Distributed Nash Equilibrium Seeking in Strongly Contractive Aggregative Population Games

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***Abstract*— This paper addresses the problem of distributed Nash equilibrium (NE) seeking in strongly contractive aggregative multi-population games subject to partial-decision information. In particular, we consider the scenario where the so-called payoff providers of the multiple populations communicate through a possibly non-complete network, and we formulate some consensus-like dynamics for the payoff providers to distributedly compute their payoff signals using local information only. Moreover, by exploiting the notions of δ-passivity and δ-antipassivity, we provide a unified analysis for several classes of evolutionary game dynamics. As the main contributions, we provide sufficient conditions to guarantee the δ-antipassivity of a class of continuous-time dynamical systems, and we exploit such results to design distributed NE seeking dynamics for strongly contractive aggregative population games, as well as for a class of merely contractive aggregative population games. To the best of our knowledge, this is the first paper to consider the problem of distributed NE seeking for such classes of population games and from a unifying passivity-based perspective.**

I. Introduction

Population games provide an evolutionary game theoretical framework to study the strategic interaction of multiple large populations of decision-making agents with bounded rationality levels [1]–[3]. Under the considered setup, the agents are payoff-driven decision makers that seek to select the strategy leading to the highest payoff. Consequently, in the context of population games one is often interested in deducing sufficient conditions for the agents to reach a Nash equilibrium (NE) of the underlying game. That is, a self-enforceable state where no agent has incentives to unilaterally deviate from her selected strategy. Namely, the motivation behind such a goal is that the convergence to an NE serves as a long-term predictor for the strategic distribution of the populations of agents.

As the number of agents is assumed to be large, the temporal evolution of the mean strategic distribution of each population can be approximated through a set of ordinary differential equations, here referred to as an evolutionary dynamics model (EDM). On the other hand, the payoff provider mechanism that generates the payoff signal is, in general, synthesized as a continuous-time dynamical system termed as the payoff dynamics model (PDM), which acts as a causal map from the populations' strategic distribution to the payoff signal [3]. Therefore, to study the convergence to an NE, one should analyze the closed-loop interconnection between the EDM and the PDM, also referred to as the EDM-PDM or the mean closed-loop system. To accomplish this goal, the authors in [3]–[6] have developed some passivity-based tools suitable for the analysis of EDM-PDMs. Namely, the authors in [4] introduce the notions of δ-passivity and δ-antipassivity, and the authors in [3] and [5] provide sufficient conditions to guarantee the convergence to an NE under EDM-PDMs comprised of a δ-passive EDM and a δ-antipassive PDM. Similarly, the authors in [6] introduce the notions of δ-dissipativity and δ-antidissipativity, and provide sufficient conditions to guarantee the convergence to an NE under EDM-PDMs comprised of a δ-dissipative EDM and a δ-antidissipative PDM. The main advantage of such passivity-based approaches is that they allow us to study several EDMs in a unified fashion. Thus, they provide a powerful toolbox to analyze various decision-making mechanisms at once.

The framework of population games is flexible enough to model several multi-agent decision-making scenarios. Some examples reported in the literature include applications to traffic assignment [6], [7], wireless networks [8], dynamic resource allocation [9], [10], and demand response [11], among others. Furthermore, some recent researches have considered networked interaction structures over the decision-making agents to further extend the scope of applications [12], [13]. However, one remaining drawback of the available theory on population games is that the payoff signal to which the agents respond is often assumed to be generated by a centralized oracle-like entity that has complete information regarding the strategic distribution of all populations. Clearly, this assumption signifies an information bottleneck that limits the flexibility and scope of application of the framework. For instance, consider the scenario where multiple populations of agents are spatially distributed over some geographical region. In such a case, it would be unpractical to regard a single payoff provider for the entire set of populations. Instead, it would be more convenient to consider multiple payoff providers that communicate over a possibly non-complete network and generate their payoff signals based on partial-decision information. Motivated by this idea, in this paper we formulate and analyze a framework for distributed NE seeking in population games under partial-decision information.

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Recently, the problem of distributed NE seeking under partial-decision information has received significant attention for classical N-player games. Typically, the reported approaches involve two components: 1) a strategy update mechanism based on local information, e.g., gradient play; and 2) a consensus-based algorithm to estimate non-local interfering information, e.g., the joint action profile of other players. For instance, the combination of gradient play and consensus methods for N-player games has been recently studied in [14] and [15] from the discrete-time perspective, as well as in [16]–[18] from the continuous-time context. Besides, such an approach has also been recently considered for N-player aggregative games in [19] and [20]. An advantage of aggregative games is that players need only to estimate the aggregate term rather than the full joint action profile. Similarly, gradient play and consensus approaches have also been proposed for distributed NE seeking in N-coalition games [21]. In the latter, multiple agents are split into N coalitions, and consensus-based methods are employed so that agents within the same coalition minimize a common objective by using partial information. On the other hand, distributed NE seeking methods have also been studied in the context of multi-population mean-field aggregative games. Namely, the authors in [22] consider the scenario where multiple large populations of players are engaged in an aggregative game. In such a context, each player responds optimally to a given mean-field signal, provided by a so-called population coordinator, while the multiple population coordinators employ a consensus-based method to compute the mean-field signal using partial-decision information. We highlight that the framework in [22] is closely related to the population games studied in this paper. However, in the evolutionary context of population games the decision-making agents follow fairly simple strategy revision rules, and thus are allowed to have significantly bounded rationality levels.

Despite the recent interest on distributed NE seeking problems under partial-decision information, such a problem has not been thoroughly studied from the aforementioned context of population games. To the best of our knowledge, such a problem has only been addressed in our previous work [23], which considered a specific class of EDMs and population games. As mentioned above, in the available literature on population games the payoff signals are often assumed to be generated by an oracle-like entity with complete information regarding the strategic distribution of all populations. In contrast, in this paper we consider the scenario where each population has an associated payoff provider subject to partial-decision information of the strategic distribution of the other populations. The multiple payoff providers communicate through a possibly non-complete network to estimate the relevant non-local information in a distributed fashion. In particular, we focus on the class of aggregative games (where the populations' payoffs are coupled through an aggregate term), and so the payoff providers need not to estimate the entire strategic distribution of all populations, but only the aggregate term. Furthermore, for the sake of generality, we focus on the classes of merely contractive and strongly contractive population games, which are relevant to study several decision-making scenarios [3], and we provide a δ-passivity-based analysis of our proposed framework, which allows us to consider a wide family of EDMs and fully generalizes the results in [23].

