mathcal{P}$ is given by $x_i^k \in \mathbb{R}_{\geq 0}$. Therefore, the vectors $\mathbf{x}^k = \text{col}\left(x_1^k, x_2^k, \ldots, x_{n^k}^k\right) \in \Delta^k$ and $\mathbf{x} = \text{col}\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right) \in \Delta$ provide the strategic distribution of population $k \in \mathcal{P}$ and of the entire society, respectively. Here, $\text{col}\,(\cdot)$ denotes the column vector stack operation, $n = \sum_{k\in\mathcal{P}} n^k$, $\Delta^k = \left\{\mathbf{x}^k \in \mathbb{R}^{n^k}_{\geq 0} : \sum_{i\in\mathcal{S}^k} x_i^k = m^k\right\}$, and $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{x}^k \in \Delta^k, \forall k \in \mathcal{P}\right\}$. Namely, Δ^k is the set of all possible strategic distributions of population $k \in \mathcal{P}$, while Δ is the set of all possible strategic distributions of the entire society.

Under the considered framework, the strategic distribution of each population evolves over time according to a stochastic decision-making process [1]. However, since the number of agents within each population is large, the temporal evolution of the strategic distribution of the society can be well approximated (see [1, Chapter 10] or [2]) by a so-called evolutionary dynamics model (EDM), as defined next.

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Definition 1: The EDM that describes the temporal evolution of the strategic distribution $\mathbf{x}(t)$ is given by

$$\rho_{ij}^k(t) = \left[p_j^k(t) - p_i^k(t)\right]_+ \tag{1a}$$

$$\dot{x}_i^k(t) = \sum_{j\in\mathcal{S}^k} x_j^k(t)\rho_{ji}^k(t) - x_i^k(t)\rho_{ij}^k(t), \tag{1b}$$

for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, with $\mathbf{x}(0) \in \Delta$. Here, $t \in \mathbb{R}_{\geq 0}$ is the continuous-time index, $\mathbf{x}(t)$ is the value of $\mathbf{x}$ at time t, $p_i^k(t) \in \mathbb{R}$ is the payoff perceived by the agents of population k choosing the strategy i at time t, and $[\cdot]_+ \triangleq \max(\cdot, 0)$.

Based on Definition 1, the EDM can be thought as a continuous-time dynamical system whose input is the payoff vector $\mathbf{p}(t) = \text{col}\left(\mathbf{p}^1(t), \mathbf{p}^2(t), \ldots, \mathbf{p}^N(t)\right) \in \mathbb{R}^n$, where $\mathbf{p}^k(t) = \text{col}\left(p_1^k(t), p_2^k(t), \ldots, p_{n^k}^k(t)\right) \in \mathbb{R}^{n^k}$, for all $k \in \mathcal{P}$. In general, the payoff vector $\mathbf{p}(t)$ is generated by a causal map of the society's strategic distribution $\mathbf{x}(t)$ [2].

Motivation: The framework of population games is suitable to model several multi-agent decision-making scenarios. Some examples include resource allocation [3], demand response [4], and congestion games [5], among others. However, one limitation of the available theory on population games is that the payoff vector $\mathbf{p}(t)$ is often assumed to be provided by an oracle-like entity with complete information about the society's strategic distribution $\mathbf{x}(t)$. Clearly, such an assumption imposes practical difficulties when the multiple populations are spatially distributed over some geographical region (as could be the case in the aforementioned examples), and complete information on the strategic distribution of the society might not be readily available at a single place, or complete information broadcasting might not be feasible. To cope with such difficulties, in this paper we consider that each population $k \in \mathcal{P}$ has an associated payoff provider, which provides the vector $\mathbf{p}^k(t)$ to the agents of population k, and has direct access only to the strategic distribution of population k. Namely, each payoff provider has only partial-decision information regarding the strategic distribution of the entire society. Yet, the payoff providers communicate through a possibly non-complete time-invariant network in order to estimate the relevant non-local information. As such, the considered framework generalizes the conventional oracle-based approach, and grants more flexibility to the design and application scope of population games.

Problem statement: Throughout, we view each population $k \in \mathcal{P}$ as a (macro) player engaged in a game with the other populations. More precisely, we consider the scenario where each population $k \in \mathcal{P}$ seeks to reach a strategic distribution $\mathbf{x}^k \in \arg\max_{\mathbf{x}^k \in \Delta^k} \psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right)$, where $\mathbf{x}^{-k} = \mathrm{col}\left(\mathbf{x}^1, \ldots, \mathbf{x}^{k-1}, \mathbf{x}^{k+1}, \ldots, \mathbf{x}^N\right)$ denotes the strategic distribution of all populations other than k, and $\psi^k : \mathbb{R}^{n^k}_{\geq 0} \times \mathbb{R}^{n-n^k}_{\geq 0} \to \mathbb{R}$ is the local utility function of population k. Thus, the considered population game can be defined in normal form as $G = \left(\mathcal{P}, \{\Delta^k\}_{k\in\mathcal{P}}, \{\psi^k(\cdot)\}_{k\in\mathcal{P}}\right)$. The key issue to keep in mind, however, is that the actual decision-making agents involved in the game G are not the populations per se, but the agents that comprise each population.

Throughout, we impose the following assumption on the utility functions of the game G.

Standing Assumption 1: For all $k \in \mathcal{P}$, it holds that

i) $\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right)$ is concave and twice continuously differentiable in $\mathbf{x}^k$.

ii) $\nabla_{\mathbf{x}^k}\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \mathbf{g}^k\left(\mathbf{x}^k\right) - \mathbf{C}^{k\top}\mathbf{C}\mathbf{x}$, where $\mathbf{g}^k : \mathbb{R}^{n^k}_{\geq 0} \to \mathbb{R}^{n^k}$ is contractive in the sense that $\left(\mathbf{x}^k - \mathbf{y}^k\right)^\top\left(\mathbf{g}^k\left(\mathbf{x}^k\right) - \mathbf{g}^k\left(\mathbf{y}^k\right)\right) \leq 0$, for all $\mathbf{x}^k, \mathbf{y}^k \in \mathbb{R}^{n^k}_{\geq 0}$, and the matrix $\mathbf{C} \in \mathbb{R}^{d\times n}$ is of the form $\mathbf{C} = \left[\mathbf{C}^1, \mathbf{C}^2, \ldots, \mathbf{C}^N\right]$, with $\mathbf{C}^k \in \mathbb{R}^{d\times n^k}$ and $d \in \mathbb{Z}_{\geq 1}$. Moreover, both $\mathbf{g}^k(\cdot)$ and $\mathbf{C}^k$ are known by the payoff provider of population k.