In summary, the contributions of this paper are threefold.

i) First, the formulation of a class of PDMs, and providing sufficient conditions to certify their δ-antipassivity property (see Theorem 1). It is worth to highlight that the considered class of PDMs is in general not captured by the so-called smoothing-anticipatory PDMs [3]–[6], which, to the best of our knowledge, are the only PDMs whose δ-antipassivity properties have been formally proven. Yet, our proposed PDM is indeed able to recover the smoothing PDM of [4] and [6] (see Example 1), as well as to model several linear systems subject to non-linear input saturation functions (see Example 2).
ii) Second, based on a Lyapunov-LaSalle argument, the establishment of sufficient conditions to guarantee the asymptotic stability of the set of equilibria of EDM-PDMs that are subject to a PDM from the aforementioned class of PDMs (see Corollary 1). We remark that, for the case where the proposed class of PDMs is considered, Corollary 1 provides alternative sufficient conditions to the ones reported in the literature (e.g., [6, Theorem 2]). Namely, Corollary 1 eliminates the need to construct a so-called informative δ-antistorage function when the proposed class of PDMs is considered and certain smoothness and contractivity conditions are satisfied.
iii) Third, the formulation of a distributed NE seeking method for aggregative population games under partial-decision information schemes. In particular, we show that the so-called proportional integral consensus algorithm [24] can also be captured by the proposed class of PDMs, and thus we exploit Corollary 1 to deduce sufficient conditions to guarantee the convergence to an NE for certain merely contractive and strongly contractive aggregative population games subject to partial-decision information, and for any Nash stationary δ-passive EDM with informative δ-storage function (see Corollary 2).

In addition, we validate our theoretical developments through some numerical simulations considering a practical scenario in the context of congestion games [25].

The remainder of this paper is organized as follows. In Section II, we introduce some preliminary concepts on population games, EDMs, and PDMs, we present the notions of δ-passive EDMs and δ-antipassive PDMs, and we formally state the problem of NE seeking in population games. In Section III, we formulate a class of PDMs and deduce sufficient conditions to certify their δ-antipassivity property. In addition, we deduce sufficient conditions for the asymptotic stability of the set of equilibria of EDM-PDMs subject to the aforementioned class of PDMs. In Section IV, we state the distributed NE seeking problem that motivates this research, and we formulate our proposed approach to solve such a problem. In Section V, we analyze the proposed distributed NE seeking method under the light of the δ-antipassivity results obtained in Section III. In particular, we deduce sufficient conditions to certify the asymptotic stability of the set of Nash equilibria under the proposed approach. In Section VI, we validate our theoretical results through some numerical simulations. Finally, Section VII concludes the paper.

II. Preliminaries

Notation: Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ are the non-negative and positive orthants of $\mathbb{R}^n$, respectively. Besides, $\mathbb{R}^{m\times n}$ is the space of $m\times n$ real matrices, and $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{\geq 1}$, and $\mathbb{Z}_{\geq 2}$ are the sets of integers greater or equal than 0, 1, and 2, respectively. We let $\text{col}\left(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\right)$ be the column vector obtained by stacking the collection of column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$. Similarly, $\text{diag}\left(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_N\right)$ denotes the block diagonal matrix, with the matrices $\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_N$ in its main diagonal. Given a symmetric matrix $\mathbf{S}$, we let $\lambda_{\max}\left(\mathbf{S}\right)$ and $\lambda_{\min}\left(\mathbf{S}\right)$ be the maximum and minimum eigenvalues of $\mathbf{S}$, respectively, and we say that $\mathbf{S}$ is negative definite if $\lambda_{\max}\left(\mathbf{S}\right) < 0$. We say that $\mathbf{S}$ is positive semi-definite if $\lambda_{\min}\left(\mathbf{S}\right) \geq 0$, and we compactly denote this fact as $\mathbf{S} \succeq 0$. Given a vector $\mathbf{v}$ and a matrix $\mathbf{M}$, we let $\left\|\mathbf{v}\right\|_\infty$, $\left\|\mathbf{v}\right\|_2$, and $\left\|\mathbf{M}\right\|_2$ denote the infinity, Euclidean, and spectral norms, respectively. In contrast, given a scalar $a \in \mathbb{R}$ and a set $\mathcal{S}$, we let $|a|$ and $|\mathcal{S}|$ denote their absolute value and cardinality, respectively. Furthermore, given a differentiable scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$, we let $\nabla_{\mathbf{x}} f\left(\mathbf{x}\right)$ denote the gradient of $f(\mathbf{x})$ with respect to $\mathbf{x}$ (we view gradients as column vectors by default). Similarly, given a differentiable vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, we let $\mathrm{D}_{\mathbf{x}}\mathbf{f}\left(\mathbf{x}\right) \in \mathbb{R}^{m\times n}$ denote the $m\times n$ Jacobian matrix of $\mathbf{f}(\mathbf{x})$ with respect to $\mathbf{x}$ (namely, if $f_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th component of $\mathbf{f}(\cdot)$, then the i-th row of $\mathrm{D}_{\mathbf{x}}\mathbf{f}\left(\mathbf{x}\right)$ is equal to $\nabla_{\mathbf{x}} f_i(\mathbf{x})^\top$). In addition, we let $\mathbf{I}_n$ be the $n \times n$ identity matrix, we let $\mathbf{1}_n$ ($\mathbf{0}_n$) be the column vector with n ones (zeros), and we let $\mathbf{0}_{n\times m}$ be the $n \times m$ matrix of zeros. Finally, in this paper we let $\otimes$ denote the Kronecker product.