It is worth to highlight that population games satisfying Standing Assumption 1 arise in various scenarios. For the sake of illustration, here we provide two examples.

Example 1 (Games with coupled concave quadratic potentials): In these games, the utility functions are of the form $\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \bar{\psi}^k\left(\mathbf{x}^k\right) - (1/2)\mathbf{x}^\top\mathbf{Q}\mathbf{x}$, where the map $\bar{\psi}^k : \mathbb{R}^{n^k}_{\geq 0} \to \mathbb{R}$ is concave and twice continuously differentiable, and $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite, for all $k \in \mathcal{P}$ (observe that every concave quadratic optimization problem subject to $\mathbf{x} \in \Delta$ can be seen as a game of this form). For these games, one can set $\mathbf{C} = \sqrt{\mathbf{\Lambda}}\mathbf{P}^\top$, where $\mathbf{\Lambda}$ and $\mathbf{P}$ satisfy the orthonormal eigendecomposition $\mathbf{Q} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top$, so that $\mathbf{Q} = \mathbf{C}^\top\mathbf{C}$. Then, setting $\mathbf{C}^k$ as the k-th block of $\mathbf{C}$ (c.f., Standing Assumption 1-ii)), it follows that $\nabla_{\mathbf{x}^k}\psi^k\left(\mathbf{x}^{-k}, \mathbf{x}\right) = \mathbf{g}^k\left(\mathbf{x}^k\right) - \mathbf{C}^{k\top}\mathbf{C}\mathbf{x}$, with $\mathbf{g}^k\left(\mathbf{x}^k\right) = \nabla_{\mathbf{x}^k}\bar{\psi}^k\left(\mathbf{x}^k\right)$, for all $k \in \mathcal{P}$.

Example 2 (Allocation games under affine congestion costs): In these games, there is a total of d possible locations and the agents of each population $k \in \mathcal{P}$ seek to distribute themselves over a subset of $n^k \leq d$ locations. Here, one can set $\mathbf{C}^k$ to define a bipartite graph between population k and the d locations. For instance, if there is a total of $d = 3$ locations and the agents of population k are allowed to choose only locations 1 and 3, then $\mathbf{C}^k = \mathbf{D}^k\left[\mathbf{e}_1, \mathbf{e}_3\right]$, where $\mathbf{D}^k \in \mathbb{R}^{d\times d}$ is a diagonal weighting matrix with positive diagonal entries, and $\mathbf{e}_1, \mathbf{e}_3$ are the first and third columns of the 3×3 identity matrix, respectively. Thus, $\mathbf{C}\mathbf{x} \in \mathbb{R}^d_{\geq 0}$ provides the overall allocation of all agents over all locations (here $\mathbf{C}$ is constructed from the $\mathbf{C}^k$ matrices as in Standing Assumption 1-ii)). Now, the locations are characterized by affine congestion costs given by the map $\mathbf{J}\left(\mathbf{C}\mathbf{x}\right) = \mathbf{C}\mathbf{x} - \bar{\mathbf{J}}$, with $\bar{\mathbf{J}} \in \mathbb{R}^d$. Namely, $\mathbf{J}(\mathbf{C}\mathbf{x}) \in \mathbb{R}^d$ provides the congestion cost for each of the d locations under the allocation $\mathbf{C}\mathbf{x}$. Consequently, the utility function for each population $k \in \mathcal{P}$ is given by $\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \bar{\psi}^k\left(\mathbf{x}^k\right) - \left(\mathbf{J}\left(\mathbf{C}\mathbf{x}\right)\right)^\top\mathbf{C}^k\mathbf{x}^k$, where $\bar{\psi}^k : \mathbb{R}^{n^k}_{\geq 0} \to \mathbb{R}$ is a concave and twice continuously differentiable local utility function for population k. Thus, $\nabla_{\mathbf{x}^k}\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \mathbf{g}^k\left(\mathbf{x}^k\right) - \mathbf{C}^{k\top}\mathbf{C}\mathbf{x}$, with $\mathbf{g}^k\left(\mathbf{x}^k\right) = \nabla_{\mathbf{x}^k}\bar{\psi}^k\left(\mathbf{x}^k\right) + \mathbf{C}^{k\top}\bar{\mathbf{J}} - \mathbf{C}^{k\top}\mathbf{C}^k\mathbf{x}^k$, for all $k \in \mathcal{P}$. We highlight that these allocation games can be used to model several practical decision-making scenarios. As an example, consider the charging coordination of N plug-in electric vehicles (PEVs) over a horizon of d time-slots. In such a context, each population $k \in \mathcal{P}$ would represent a PEV, m^k would be the total energy to be charged to PEV k, $\mathcal{S}^k$ would be the charging time-slots available to PEV k, $\mathbf{x}^k$ would be the scheduled charging profile of PEV k, and $\mathbf{J}\left(\mathbf{C}\mathbf{x}\right)$ would represent the price of electricity under the allocation $\mathbf{C}\mathbf{x}$.

Based on the considered framework, the technical problem that we study in this paper is the evolutionary convergence to a Nash equilibrium (NE) of the game G, as defined next.

Definition 2: An NE of the game G is a strategic distribution $\mathbf{x}^* = \mathrm{col}\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \ldots, \mathbf{x}^{N*}\right) \in \Delta$ such that $\psi^k\left(\mathbf{x}^{k*}, \mathbf{x}^{-k*}\right) \geq \psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k*}\right), \quad \forall \mathbf{x}^k \in \Delta^k$, for all $k \in \mathcal{P}$. Here, $\mathbf{x}^{-k*} = \mathrm{col}\left(\mathbf{x}^{1*}, \ldots, \mathbf{x}^{(k-1)*}, \mathbf{x}^{(k+1)*}, \ldots, \mathbf{x}^{N*}\right)$.

By Standing Assumption 1-i), it holds that an NE of the game G exists (see Lemma 1). Moreover, according to the EDM, the only way to steer the strategic distributions $\mathbf{x}^k(t)$ towards an NE of the game G is through the payoff signals $\mathbf{p}^k(t)$. Hence, the task is to design appropriate payoff providers to guarantee the evolutionary convergence of the strategic distribution $\mathbf{x}(t)$ to an NE of the game G.

As mentioned above and shown in Fig. 1, we consider that the multiple payoff providers communicate through a possibly non-complete time-invariant network. More formally, we let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{W})$ be the directed graph (digraph) of the network, where $\mathcal{P}$ is the set of nodes corresponding to the payoff providers, $\mathcal{L} \subseteq \{(k,\ell) : k,\ell \in \mathcal{P}\}$ is the set of communication links, and $\mathbf{W} \in \mathbb{R}^{N\times N}_{\geq 0}$ is the corresponding weighted adjacency matrix. Here, we let $w_{k\ell} \in \mathbb{R}_{\geq 0}$ be the (k,ℓ)-th element of $\mathbf{W}$, and it is assumed that $w_{k\ell} > 0$ if and only if node k can receive information from node ℓ (by convention $w_{kk} = 0$). In addition, we let $\mathcal{N}^k = \{\ell \in \mathcal{P} : w_{k\ell} > 0\}$ be the set of in-neighbors of node k, for all $k \in \mathcal{P}$, and we impose the following assumptions.

Standing Assumption 2: The digraph $\mathcal{G}$ is strongly connected and weight-balanced.

Standing Assumption 3: All the payoff providers know the total number of populations N. Also, for all $k \in \mathcal{P}$, the k-th payoff provider knows the k-th row and column of $\mathbf{W}$.

Contributions of this paper: Based on the considered framework, the main contribution of this paper is the design and formulation of appropriate payoff providers (synthesized as continuous-time dynamical systems that comprise a so-called payoff dynamics model [2]), which effectively steer the strategic distribution of the entire society to an NE of the population game G under partial-decision information constraints ruled by the digraph $\mathcal{G}$. Namely, our proposed approach solves the distributed NE seeking problem for the game G under the digraph $\mathcal{G}$. To the best of our knowledge,

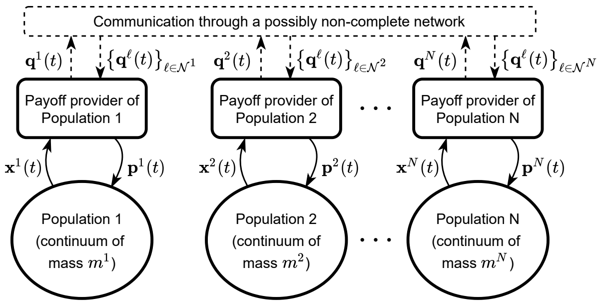

Fig. 1. Considered framework. Note that the payoff providers exchange certain auxiliary variables $\mathbf{q}^k(t)$ (defined in Section II) by communicating with their neighbors over a possibly non-complete network.

this is the first paper to address such a distributed NE seeking problem from the considered perspective of population games and evolutionary dynamics. As the main technical contribution, we formally prove the asymptotic stability of the set of Nash equilibria of the game G under the proposed continuous-time dynamics. Moreover, to validate our results, we provide a numerical illustration regarding an allocation game as the one in Example 2.

Related work: The problem of distributed NE seeking has been recently studied from various perspectives. Perhaps, the most popular one regards the combination of gradient play and consensus algorithms. Namely, a gradient-based mechanism is used to update the players' actions, and consensus methods are applied to estimate the joint action profile of all players. Such a scheme has been recently considered both in discrete time [6], [7] and continuous time [8], [9]. Similarly, in this paper we design each payoff provider so that $\mathbf{p}^k(t)$ corresponds to an estimate of $\nabla_{\mathbf{x}^k}\psi^k\left(\mathbf{x}^k(t), \mathbf{x}^{-k}(t)\right)$, for all $k \in \mathcal{P}$. Moreover, to handle the partial-decision information scenario, we ascribe the payoff providers within the so-called proportional integral consensus algorithm [10], [11] (c.f., Section II). Our considered problem is closely related to the contexts of N-coalition games [12] and multi-population mean-field games [13]. Namely, in N-coalition games a group of decision-making agents is split into multiple coalitions, each of them behaving as a virtual (macro) player engaged in a non-cooperative game with the other coalitions. While such a setup resembles our considered framework, the perspective of population games is more suitable when considering a large number of agents that make decisions in a sporadic and stochastic manner. On the other hand, multi-population mean-field games regard a large society of non-cooperative players that optimally respond to a mean-field signal. To reduce the communication burden, the players are partitioned into multiple populations and the so-called population coordinators employ a consensus-like protocol to estimate the mean-field signal using partial-decision information. In contrast with such a context, in our framework the population agents are payoff-driven decision-makers that choose strategies based on quite simple pairwise comparison protocols, and thus are allowed to have significantly bounded rationality levels. Nevertheless, similar to the population coordinators, our payoff providers also employ a consensus-like method to estimate the relevant non-local information to compute the payoff vectors. Finally, it is worth to highlight that even though some non-complete interaction scenarios have been considered in the context of NE seeking in population games, e.g., [14], [15], to the best of our knowledge all of these approaches have only considered the scenario with a centralized oracle-like payoff provider which has complete information regarding the strategic distribution of the entire society. Therefore, such approaches are not applicable to the distributed NE seeking problem under partial-decision information that is studied in this paper.

II. Proposed Approach

In this section, we formulate our proposed approach for distributed NE seeking under the framework of Section I.

As mentioned above, we formulate the payoff provider of each population $k \in \mathcal{P}$ as a dynamical system whose output $\mathbf{p}^k(t)$ corresponds to an estimate of $\nabla_{\mathbf{x}^k}\psi^k\left(\mathbf{x}^k(t), \mathbf{x}^{-k}(t)\right)$. Furthermore, to cope with the considered partial-decision information scenario, in this paper we ascribe the payoff providers within the so-called proportional integral consensus algorithm [10], [11]. More precisely, we let the payoff provider of each population $k \in \mathcal{P}$ be characterized by

$$\begin{aligned}\dot{\boldsymbol{\mu}}^k(t) = &-\boldsymbol{\mu}^k(t) - \sum_{\ell\in\mathcal{P}} w_{k\ell}\left(\boldsymbol{\mu}^k(t) - \boldsymbol{\mu}^\ell(t)\right)\\ &- \sum_{\ell\in\mathcal{P}} w_{\ell k}\Big(\boldsymbol{\lambda}^k(t) - \boldsymbol{\lambda}^\ell(t)\Big) + \mathbf{C}^k\mathbf{x}^k(t), \end{aligned} \tag{2a}$$

$$\dot{\boldsymbol{\lambda}}^k(t) = \sum_{\ell\in\mathcal{P}} w_{k\ell}\left(\boldsymbol{\mu}^k(t) - \boldsymbol{\mu}^\ell(t)\right), \tag{2b}$$

$$\mathbf{p}^k(t) = \mathbf{g}^k\left(\mathbf{x}^k(t)\right) - N\mathbf{C}^{k\top}\boldsymbol{\mu}^k(t), \tag{2c}$$

where $\boldsymbol{\mu}^k(t), \boldsymbol{\lambda}^k(t) \in \mathbb{R}^d$. Under such a model, it follows that the information exchange between the different payoff providers regards only the vectors $\mathbf{q}^k(t) = \mathrm{col}\Big(\boldsymbol{\mu}^k(t), \boldsymbol{\lambda}^k(t)\Big) \in \mathbb{R}^{2d}$, and such information exchange is ruled by the digraph $\mathcal{G}$ (see Fig. 1). Hence, under Standing Assumptions 1, 2, and 3, the dynamics in (2) can be computed locally at every payoff provider $k \in \mathcal{P}$.