A. Population Games

Population games provide a game theoretical framework to study the strategic interaction of large populations of decision-making agents [2]. In this paper, we adopt the deterministic model approximation described in [3] and [5], and so our analyses are subject to the assumption that the number of agents within each population tends to infinity[1]. More precisely, we consider a set of $N \in \mathbb{Z}_{\geq 2}$ populations, each comprised of a continuum of agents modeled as a mass $m^k \in \mathbb{R}_{>0}$, for all $k \in \mathcal{P}$. Here, m^k denotes the mass of agents of population k, and $\mathcal{P} = \{1, 2, \dots, N\}$ is the set indexing the populations. Besides, we refer to the whole set of populations as the society.

[1]We refer the interested reader to [2, Chapter 10] and [5, Section V] for the finite-agent description of population games.

Under the considered framework, the strategies available to the agents of each population $k \in \mathcal{P}$ are indexed by the set $\mathcal{S}^k = \{1, 2, \dots, n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$. Therefore, to describe the strategic distribution within population k, we let $x^k_i \in \mathbb{R}_{\geq 0}$ denote the mass of agents of population k choosing strategy $i \in \mathcal{S}^k$. Hence, the set of possible strategic distributions for each population k is given by $\Delta^k = \left\{\mathbf{x}^k \in \mathbb{R}^{n^k}_{\geq 0} : \sum_{i\in\mathcal{S}^k} x^k_i = m^k\right\}$, while the set of possible strategic distributions for the entire society is given by $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{x}^k \in \Delta^k, \forall k \in \mathcal{P}\right\}$. Here, and through the remaining of this paper, we let $\mathbf{x}^k = \text{col}\left(x^k_1, x^k_2, \dots, x^k_{n^k}\right)$, $\mathbf{x} = \text{col}\left(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\right)$, and $n = \sum_{k\in\mathcal{P}} n^k$. Thus, $\mathbf{x}^k$ is the strategic distribution of population k, whilst $\mathbf{x}$ is the strategic distribution of the entire society. Furthermore, each strategy $i \in \mathcal{S}^k$ is characterized by a continuously differentiable and Lipschitz continuous fitness function $f^k_i : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$. Namely, $f^k_i(\mathbf{x})$ provides the fitness value of strategy $i \in \mathcal{S}^k$ at the strategic distribution $\mathbf{x} \in \Delta$. Throughout, we let $\mathbf{f}(\cdot) = \text{col}\left(\mathbf{f}^1(\cdot), \mathbf{f}^2(\cdot), \dots, \mathbf{f}^N(\cdot)\right)$ be the overall fitness vector, where $\mathbf{f}^k(\cdot) = \text{col}\left(f^k_1(\cdot), f^k_2(\cdot), \dots, f^k_{n^k}(\cdot)\right)$, for all $k \in \mathcal{P}$. Based on the overall fitness vector $\mathbf{f}(\cdot)$, a population game can then be defined in normal form as the tuple $G = (\mathcal{P}, \Delta, \mathbf{f}(\cdot))$, which captures the involved populations ($\mathcal{P}$), the set of possible strategic distributions (Δ), and the fitness vector ($\mathbf{f}(\cdot)$).

Remark 1: We remark that fitness functions depend on the problem under consideration and they determine the strategic environment for the population game. For example, in the scenario where each population seeks to maximize some (smooth) utility function, the populations' fitness vectors might be taken as the gradients of such utility functions. In such a case, the overall fitness vector plays the role of the so-called pseudo-gradient mapping of the game [26, Section 6].

B. Evolutionary Dynamics and Payoff Dynamics Models

To establish how the strategic distribution of the society evolves over time, let $t \in \mathbb{R}_{\geq 0}$ denote the continuous-time index, and let $\mathbf{x}(t)$ be the value of $\mathbf{x}$ at time t. Moreover, let $p^k_i(t) \in \mathbb{R}$ be the (time-varying) payoff received by the agents choosing strategy i in population k at time t, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. Accordingly, $\mathbf{p}^k(t) = \text{col}\left(p^k_1(t), p^k_2(t), \dots, p^k_{n^k}(t)\right) \in \mathbb{R}^{n^k}$ is the payoff vector of population k, and $\mathbf{p}(t) = \text{col}\left(\mathbf{p}^1(t), \mathbf{p}^2(t), \dots, \mathbf{p}^N(t)\right) \in \mathbb{R}^n$ is the payoff vector of the entire society.

In the finite-agent description of population games, it is assumed that each agent is equipped with a stochastic alarm clock and a so-called revision protocol. Alarm clocks provide (independent) strategic revision opportunities that follow a rate R exponential distribution, while the revision protocols are maps of the form $\rho^k_{ij} : \Delta^k \times \mathbb{R}^{n^k} \to \mathbb{R}_{\geq 0}$ which define the probability distribution that agents use to update their strategies. More precisely, if at time t an agent choosing strategy $i \in \mathcal{S}^k$ in population $k \in \mathcal{P}$ receives a revision opportunity, then such an agent switches to strategy $j \in \mathcal{S}^k \setminus \{i\}$ with probability $\rho_{ij}^{k}\left(\mathbf{x}^{k}\left(\tilde{t}\right), \mathbf{p}^{k}\left(\tilde{t}\right)\right)/R$, or remains at strategy i with probability $1-(1/R)\sum_{j\in\mathcal{S}^{k}\setminus\{i\}}\rho_{ij}^{k}\left(\mathbf{x}^{k}\left(\tilde{t}\right), \mathbf{p}^{k}\left(\tilde{t}\right)\right)$, where $\tilde{t}<t$ is an arbitrary time instant between the previous revision time of any agent of the society and time t (as in [2, Section 4.1], it is assumed that R is large enough so that these probabilities are well-defined for all times). As discussed in [2, Chapter 10], [3], and [5], in the limit of an infinite number of agents within each population k, the aforementioned stochastic decision-making process can be arbitrarily well described by a set of deterministic ordinary differential equations. Throughout, we refer to such a deterministic model as the evolutionary dynamics model, which is defined as follows [3].