We highlight that the collection of all payoff providers comprise a dynamical system, with input $\mathbf{x}(t)$ and output $\mathbf{p}(t)$, here referred to as the payoff dynamics model (PDM).

Definition 3: The PDM that describes the temporal evolution of the overall payoff vector $\mathbf{p}(t)$ is given by

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{\mu}}(t)\\ \dot{\boldsymbol{\lambda}}(t)\end{bmatrix} = &\begin{bmatrix}-\mathbf{I}_{Nd} - \mathbf{L}\otimes\mathbf{I}_d, & -\mathbf{L}^\top\otimes\mathbf{I}_d\\ \mathbf{L}\otimes\mathbf{I}_d, & \mathbf{0}_{Nd\times Nd}\end{bmatrix}\begin{bmatrix}\boldsymbol{\mu}(t)\\ \boldsymbol{\lambda}(t)\end{bmatrix}\\ &+\begin{bmatrix}\mathbf{E}\\ \mathbf{0}_{Nd\times n}\end{bmatrix}\mathbf{x}(t),\end{aligned} \tag{3a}$$

$$\mathbf{p}(t) = \mathbf{g}\left(\mathbf{x}(t)\right) - N\mathbf{E}^\top\boldsymbol{\mu}(t), \tag{3b}$$

with $\boldsymbol{\mu}(0), \boldsymbol{\lambda}(0) \in \mathbb{R}^{Nd}$, where $\otimes$ denotes the Kronecker product, $\mathbf{I}_a$ is the $a\times a$ identity matrix, $\mathbf{0}_{a\times b}$ is the $a\times b$ matrix of zeros, $\mathbf{L} \in \mathbb{R}^{N\times N}$ is the Laplacian matrix of the digraph $\mathcal{G}$, and

$$\begin{aligned}\mathbf{E} &= \mathrm{diag}\left(\mathbf{C}^1, \mathbf{C}^2, \ldots, \mathbf{C}^N\right) \in \mathbb{R}^{Nd\times n},\\ \boldsymbol{\mu}(t) &= \mathrm{col}\left(\boldsymbol{\mu}^1(t), \boldsymbol{\mu}^2(t), \ldots, \boldsymbol{\mu}^N(t)\right) \in \mathbb{R}^{Nd},\\ \boldsymbol{\lambda}(t) &= \mathrm{col}\left(\boldsymbol{\lambda}^1(t), \boldsymbol{\lambda}^2(t), \ldots, \boldsymbol{\lambda}^N(t)\right) \in \mathbb{R}^{Nd},\\ \mathbf{g}\left(\mathbf{x}(t)\right) &= \mathrm{col}\left(\mathbf{g}^1\left(\mathbf{x}^1(t)\right), \ldots, \mathbf{g}^N\left(\mathbf{x}^N(t)\right)\right) \in \mathbb{R}^n.\end{aligned}$$

Here, $\mathrm{diag}(\cdot)$ is the block diagonal matrix stack operation.

Under the considered framework, the EDM and PDM systems are interconnected in a feedback loop where the EDM takes as input the overall payoff vector $\mathbf{p}(t)$ and provides as output the strategic distribution $\mathbf{x}(t)$, while the PDM takes $\mathbf{x}(t)$ as input and provides $\mathbf{p}(t)$ as output. Throughout, we refer to such an interconnected system as the EDM-PDM system, and our main theoretical result is summarized next.

Theorem 1: The set of Nash equilibria of the game G is asymptotically stable under the EDM-PDM system.

We now proceed to formally prove Theorem 1.

III. Proof of Theorem 1

To start the discussion, we first characterize the set of Nash equilibria of G.

Lemma 1: Consider the pseudo-gradient given by

$$\mathbf{f}(\mathbf{x}) = \mathrm{col}\left(\nabla_{\mathbf{x}^1}\psi^1\left(\mathbf{x}^1, \mathbf{x}^{-1}\right), \cdots, \nabla_{\mathbf{x}^N}\psi^N\left(\mathbf{x}^N, \mathbf{x}^{-N}\right)\right),$$

and the set $\mathrm{NE}\left(\mathbf{f}\right) = \left\{\mathbf{x} \in \Delta : \mathbf{x} \in \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top \mathbf{f}\left(\mathbf{x}\right)\right\}$. A strategic distribution $\mathbf{x}^* \in \Delta$ is an NE of the game G if and only if $\mathbf{x}^* \in \mathrm{NE}\left(\mathbf{f}\right)$. Moreover, the set $\mathrm{NE}\left(\mathbf{f}\right)$ is nonempty and compact. Furthermore, if $\mathbf{f}(\cdot)$ is strictly contractive in the sense that $\left(\mathbf{x} - \mathbf{y}\right)^\top \left(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\right) < 0$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\geq 0}$ with $\mathbf{x} \neq \mathbf{y}$, then there exists a unique $\mathbf{x}^* \in \mathrm{NE}(\mathbf{f})$.

Proof: The proof follows immediately from [16, Proposition 1.4.2] and [16, Corollary 2.2.5] by considering the variational inequality $\mathrm{VI}\left(\Delta, -\mathbf{f}(\cdot)\right)$. Similarly, the uniqueness claim follows from [16, Theorem 2.3.3]. ■

Now, we proceed to highlight some properties of the EDM.

Lemma 2: The set Δ is positively invariant under the EDM, i.e., $\mathbf{x}(0) \in \Delta \Rightarrow \mathbf{x}(t) \in \Delta$, for all $t \geq 0$.

Proof: The proof follows from [17, Proposition 1]. ■

Lemma 3: Consider the EDM. Then, $\dot{\mathbf{x}}(t) = \mathbf{0}_n$ if and only if $\mathbf{x}(t) \in \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top \mathbf{p}(t)$. Here, $\mathbf{0}_n \in \mathbb{R}^n$ is the vector of n zeros.

Proof: By Lemma 2, $\mathbf{x}(t) \in \Delta \subset \mathbb{R}^n_{\geq 0}$, for all $t \geq 0$.

(Sufficiency) Suppose that $\mathbf{x}(t) \in \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top \mathbf{p}(t)$. Then, $x_i^k(t) > 0 \Rightarrow p_i^k(t) \geq p_j^k(t)$, for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Hence, from (1), $x_i^k(t)\rho_{ij}^k(t) = 0$, for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, and $\dot{x}_i^k(t) = 0$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$.

(Necessity) Suppose that $\mathbf{x}(t) \notin \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top \mathbf{p}(t)$. Then, there are some $\ell \in \mathcal{P}$ and $i \in \mathcal{S}^\ell$ such that $x_i^\ell(t) > 0$ and $p_i^\ell(t) < \max_{j\in\mathcal{S}^\ell} p_j^\ell(t)$. Now, let $z \in \mathcal{S}^\ell$ be such that $p_z^\ell(t) = \max_{j\in\mathcal{S}^\ell} p_j^\ell(t)$. It follows that $x_z^\ell(t)\rho_{zj}^\ell(t) = 0$, for all $j \in \mathcal{S}^\ell$, and thus $\dot{x}_z^\ell(t) \geq 0$. In fact, by construction we have that $x_i^\ell(t)\rho_{iz}^\ell(t) > 0$, which implies that $\dot{x}_z^\ell(t) > 0$. Therefore, $\mathbf{x}(t) \notin \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top \mathbf{p}(t)$ implies that $\dot{\mathbf{x}}(t) \neq \mathbf{0}_n$. ■

Namely, Lemma 2 provides some invariance properties of the EDM, and Lemma 3 characterizes the set of equilibria of the EDM as a function of the overall payoff vector $\mathbf{p}(t)$.

To analyze the EDM-PDM system, we reformulate the PDM in an equivalent form as follows. Take $\mathbf{v} = (1/\sqrt{N})\mathbf{1}_N$, where $\mathbf{1}_N \in \mathbb{R}^N$ is the vector of N ones (hence $\mathbf{L}\mathbf{v} = \mathbf{0}_N$ and $\mathbf{v}^\top\mathbf{L} = \mathbf{0}_N^\top$), and let $\mathbf{U} = \left[\tilde{\mathbf{U}}, \mathbf{v}\right] \in \mathbb{R}^{N\times N}$ be an orthonormal matrix. Using such a matrix $\mathbf{U}$, we replace $\boldsymbol{\lambda}(t)$ in (3a) by $\boldsymbol{\lambda}(t) = \left(\mathbf{U}\otimes\mathbf{I}_d\right)\begin{bmatrix}\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)\\ \boldsymbol{\lambda}_{\mathbf{v}}(t)\end{bmatrix}$, with $\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t) \in \mathbb{R}^{Nd-d}$ and $\boldsymbol{\lambda}_{\mathbf{v}}(t) \in \mathbb{R}^d$. Thus, by the identities $\left(\mathbf{U}\otimes\mathbf{I}_d\right)^{-1} = \mathbf{U}^{-1}\otimes\mathbf{I}_d$ and $\mathbf{U}^{-1} = \mathbf{U}^\top$, one obtains that $\begin{bmatrix}\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}(t)\\ \dot{\boldsymbol{\lambda}}_{\mathbf{v}}(t)\end{bmatrix} = \left(\mathbf{U}^\top\otimes\mathbf{I}_d\right)\dot{\boldsymbol{\lambda}}(t)$. Noting that $\dot{\boldsymbol{\lambda}}(t) = \left(\mathbf{L}\otimes\mathbf{I}_d\right)\boldsymbol{\mu}(t)$ [c.f., (3a)], and applying the mixed-product property $\left(\mathbf{M}_1\otimes\mathbf{M}_2\right)\left(\mathbf{M}_3\otimes\mathbf{M}_4\right) = \left(\mathbf{M}_1\mathbf{M}_3\right)\otimes\left(\mathbf{M}_2\mathbf{M}_4\right)$ (whenever $\mathbf{M}_1\mathbf{M}_3$ and $\mathbf{M}_2\mathbf{M}_4$ are both valid matrix products), it follows that the dynamics in (3a) can be equivalently written as $\dot{\tilde{\mathbf{q}}}(t) = \mathbf{A}\tilde{\mathbf{q}}(t) + \mathbf{B}\mathbf{x}(t)$, with $\tilde{\mathbf{q}}(t) = \mathrm{col}\left(\boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t), \boldsymbol{\lambda}_{\mathbf{v}}(t)\right)$, and

$$\mathbf{A} = \begin{bmatrix} -\mathbf{I}_{Nd} - \mathbf{L}\otimes\mathbf{I}_d, & -\left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_d, & \mathbf{0}_{Nd\times d}\\ \left(\tilde{\mathbf{U}}^\top\mathbf{L}\right)\otimes\mathbf{I}_d, & \mathbf{0}_{(Nd-d)\times(Nd-d)}, & \mathbf{0}_{(Nd-d)\times d}\\ \mathbf{0}_{d\times Nd}, & \mathbf{0}_{d\times(Nd-d)}, & \mathbf{0}_{d\times d}\end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{E}\\ \mathbf{0}_{(Nd-d)\times n}\\ \mathbf{0}_{d\times n}\end{bmatrix}.$$

Here, we highlight that $\boldsymbol{\lambda}_{\mathbf{v}}(t)$ comprises a state variable that does not interact with the rest of the system, and so one can eliminate $\boldsymbol{\lambda}_{\mathbf{v}}(t)$ and consider only the resulting reduced-order system. For the remainder of this section, we refer to such a reduced-order system as the reduced PDM (rPDM).

Definition 4: Let $\mathbf{A}_r \in \mathbb{R}^{(2Nd-d)\times(2Nd-d)}$ be the upper-left block of the matrix $\mathbf{A}$, and let $\mathbf{B}_r \in \mathbb{R}^{(2Nd-d)\times n}$ be the upper block of the matrix $\mathbf{B}$. The rPDM is given by

$$\begin{bmatrix}\dot{\boldsymbol{\mu}}(t)\\ \dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}(t)\end{bmatrix} = \mathbf{A}_r\begin{bmatrix}\boldsymbol{\mu}(t)\\ \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)\end{bmatrix} + \mathbf{B}_r\mathbf{x}(t), \tag{4a}$$

$$\mathbf{p}(t) = \mathbf{g}\left(\mathbf{x}(t)\right) - N\mathbf{E}^\top\boldsymbol{\mu}(t), \tag{4b}$$

with $\boldsymbol{\mu}(0) \in \mathbb{R}^{Nd}$ and $\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(0) \in \mathbb{R}^{Nd-d}$.

The following are key properties of the rPDM.

Lemma 4: The matrix $\mathbf{A}_r$ is Hurwitz.

Proof: See [10, Lemma 9] or [11, Lemma 2.2]. ■

Lemma 5: (Adapted from [10, Theorem 5]) Consider the rPDM under a constant input $\mathbf{x}(t) = \mathbf{x}^* \in \mathbb{R}^n$, for all $t \geq 0$. Such a system has a unique globally exponentially stable equilibrium point $\left(\boldsymbol{\mu}^*, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}^*\right)$, with $\boldsymbol{\mu}^* = (1/N)\left(\mathbf{1}_N\otimes\mathbf{C}\mathbf{x}^*\right)$.