Definition 1: The temporal evolution of the strategic distribution $\mathbf{x}(t)$ is described by an evolutionary dynamics model (EDM) of the form

$$\dot{\mathbf{x}}(t)=\boldsymbol{\mathcal{V}}\left(\mathbf{x}(t),\mathbf{p}(t)\right), \quad \mathbf{x}(0)\in\Delta,$$

where $\boldsymbol{\mathcal{V}}:\Delta\times\mathbb{R}^{n}\to\mathbb{R}^{n}$ is Lipschitz continuous and satisfies that $\boldsymbol{\mathcal{V}}\left(\mathbf{x}(t),\mathbf{p}(t)\right)\in\mathrm{T}\Delta\left(\mathbf{x}(t)\right)$ and $\left\|\boldsymbol{\mathcal{V}}\left(\mathbf{x}(t),\mathbf{p}(t)\right)\right\|_{\infty}<\infty$, for all $t\geq 0$. Here, $\mathrm{T}\Delta\left(\mathbf{x}(t)\right)$ denotes the tangent cone of Δ at $\mathbf{x}(t)$, and the input space of an EDM regards all differentiable and bounded $\mathbf{p}(t)\in\mathbb{R}^{n}$, such that $\left\|\mathbf{p}(t)\right\|_{\infty}<\infty$ and $\left\|\dot{\mathbf{p}}(t)\right\|_{\infty}<\infty$, for all $t\geq 0$. In addition, an EDM is said to be Nash stationary, if for every $t\geq 0$, it satisfies that

$$\boldsymbol{\mathcal{V}}\left(\mathbf{x}(t),\mathbf{p}(t)\right)=\mathbf{0}_{n}\Leftrightarrow\mathbf{x}(t)\in\arg\max_{\mathbf{y}\in\Delta}\mathbf{y}^{\top}\mathbf{p}(t).$$

Some examples of EDMs include the Brown-von Neumann-Nash (BNN) dynamics [27], the replicator dynamics [28], and the Smith dynamics [7], among many others [2], [3]. For instance, the so-called Smith dynamics are characterized by

$$\dot{x}_{i}^{k}(t)=\sum_{j\in\mathcal{S}^{k}}x_{j}^{k}(t)\left[p_{i}^{k}(t)-p_{j}^{k}(t)\right]_{+}-x_{i}^{k}(t)\left[p_{j}^{k}(t)-p_{i}^{k}(t)\right]_{+}, \tag{1}$$

for all $i\in\mathcal{S}^{k}$ and all $k\in\mathcal{P}$, with $[\cdot]_{+}\triangleq\max(\cdot,0)$. In fact, the Smith dynamics in (1) are a Nash stationary EDM [3].

According to Definition 1, an EDM can be viewed as a continuous-time dynamical system with input $\mathbf{p}(t)$ and output $\mathbf{x}(t)$. Hence, in the study of EDMs and population games, it is paramount to establish how the payoff vector $\mathbf{p}(t)$ is generated. In the nominal framework of [2], the payoff vector $\mathbf{p}(t)$ is provided by a memoryless map $\mathbf{x}(t)\mapsto\mathbf{p}(t)$, from the society state to the payoff vector. Conventionally, such a memoryless map is precisely the overall fitness vector $\mathbf{f}(\cdot)$. That is, $\mathbf{p}(t)=\mathbf{f}\left(\mathbf{x}(t)\right)$. Therefore, in the nominal framework of [2], the notions of payoffs and fitness values are equivalent. Nevertheless, as highlighted in [3], payoffs mechanisms based solely on memoryless maps cannot account for dynamical effects such as anticipation, delay, and inertia, which are inherent to learning processes. Consequently, for the sake of generality, it is convenient to consider that payoffs are provided by a dynamical system rather than by a memoryless map. Thus, following the ideas in [3] and [5], in this paper we consider that the payoff vector $\mathbf{p}(t)$ is given by a so-called payoff dynamics model, which is defined as follows.

Definition 2: The payoff vector $\mathbf{p}(t)$ is specified by a payoff dynamics model (PDM) of the form

$$\begin{aligned}\dot{\mathbf{q}}(t)&=\boldsymbol{\mathcal{W}}\left(\mathbf{q}(t),\mathbf{x}(t)\right), \quad \mathbf{q}(0)\in\mathbb{R}^{d}\\ \mathbf{p}(t)&=\boldsymbol{\mathcal{H}}\left(\mathbf{q}(t),\mathbf{x}(t)\right),\end{aligned}$$

where $\mathbf{q}(t)\in\mathbb{R}^{d}$ is the (internal) state of the PDM at time t, $\boldsymbol{\mathcal{W}}:\mathbb{R}^{d}\times\Delta\to\mathbb{R}^{d}$ is Lipschitz continuous, and the map $\boldsymbol{\mathcal{H}}:\mathbb{R}^{d}\times\mathbb{R}_{\geq 0}^{n}\to\mathbb{R}^{n}$ is continuously differentiable and Lipschitz continuous. Moreover, the input space of a PDM regards all differentiable and bounded $\mathbf{x}(t)$, such that $\mathbf{x}(t)\in\Delta$ and $\left\|\dot{\mathbf{x}}(t)\right\|_{\infty}<\infty$, for all $t\geq 0$. In addition, a PDM is said to be bounded if it satisfies that $\left\|\mathbf{q}(t)\right\|_{\infty}<\infty$, for all $t\geq 0$.

Furthermore, to align the PDM to the fitness vector of the underlying population game G, the following technical condition is often imposed [3], [6].

Assumption 1: The PDM recovers the overall fitness vector in steady state, i.e., for every $(\mathbf{x}^{*},\mathbf{q}^{*})\in\Delta\times\mathbb{R}^{d}$ it holds that

$$\boldsymbol{\mathcal{W}}\left(\mathbf{q}^{*},\mathbf{x}^{*}\right)=\mathbf{0}_{d} \quad\Rightarrow\quad \boldsymbol{\mathcal{H}}\left(\mathbf{q}^{*},\mathbf{x}^{*}\right)=\mathbf{f}\left(\mathbf{x}^{*}\right).$$