Proof: The proof follows from [10, Theorem 5]. ■

Lemma 6: Consider the rPDM under a bounded input $\mathbf{x}(t) \in \mathbb{R}^n$, for all $t \geq 0$, i.e., $\left\|\mathbf{x}(t)\right\|_2 < \infty, \forall t \geq 0$. Then,

i) $\left\|\boldsymbol{\mu}(t)\right\|_2 < \infty$ and $\left\|\boldsymbol{\lambda}(t)\right\|_2 < \infty$, for all $t \geq 0$.

ii) $\lim_{t\to\infty}\left\|\dot{\mathbf{x}}(t)\right\|_2 = 0 \Rightarrow \lim_{t\to\infty}\left\|\mathbf{p}(t) - \mathbf{f}\left(\mathbf{x}(t)\right)\right\|_2 = 0$, where $\mathbf{f} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is the map defined in Lemma 1.

Here, $\|\cdot\|_2$ is the Euclidean norm.

Proof: i) From Lemma 5 and [18, Lemma 4.6], we conclude that the dynamics in (4a) are input-to-state stable. Thus, under any bounded input $\mathbf{x}(t) \in \mathbb{R}^n$ it holds that $\|\boldsymbol{\mu}(t)\|_2 < \infty$ and $\|\boldsymbol{\lambda}(t)\|_2 < \infty$, for all $t \geq 0$.

ii) From Lemma 5, it holds that $\lim_{t\to\infty} \|\dot{\mathbf{x}}(t)\|_2 = 0$ implies that $\lim_{t\to\infty} \|\boldsymbol{\mu}(t) - (1/N)(\mathbf{1}_N \otimes \mathbf{C}\mathbf{x}(t))\|_2 = 0$. Observing that $N\mathbf{E}^\top ((1/N)(\mathbf{1}_N \otimes \mathbf{C}\mathbf{x}(t))) = \mathbf{C}^\top\mathbf{C}\mathbf{x}(t)$ leads to the desired result $\mathbf{p}(t) \to \mathbf{f}(\mathbf{x}(t))$. ■

Remark 1: Following the same argument in [11, Lemma 2.1], to study the EDM-PDM system we can equivalently analyze the EDM-rPDM system that results from the feedback interconnection between the EDM and the rPDM (i.e., the EDM takes $\mathbf{p}(t)$ as input and provides $\mathbf{x}(t)$ as output, and the rPDM takes $\mathbf{x}(t)$ as input and provides $\mathbf{p}(t)$ as output). Hence, without loss of generality, we analyze the EDM-rPDM system in the remainder of this section.

Based on the above results, we now characterize the set of equilibria of the EDM-rPDM system.

Proposition 1: Consider the EDM-rPDM system, and the map $V : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^{Nd} \times \mathbb{R}^{Nd-d} \to \mathbb{R}_{\geq 0}$ given by $V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) = V_1(\mathbf{x}, \boldsymbol{\mu}) + V_2(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}})$, with

$$V_1(\mathbf{x}, \boldsymbol{\mu}) = \sum_{k\in\mathcal{P}} \sum_{j\in\mathcal{S}^k} \sum_{i\in\mathcal{S}^k} x_i^k \int_0^{p_j^k - p_i^k} [\sigma]_+ \, d\sigma,$$

$$V_2(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) = \frac{N}{2} \left\| \mathbf{A}_r \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{U}}} \end{bmatrix} + \mathbf{B}_r\mathbf{x} \right\|_2^2,$$

where $p_i^k \triangleq p_i^k(\mathbf{x}^k, \boldsymbol{\mu}^k)$ is defined as the i-th element of the vector $\mathbf{p}^k(\mathbf{x}^k, \boldsymbol{\mu}^k) = \mathbf{g}^k(\mathbf{x}^k) - N\mathbf{C}^{k\top}\boldsymbol{\mu}^k$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Furthermore, consider the set $\mathcal{E} = \{(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) \in \Delta \times \mathbb{R}^{Nd} \times \mathbb{R}^{Nd-d} : V(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) = 0\}$. Then,

i) $\mathcal{E}$ is the set of equilibria of the EDM-rPDM system.
ii) If $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)) \in \mathcal{E}$, then $\mathbf{x}(t) \in \mathrm{NE}(\mathbf{f})$.
iii) $\mathcal{E}$ is nonempty and compact.
iv) $\mathcal{E}$ is asymptotically stable.

Proof: i) From [1, Theorem 7.2.9], we conclude that $V_1(\mathbf{x}(t), \boldsymbol{\mu}(t)) = 0 \Leftrightarrow \mathbf{x}(t) \in \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top\mathbf{p}(t)$. Thus, from Lemma 3, $V_1(\mathbf{x}(t), \boldsymbol{\mu}(t)) = 0 \Leftrightarrow \dot{\mathbf{x}}(t) = \mathbf{0}_n$. Similarly, from (4a) it holds that $V_2(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)) = 0$ if and only if $\dot{\boldsymbol{\mu}}(t) = \mathbf{0}_{Nd}$ and $\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}(t) = \mathbf{0}_{Nd-d}$.

ii) From i) and Lemmas 3 and 6-ii), we conclude that $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)) \in \mathcal{E} \Rightarrow \mathbf{x}(t) \in \arg\max_{\mathbf{y}\in\Delta} \mathbf{y}^\top\mathbf{p}(t)$ with $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t))$. Therefore, from Lemma 1, $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)) \in \mathcal{E} \Rightarrow \mathbf{x}(t) \in \mathrm{NE}(\mathbf{f})$.

iii) First, we prove the nonemptiness of $\mathcal{E}$. From Lemma 1, $\mathrm{NE}(\mathbf{f}) \neq \emptyset$. Thus, pick an arbitrary $\mathbf{x}^* \in \mathrm{NE}(\mathbf{f})$. Now, consider the rPDM under the constant input $\mathbf{x}(t) = \mathbf{x}^*$, for all $t \geq 0$. From Lemma 5, it holds that such an rPDM converges exponentially to $(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}})$ with $\boldsymbol{\mu}^* = (1/N)(\mathbf{1}_N \otimes \mathbf{C}\mathbf{x}^*)$. By construction, $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}}) \in \mathcal{E}$, and thus $\mathcal{E} \neq \emptyset$.

Now, we prove the compactness of $\mathcal{E}$. First, note that from the definition of $\mathcal{E}$ it follows that $\mathcal{E}$ is the preimage of the closed set $\{0\}$ under the continuous map $V(\cdot)$. Hence, $\mathcal{E}$ is closed. On the other hand, to prove the boundedness of $\mathcal{E}$, pick an arbitrary $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}}) \in \mathcal{E}$. By the definition of $\mathcal{E}$, $\mathbf{x}^* \in \Delta$ and so $\mathbf{x}^*$ is bounded, i.e., $\|\mathbf{x}^*\|_2 < \infty$. In contrast, from i) and Lemma 4, $\begin{bmatrix} \boldsymbol{\mu}^* \\ \boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}} \end{bmatrix} = -\mathbf{A}_r^{-1}\mathbf{B}_r\mathbf{x}^*$. Therefore, since $\mathbf{x}^*$ is bounded, it holds that $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}}$ are bounded as well. Since the considered $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*_{\tilde{\mathbf{U}}}) \in \mathcal{E}$ is arbitrary, we conclude that $\mathcal{E}$ must be bounded.

iv) From i), it follows that $V(\cdot)$ is a valid Lyapunov function candidate to analyze the stability properties of $\mathcal{E}$ [19, Corollary 4.7]. Throughout, let $\mathbf{x} \triangleq \mathbf{x}(t)$, $\boldsymbol{\mu} \triangleq \boldsymbol{\mu}(t)$, and $\boldsymbol{\lambda}_{\tilde{\mathbf{U}}} \triangleq \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)$, for all $t \geq 0$, and let $V_1 \triangleq V_1(\mathbf{x}, \boldsymbol{\mu})$, and $P_{ij}^k \triangleq \int_0^{p_j^k - p_i^k} [\sigma]_+ d\sigma$, for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Then,

$$\frac{\partial V_1}{\partial x_s^z} = \sum_{j\in\mathcal{S}^z} P_{sj}^z + \sum_{j\in\mathcal{S}^z}\sum_{i\in\mathcal{S}^z} x_i^z \left[p_j^z - p_i^z\right]_+ \left(\frac{\partial p_j^z}{\partial x_s^z} - \frac{\partial p_i^z}{\partial x_s^z}\right)$$
$$= \sum_{j\in\mathcal{S}^z} P_{sj}^z + \sum_{j\in\mathcal{S}^z} \dot{x}_j^z \frac{\partial p_j^z}{\partial x_s^z}, \quad \text{[using (1)]},$$

for all $s \in \mathcal{S}^z$ and all $z \in \mathcal{P}$. Similarly, by letting μ_a^z be the a-th entry of $\boldsymbol{\mu}^z$, for all $a = 1, 2, \ldots, d$, it follows that $\partial V_1/\partial \mu_a^z = \sum_{j\in\mathcal{S}^z} \dot{x}_j^z \partial p_j^z/\partial \mu_a^z$. Hence, $\nabla_{\mathbf{x}} V_1 = \boldsymbol{\Gamma} + (\mathbf{D}\mathbf{g}(\mathbf{x}))^\top \dot{\mathbf{x}}$ and $\nabla_{\boldsymbol{\mu}} V_1 = -N\mathbf{E}\dot{\mathbf{x}}$. Here, $\boldsymbol{\Gamma} = \mathrm{col}\left(\sum_{j\in\mathcal{S}^1} P_{1j}^1, \sum_{j\in\mathcal{S}^1} P_{2j}^1, \ldots, \sum_{j\in\mathcal{S}^N} P_{n^N j}^N\right) \in \mathbb{R}^n$, and $\mathbf{D}\mathbf{g}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of $\mathbf{g}(\cdot)$ at $\mathbf{x}$. On the other hand, let $V_2 \triangleq V_2(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}})$, and note that

$$\nabla_{\mathbf{x}} V_2 = N\mathbf{B}_r^\top \left[\mathbf{A}_r \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{U}}} \end{bmatrix} + \mathbf{B}_r\mathbf{x}\right],$$

$$\nabla_{\boldsymbol{\mu}} V_2 = N \begin{bmatrix} -\mathbf{I}_{Nd} - \mathbf{L}\otimes\mathbf{I}_d \\ \left(\tilde{\mathbf{U}}^\top\mathbf{L}\right)\otimes\mathbf{I}_d \end{bmatrix}^\top \left[\mathbf{A}_r \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{U}}} \end{bmatrix} + \mathbf{B}_r\mathbf{x}\right],$$

$$\nabla_{\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}} V_2 = N \begin{bmatrix} -\left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_d \\ \mathbf{0}_{(Nd-d)\times(Nd-d)} \end{bmatrix}^\top \left[\mathbf{A}_r \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{U}}} \end{bmatrix} + \mathbf{B}_r\mathbf{x}\right].$$

Therefore, from (4) it holds that: $\nabla_{\mathbf{x}} V_2 = N\mathbf{E}^\top\dot{\boldsymbol{\mu}}$, $\nabla_{\boldsymbol{\mu}} V_2 = -N(\mathbf{I}_{Nd} + \mathbf{L}\otimes\mathbf{I}_d)^\top \dot{\boldsymbol{\mu}} + N\left(\left(\tilde{\mathbf{U}}^\top\mathbf{L}\right)\otimes\mathbf{I}_d\right)^\top \dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}$, and $\nabla_{\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}} V_2 = -N\left(\left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_d\right)^\top \dot{\boldsymbol{\mu}}$. Thus,

$$\frac{dV}{dt} = \nabla_{\mathbf{x}} V^\top\dot{\mathbf{x}} + \nabla_{\boldsymbol{\mu}} V^\top\dot{\boldsymbol{\mu}} + \nabla_{\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}} V^\top\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}$$
$$= \boldsymbol{\Gamma}^\top\dot{\mathbf{x}} + \dot{\mathbf{x}}^\top\mathbf{D}\mathbf{g}(\mathbf{x})\dot{\mathbf{x}} - N\dot{\boldsymbol{\mu}}^\top\dot{\boldsymbol{\mu}} - N\dot{\boldsymbol{\mu}}^\top(\mathbf{L}\otimes\mathbf{I}_d)\dot{\boldsymbol{\mu}}.$$

Here, we have used the facts that $\dot{\boldsymbol{\mu}}^\top\mathbf{E}\dot{\mathbf{x}} = \dot{\mathbf{x}}^\top\mathbf{E}^\top\dot{\boldsymbol{\mu}}$, and that $\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}^\top\left(\left(\tilde{\mathbf{U}}^\top\mathbf{L}\right)\otimes\mathbf{I}_d\right)\dot{\boldsymbol{\mu}} = \dot{\boldsymbol{\mu}}^\top\left(\left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_d\right)\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}$.