Notice that Under Definition 2 and Assumption 1, the notions of payoff and fitness values are equivalent at every equilibrium point of the PDM, yet they might differ at the transient state of the PDM. Moreover, we highlight that while the considered PDM framework is flexible enough to capture the nominal memoryless approach of [2] (i.e., one might simply consider a PDM with $d=0$, neglect the internal dynamics, and set $\boldsymbol{\mathcal{H}}(\cdot)=\mathbf{f}(\cdot)$), the PDM framework also allows us to include dynamics on the payoff generation process (an impossibility under the memoryless approach). As an example, consider the so-called smoothing-anticipatory PDM [3, Section V], [6, Section VI] given by

$$\dot{\mathbf{q}}(t)=\epsilon_{0}\left(\tilde{\boldsymbol{\mathcal{W}}}\left(\mathbf{x}(t)\right)-\mathbf{q}(t)\right), \quad \mathbf{q}(0)\in\mathbb{R}^{d} \tag{2a}$$

$$\mathbf{p}(t)=\epsilon_{1}\tilde{\boldsymbol{\mathcal{H}}}\left(\mathbf{x}(t)\right)+\epsilon_{2}\mathbf{H}\mathbf{q}(t)+\epsilon_{3}\dot{\mathbf{q}}(t), \tag{2b}$$

where $\tilde{\boldsymbol{\mathcal{W}}}:\mathbb{R}_{\geq 0}^{n}\to\mathbb{R}^{d}$ and $\tilde{\boldsymbol{\mathcal{H}}}:\mathbb{R}_{\geq 0}^{n}\to\mathbb{R}^{n}$ are continuously differentiable and Lipschitz continuous, $\mathbf{H}\in\mathbb{R}^{n\times d}$, $\epsilon_{0}\in\mathbb{R}_{>0}$, and $\epsilon_{1},\epsilon_{2},\epsilon_{3}\in\mathbb{R}_{\geq 0}$, with $\epsilon_{1}+\epsilon_{2}=1$. The PDM in (2) comprises a bounded PDM and, depending on its parameters, it might capture different learning dynamics [3]. For instance, by setting $\epsilon_{0}>0$, $\epsilon_{1}=1$, $\epsilon_{2}=0$, $\epsilon_{3}>0$, $\tilde{\boldsymbol{\mathcal{W}}}(\cdot)=\tilde{\boldsymbol{\mathcal{H}}}(\cdot)=\mathbf{f}(\cdot)$, and $\mathbf{H}=\mathbf{I}_{n}$, one obtains an anticipatory PDM, as in [29]; by setting $\epsilon_{0}>0$, $\epsilon_{1}=0$, $\epsilon_{2}=1$, $\epsilon_{3}=0$, $\tilde{\boldsymbol{\mathcal{W}}}(\cdot)=\tilde{\boldsymbol{\mathcal{H}}}(\cdot)=\mathbf{f}(\cdot)$, and $\mathbf{H}=\mathbf{I}_{n}$, one obtains a smoothing PDM, as in [4]; and, if the fitness vector is of the form $\mathbf{f}(\mathbf{x})=\mathbf{R}\tilde{\mathbf{f}}(\mathbf{x})$, with $\mathbf{R}^{n\times d}$ and $\tilde{\mathbf{f}}:\mathbb{R}_{\geq 0}^{n}\to\mathbb{R}^{d}$, then by setting $\epsilon_{0}>0$, $\epsilon_{1}=0$, $\epsilon_{2}=1$, $\epsilon_{3}=0$, $\tilde{\boldsymbol{\mathcal{W}}}(\cdot)=\tilde{\mathbf{f}}(\cdot)$, $\tilde{\boldsymbol{\mathcal{H}}}(\cdot)=\mathbf{0}_{n}$, and $\mathbf{H}=-\mathbf{R}$, one obtains an alternative smoothing PDM, as in [6].

According to Definition 2, a PDM can be viewed as a continuous-time dynamical system with input $\mathbf{x}(t)$ and output $\mathbf{p}(t)$. Consequently, in the context of population games one is often interested in studying the closed-loop

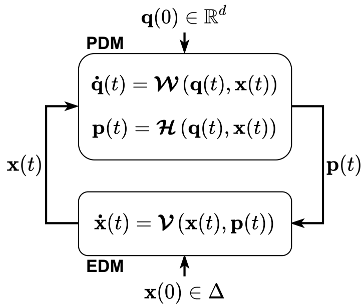

Fig. 1. Feedback interconnection between an EDM and a PDM.

interconnection between the EDM and the PDM, which is depicted in Fig. 1 and formally defined as follows [3].

Definition 3: An EDM-PDM is the feedback interconnection between an EDM and a PDM, where the EDM takes $\mathbf{p}(t)$ as input and provides $\mathbf{x}(t)$ as output, while the PDM takes $\mathbf{x}(t)$ as input and provides $\mathbf{p}(t)$ as output (see Fig. 1). More precisely, an EDM-PDM is characterized by the dynamics

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \boldsymbol{\mathcal{H}}\left(\mathbf{q}(t), \mathbf{x}(t)\right)\right), & \mathbf{x}(0) &\in \Delta \\ \dot{\mathbf{q}}(t) &= \boldsymbol{\mathcal{W}}\left(\mathbf{q}(t), \mathbf{x}(t)\right), & \mathbf{q}(0) &\in \mathbb{R}^d. \end{aligned}$$

Consequently, the set of equilibria of an EDM-PDM is

$$\mathcal{E} = \left\{ (\mathbf{x}^*, \mathbf{q}^*) \in \Delta \times \mathbb{R}^d : \begin{array}{l} \boldsymbol{\mathcal{V}}\left(\mathbf{x}^*, \boldsymbol{\mathcal{H}}(\mathbf{q}^*, \mathbf{x}^*)\right) = \mathbf{0}_n \\ \boldsymbol{\mathcal{W}}(\mathbf{q}^*, \mathbf{x}^*) = \mathbf{0}_d \end{array} \right\}.$$

Furthermore, an EDM-PDM is said to be admissible if, for every initial condition $(\mathbf{x}(0), \mathbf{q}(0)) \in \Delta \times \mathbb{R}^d$, it satisfies that $\|\mathbf{q}(t)\|_\infty < \infty$, for all $t \geq 0$.

Remark 2: From Definitions 1-3 and [5, Proposition 1], it follows that if the PDM is bounded, then the resulting EDM-PDM is admissible. Moreover, if the EDM-PDM is admissible, then for every initial condition $(\mathbf{x}(0), \mathbf{q}(0)) \in \Delta \times \mathbb{R}^d$ there exists a unique solution $\{(\mathbf{x}(t), \mathbf{q}(t))\}_{t \geq 0}$, and the solution is such that $\mathbf{x}(t)$ belongs to the input space of the PDM while $\mathbf{p}(t)$ belongs to the input space of the EDM, for all $t \geq 0$.