Now, following the same analysis as in [1, Theorem 7.2.9] or [17, Theorem 1], it is straightforward to show that $\boldsymbol{\Gamma}^\top\dot{\mathbf{x}} \leq 0$, for all $t \geq 0$, and that $\boldsymbol{\Gamma}^\top\dot{\mathbf{x}} = 0$ if and only if $\dot{\mathbf{x}} = \mathbf{0}_n$ (c.f., Lemma 3). On the other hand, due to Standing Assumption 1-ii) and [1, Theorem 3.3.1], it follows that $\dot{\mathbf{x}}^\top\mathbf{D}\mathbf{g}(\mathbf{x})\dot{\mathbf{x}} \leq 0$, for all $t \geq 0$. In addition, it is clear that $-N\dot{\boldsymbol{\mu}}^\top\dot{\boldsymbol{\mu}} \leq 0$, for all $t \geq 0$, and $-N\dot{\boldsymbol{\mu}}^\top\dot{\boldsymbol{\mu}} = 0 \Leftrightarrow \dot{\boldsymbol{\mu}} = \mathbf{0}_{Nd}$. Finally, due to Standing Assumption 2, it follows that $-N\dot{\boldsymbol{\mu}}^\top(\mathbf{L}\otimes\mathbf{I}_d)\dot{\boldsymbol{\mu}} \leq 0$, for all $t \geq 0$, and $-N\dot{\boldsymbol{\mu}}^\top(\mathbf{L}\otimes\mathbf{I}_d)\dot{\boldsymbol{\mu}} = 0 \Leftrightarrow \dot{\boldsymbol{\mu}} \in \mathrm{span}(\mathbf{1}_{Nd})$. Putting all these facts together, we conclude that $dV/dt \leq 0$, for all $t \geq 0$, and

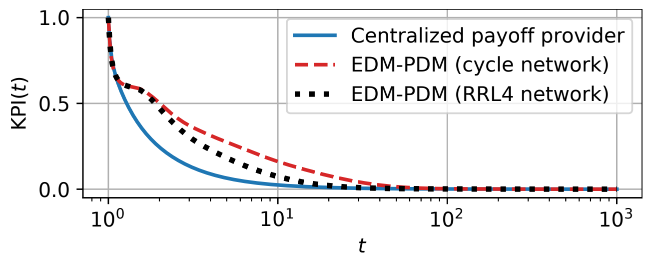

Fig. 2. Simulation of the allocation game. Without loss of generality, in all cases we let $x_i^k(0) = m^k/n^k$, and $\boldsymbol{\mu}^k(0) = \boldsymbol{\lambda}^k(0) = \mathbf{0}_d$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Moreover, the key performance index (KPI) is taken as $\text{KPI}(t) = \|\mathbf{x}(t) - \mathbf{x}^*\|_2 \,/\, \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^*$ is the NE of G.

therefore $\mathcal{E}$ is stable in the sense of Lyapunov. Furthermore, note that $dV/dt = 0 \Leftrightarrow (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) \in \mathcal{R}$, with

$$\mathcal{R} = \left\{ (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}) \in \Delta \times \mathbb{R}^{Nd} \times \mathbb{R}^{Nd-d} : \begin{array}{l} \dot{\mathbf{x}} = \mathbf{0}_n, \\ \dot{\boldsymbol{\mu}} = \mathbf{0}_{Nd} \end{array} \right\}.$$

Moreover, $\mathcal{E}$ is the largest invariant set of the EDM-rPDM system within $\mathcal{R}$. To see the latter, let $\mathcal{I} \subseteq \mathcal{R}$ be the largest invariant set of the EDM-rPDM system within $\mathcal{R}$, and let $\mathcal{T} = \mathcal{I} \backslash \mathcal{E}$. Now, suppose that $\mathcal{T} \neq \emptyset$. Then, there exists some point $(\mathbf{x}(t), \boldsymbol{\mu}(t), \boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)) \in \mathcal{T}$ such that $\dot{\mathbf{x}}(t) = \mathbf{0}_n$, $\dot{\boldsymbol{\mu}}(t) = \mathbf{0}_{Nd}$, and $\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{U}}}(t) \neq \mathbf{0}_{Nd-d}$, for all $t \geq 0$. Consequently, from (4a), at such a point $\lim_{t\to\infty} \left\|\boldsymbol{\lambda}_{\tilde{\mathbf{U}}}(t)\right\|_2 = \infty$, which is a contradiction of Lemma 6-i). Hence, $\mathcal{T}$ must be the empty set and $\mathcal{I} = \mathcal{E}$. Therefore, from LaSalle's Theorem [19, Theorem 3.3], we conclude that $\mathcal{E}$ is asymptotically stable. ■

Proposition 1 and Remark 1 lead to the desired result.

IV. An Illustrative Example

In this section, we validate our theoretical results through some numerical simulations regarding the allocation game of Example 2. Without loss of generality, we consider the setup described in Section I with $N = 10$, $d = 7$, $n^k = d$, $m^k = 1$, and $\mathbf{C}^k = \mathbf{I}_d$, for all $k \in \mathcal{P}$. Moreover, we set $\bar{\psi}^k\left(\mathbf{x}^k\right) = -\mathbf{x}^{k\top}\bar{\mathbf{Q}}^k\mathbf{x}^k - \bar{\mathbf{h}}^{k\top}\mathbf{x}^k$, where $\bar{\mathbf{Q}}^k \in \mathbb{R}^{n^k \times n^k}$ is a random symmetric and positive definite matrix with elements within $[-1, 1]$, and $\bar{\mathbf{h}}^k \in \mathbb{R}^{n^k}$ is a random vector with elements within $[-2, 2]$, for all $k \in \mathcal{P}$. Finally, we randomly sample the elements of $\bar{\mathbf{J}} \in \mathbb{R}^d$ from $[-5, 5]$. Using Lemma 1, it can be shown that, in this case, the game G has a unique NE.

In Fig. 2, we depict the simulation of the EDM-PDM system for the considered allocation game under two communication digraphs with unitary weights, a directed cycle, and a clockwise directed regular ring lattice of degree 4 (RRL4). In addition, for the sake of comparison, we also simulate the EDM under a centralized oracle-like payoff provider with full-decision information (this corresponds to the conventional framework of [1]). Clearly, in all cases the EDM effectively converges to the unique NE of the allocation game G.

V. Concluding Remarks

This paper has formulated a novel approach for distributed Nash equilibrium seeking in a class of contractive population games under partial-decision information and directed time-invariant communication networks. Using Lyapunov stability theory, we have provided sufficient conditions to guarantee the asymptotic stability of the set of Nash equilibria of the underlying game. Moreover, the theoretical results have been validated through numerical simulations of an allocation game. Future work should extend the results to more general games and evolutionary dynamics, as well as to time-varying networks.

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