In order to study the temporal evolution of admissible EDM-PDMs from a generalized point of view, we now proceed to introduce some notions of passivity that result useful in the analysis of such interconnected systems.

C. δ-Passive EDMs and δ-Antipassive PDMs

In this section, we introduce the concepts of δ-passive EDMs and δ-antipassive PDMs. The following definitions are adapted from [3], [5], and [6].

Definition 4: An EDM is said to be δ-passive if there exist a continuously differentiable non-negative δ-storage function $S : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a non-negative auxiliary function $\zeta : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that, for all $t \geq 0$,

$$S\left(\mathbf{x}(t), \mathbf{p}(t)\right) = 0 \Leftrightarrow \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \mathbf{p}(t)\right) = \mathbf{0}_n \tag{3a}$$

$$\dot{S}(t) \leq -\zeta\left(\mathbf{x}(t), \mathbf{p}(t)\right) + \dot{\mathbf{x}}(t)^\top \dot{\mathbf{p}}(t). \tag{3b}$$

Here, $\dot{\mathbf{p}}(t)$ is the time-derivative of the payoff vector $\mathbf{p}(t)$, and $\dot{S}(t) = \nabla_{\mathbf{x}} S\left(\mathbf{x}(t), \mathbf{p}(t)\right)^\top \dot{\mathbf{x}}(t) + \nabla_{\mathbf{p}} S\left(\mathbf{x}(t), \mathbf{p}(t)\right)^\top \dot{\mathbf{p}}(t)$. Furthermore, the auxiliary function $\zeta(\cdot,\cdot)$ is said to be informative if, for all $t \geq 0$, it satisfies that

$$\zeta\left(\mathbf{x}(t), \mathbf{p}(t)\right) = 0 \Leftrightarrow \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \mathbf{p}(t)\right) = \mathbf{0}_n.$$

Definition 5: A PDM is said to be δ-antipassive if there exist a continuously differentiable non-negative δ-antistorage function $Q : \mathbb{R}^d \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a non-negative auxiliary function $\nu : \mathbb{R}^d \times \Delta \to \mathbb{R}_{\geq 0}$, such that, for all $t \geq 0$,

$$Q\left(\mathbf{q}(t), \mathbf{x}(t)\right) = 0 \Leftrightarrow \boldsymbol{\mathcal{W}}\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \mathbf{0}_d \tag{4a}$$

$$\dot{Q}(t) \leq -\nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) - \dot{\mathbf{x}}(t)^\top \dot{\mathbf{p}}(t). \tag{4b}$$

Here, $\dot{\mathbf{p}}(t) = \mathrm{D}_{\mathbf{q}} \boldsymbol{\mathcal{H}}\left(\mathbf{q}(t), \mathbf{x}(t)\right) \dot{\mathbf{q}}(t) + \mathrm{D}_{\mathbf{x}} \boldsymbol{\mathcal{H}}\left(\mathbf{q}(t), \mathbf{x}(t)\right) \dot{\mathbf{x}}(t)$, and $\dot{Q}(t) = \nabla_{\mathbf{q}} Q\left(\mathbf{q}(t), \mathbf{x}(t)\right)^\top \dot{\mathbf{q}}(t) + \nabla_{\mathbf{x}} Q\left(\mathbf{q}(t), \mathbf{x}(t)\right)^\top \dot{\mathbf{x}}(t)$. Furthermore, the auxiliary function $\nu(\cdot,\cdot)$ is said to be informative if, for all $t \geq 0$, it satisfies that

$$\nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) = 0 \Leftrightarrow \boldsymbol{\mathcal{W}}\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \mathbf{0}_d.$$

As with other notions of passivity, it is expected for an EDM-PDM comprised of a δ-passive EDM and a δ-antipassive PDM to exhibit certain stability properties. Such a result is formalized in Proposition 1, which is adapted from [6].

Proposition 1: Consider an admissible EDM-PDM comprised of a δ-passive EDM with informative $\zeta(\cdot,\cdot)$, and a δ-antipassive PDM with informative $\nu(\cdot,\cdot)$. Moreover, suppose that the set of equilibria $\mathcal{E}$ of such an EDM-PDM is nonempty and compact. Then, the set $\mathcal{E}$ is asymptotically stable under the considered EDM-PDM.

Proof: Since $\mathcal{E}$ is nonempty and compact, we can apply standard Lyapunov stability theory to investigate the stability properties of $\mathcal{E}$ [30, Corollary 4.7]. Namely, consider the function $V\left(\mathbf{x}(t), \mathbf{q}(t)\right) = S\left(\mathbf{x}(t), \mathbf{p}(t)\right) + Q\left(\mathbf{q}(t), \mathbf{x}(t)\right)$, where $\mathbf{p}(t) = \boldsymbol{\mathcal{H}}\left(\mathbf{q}(t), \mathbf{x}(t)\right)$. From (3a) and (4a), it follows that $V(\cdot,\cdot)$ is a valid Lyapunov function candidate. Furthermore, from (3b) and (4b) we have that, for all $t \geq 0$,

$$\begin{aligned} \dot{V}(t) &= \dot{S}(t) + \dot{Q}(t) \\ &\leq -\zeta\left(\mathbf{x}(t), \mathbf{p}(t)\right) - \nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) \\ &\leq 0, \end{aligned}$$

where the last inequality follows from the non-negativity of $\zeta(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$. Finally, given that $\zeta(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are informative, we conclude that $\dot{V}(t) = 0$ if and only if $(\mathbf{x}(t), \mathbf{q}(t)) \in \mathcal{E}$. Hence, the set $\mathcal{E}$ is asymptotically stable under the considered EDM-PDM. ■

Proposition 1 provides sufficient conditions to certify the stability of admissible EDM-PDMs. It is worth to highlight that there are several EDMs in the literature which have been shown to be δ-passive with informative $\zeta(\cdot,\cdot)$. Some examples include the impartial pairwise comparison EDM [6], excess payoff target EDM [3], and perturbed best response EDM [5]. As a punctual example, recall the Smith dynamics in (1). Such dynamics can be shown to be δ-passive with informative $\zeta(\cdot,\cdot)$ by taking [6] $S\left(\mathbf{x}, \mathbf{p}\right) = \sum_{k \in \mathcal{P}} \sum_{i \in \mathcal{S}^k} x_i^k \Gamma_i^k\left(\mathbf{p}^k\right)$ and $\zeta\left(\mathbf{x}, \mathbf{p}\right) = -\sum_{k \in \mathcal{P}} \sum_{i \in \mathcal{S}^k} \mathcal{V}_i^k\left(\mathbf{x}^k, \mathbf{p}^k\right) \Gamma_i^k\left(\mathbf{p}^k\right)$, with

$\mathcal{V}_i^k\left(\mathbf{x}^k,\mathbf{p}^k\right)=\sum_{j\in\mathcal{S}^k} x_j^k\left[p_i^k-p_j^k\right]_+ - x_i^k\left[p_j^k-p_i^k\right]_+$, and $\Gamma_i^k\left(\mathbf{p}^k\right)=\sum_{\ell\in\mathcal{S}^k}\int_0^{p_\ell^k-p_i^k}[z]_+\,dz$. On the other hand, the δ-antipassivity of PDMs has been significantly less studied. To the best of our knowledge, the only PDM whose δ-antipassivity properties have been formally characterized is the smoothing-anticipatory PDM in (2). For certain particular parameters, such a PDM has been shown to be δ-antipassive with informative $\nu(\cdot,\cdot)$. The corresponding analyses are primarily based on the Legendre transform (see for instance [6, Section VI.A]).

To further contribute to the field of δ-antipassive PDMs, in this paper we characterize the δ-antipassivity properties of a class of (bounded) PDMs, which captures various dynamical systems, including the smoothing PDM in (2) when $\epsilon_3=0$. Such a topic is the focus of Section III.

D. Nash Equilibrium Seeking in Population Games

As presented in Section II-B, in the context of population games the agents are payoff-driven decision-makers that update their strategies usually seeking the highest payoff. Hence, it is important to characterize the set of strategic distributions where no agent can improve her payoff by unilaterally changing her strategy. Such a set corresponds to the set of Nash equilibria for the society of agents, and it is formally defined as follows [2], [3].

Definition 6: Given a (fixed) payoff vector $\mathbf{p}^*\in\mathbb{R}^n$, the set of Nash equilibria for the society of agents is given by

$$\mathrm{NE}(\mathbf{p}^*)=\left\{\mathbf{x}^*\in\Delta\,:\,\mathbf{x}^*\in\arg\max_{\mathbf{x}\in\Delta}\mathbf{x}^\top\mathbf{p}^*\right\}.$$

Moreover, if $\mathbf{p}^*$ is the output of a PDM in steady state and Assumption 1 holds, then $\mathrm{NE}(\mathbf{p}^*)=\mathrm{NE}(\mathbf{f})$, where

$$\mathrm{NE}(\mathbf{f})=\left\{\mathbf{x}^*\in\Delta\,:\,\mathbf{x}^*\in\arg\max_{\mathbf{x}\in\Delta}\mathbf{x}^\top\mathbf{f}(\mathbf{x}^*)\right\}.$$

Here, $\mathrm{NE}(\mathbf{f})$ is the set of Nash equilibria of the population game G characterized by the fitness vector $\mathbf{f}(\cdot)$.

Definition 6 reveals the following facts. First, it follows that $\mathrm{NE}(\mathbf{f})$ is aligned to the set of solutions of the variational inequality $\mathrm{VI}(\Delta,-\mathbf{f}(\cdot))$ (see [31, Definition 1.1.1]). Thus, since by definition $\mathbf{f}(\cdot)$ is continuous and Δ is nonempty, compact, and convex, it holds from [31, Corollary 2.2.5] that the set $\mathrm{NE}(\mathbf{f})$ is nonempty and compact. We formalize this observation in Lemma 1, which is adapted from [31].

Lemma 1: The set $\mathrm{NE}(\mathbf{f})$ is nonempty and compact. Moreover, if $\mathbf{f}(\cdot)$ is strictly contractive in the sense that $(\mathbf{x}-\mathbf{y})^\top(\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y}))<0$, for all $\mathbf{x},\mathbf{y}\in\Delta$ with $\mathbf{x}\neq\mathbf{y}$, then there exists a unique $\mathbf{x}^*\in\mathrm{NE}(\mathbf{f})$.

Proof: See [31, Corollary 2.2.5 and Theorem 2.3.3]. ■

Second, if the society of agents is characterized by an EDM-PDM comprised of a Nash stationary EDM and a PDM satisfying Assumption 1, then the set of equilibria $\mathcal{E}$ of the EDM-PDM is aligned to the set of Nash equilibria of the population game G. This result is formally stated as follows.

Lemma 2: Consider an EDM-PDM comprised of a Nash stationary EDM and a PDM satisfying Assumption 1. Then, it holds that $(\mathbf{x}^*,\mathbf{q}^*)\in\mathcal{E}\Rightarrow\mathbf{x}^*\in\mathrm{NE}(\mathbf{f})$.

Proof: Directly from Definitions 1, 3, and 6. ■

Lemma 2 asserts the correspondence between the set of equilibria of the EDM-PDM and the set of Nash equilibria of the population game G. Such a result is paramount when considering the problem of Nash equilibrium seeking in population games, as defined next.

Definition 7: Consider a population game G characterized by the fitness vector $\mathbf{f}(\cdot)$, and a society characterized by any Nash stationary EDM. The NE seeking problem consists of designing a PDM that renders the set $\mathrm{NE}(\mathbf{f})$ asymptotically stable under the resulting EDM-PDM.

In this paper, we consider the NE seeking problem of Definition 7 under the additional assumption that the PDM is comprised of multiple payoff providers, one for each population, that are subject to partial-decision information regarding the strategic distribution of the society, and that are allowed to communicate with each other over a possibly non-complete network. Consequently, in this paper we deal with a distributed NE seeking problem under a partial-decision information scheme. Further details on such a problem are given in Section IV.

III. On the δ-Antipassivity of a Class of PDMs

In this section, we provide sufficient conditions to guarantee the δ-antipassivity of a class of PDMs (Theorem 1). Moreover, we provide a result analogous to Proposition 1 for the case where such PDMs are considered (Corollary 1). The results provided in this section comprise the building blocks for the forthcoming analyses of the proposed distributed NE seeking dynamics in Section V.

Consider a PDM of the form

$$\dot{\mathbf{q}}_1(t)=\frac{1}{\tau}\left(\mathbf{A}\mathbf{q}_1(t)-\mathbf{B}\mathbf{q}_2(t)+\mathbf{\Phi}\left(\mathbf{x}(t)\right)\right) \tag{5a}$$

$$\dot{\mathbf{q}}_2(t)=\frac{1}{\tau}\mathbf{B}^\top\mathbf{q}_1(t) \tag{5b}$$

$$\mathbf{p}(t)=\boldsymbol{\mathcal{F}}\left(\mathbf{x}(t),\mathbf{q}_1(t)\right), \tag{5c}$$

where $\tau\in\mathbb{R}_{>0}$, $\mathbf{q}_1(0)\in\mathbb{R}^{d_1}$, $\mathbf{q}_2(0)\in\mathbb{R}^{d_2}$, $\mathbf{A}\in\mathbb{R}^{d_1\times d_1}$, $\mathbf{B}\in\mathbb{R}^{d_1\times d_2}$, $\mathbf{\Phi}:\mathbb{R}^n_{\geq 0}\to\mathbb{R}^{d_1}$, $\boldsymbol{\mathcal{F}}:\mathbb{R}^n_{\geq 0}\times\mathbb{R}^{d_1}\to\mathbb{R}^n$, with $d_1\in\mathbb{Z}_{\geq 1}$ and $d_2\in\mathbb{Z}_{\geq 0}$. Besides, both $\mathbf{\Phi}(\cdot)$ and $\boldsymbol{\mathcal{F}}(\cdot,\cdot)$ are continuously differentiable and Lipschitz continuous. Thus, the PDM in (5) agrees with Definition 2, with $\mathbf{q}(t)=\mathrm{col}\left(\mathbf{q}_1(t),\mathbf{q}_2(t)\right)$, $d=d_1+d_2$, and $\boldsymbol{\mathcal{H}}\left(\mathbf{q}(t),\mathbf{x}(t)\right)=\boldsymbol{\mathcal{F}}\left(\mathbf{x}(t),\mathbf{q}_1(t)\right)$. In addition, we often impose Assumption 2.

Assumption 2: The following conditions hold.

i) The matrix $\left(\mathbf{A}+\mathbf{A}^\top\right)$ is negative definite.

ii) The matrix $\tilde{\mathbf{A}}=\begin{bmatrix}\mathbf{A} & -\mathbf{B}\\ \mathbf{B}^\top & \mathbf{0}_{d_2\times d_2}\end{bmatrix}$ is Hurwitz[2].

iii) The function $\mathbf{\Phi}(\cdot)$ is $L_{\mathbf{\Phi}}$-Lipschitz continuous under the Euclidean norm, i.e., there is an $L_{\mathbf{\Phi}}\in\mathbb{R}_{>0}$ such that $\left\|\mathbf{\Phi}(\mathbf{x})-\mathbf{\Phi}(\tilde{\mathbf{x}})\right\|_2\leq L_{\mathbf{\Phi}}\left\|\mathbf{x}-\tilde{\mathbf{x}}\right\|_2$, for all $\mathbf{x},\tilde{\mathbf{x}}\in\mathbb{R}^n_{\geq 0}$.

iv) The function $\boldsymbol{\mathcal{F}}(\mathbf{x},\cdot)$ is $L_{\boldsymbol{\mathcal{F}}}$-Lipschitz continuous under the Euclidean norm, i.e., there is an $L_{\boldsymbol{\mathcal{F}}}\in\mathbb{R}_{>0}$ such

[2] If $\mathbf{A}$ satisfies Assumption 2-i) and $d_1\geq d_2$, then a sufficient condition for $\tilde{\mathbf{A}}$ to be Hurwitz is $\mathrm{rank}(\mathbf{B})=d_2$ [24, Lemma 9], [32, Lemma 2.2].

that $\|\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)-\boldsymbol{\mathcal{F}}(\mathbf{x},\tilde{\mathbf{q}}_1)\|_2 \leq L_{\boldsymbol{\mathcal{F}}}\|\mathbf{q}_1-\tilde{\mathbf{q}}_1\|_2$, for all $\mathbf{x}\in\Delta$, and all $\mathbf{q}_1,\tilde{\mathbf{q}}_1\in\mathbb{R}^{d_1}$.

v) The function $\boldsymbol{\mathcal{F}}(\cdot,\mathbf{q}_1)$ is μ-strongly contractive in the following sense. There is a $\mu\in\mathbb{R}_{>0}$ such that

$$\mathbf{z}^\top \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)\mathbf{z} \leq -\mu\mathbf{z}^\top\mathbf{z},$$

for all $\mathbf{x}\in\Delta$, all $\mathbf{q}_1\in\mathbb{R}^{d_1}$, and all $\mathbf{z}\in\mathbb{R}^n$.

We now provide two examples to show that the PDM in (5) is flexible enough to capture several dynamical systems.

Example 1: Consider the smoothing PDM in (2) subject to $\epsilon_3 = 0$. By setting $d_1 = d$, $d_2 = 0$, $\mathbf{q}_1(t) \triangleq \mathbf{q}(t)$, $\tau = 1/\epsilon_0$, $\mathbf{A} = -\mathbf{I}_d$, $\boldsymbol{\Phi}(\cdot) = \tilde{\boldsymbol{\mathcal{W}}}(\cdot)$, and $\boldsymbol{\mathcal{F}}(\mathbf{x}(t),\mathbf{q}_1(t)) = \epsilon_1\tilde{\boldsymbol{\mathcal{H}}}(\mathbf{x}(t))+\epsilon_2\mathbf{H}\mathbf{q}_1(t)$, it follows that our proposed PDM in (5) is able to capture the considered smoothing PDM (yet the converse is in general not true). Furthermore, if in addition $\epsilon_1\tilde{\boldsymbol{\mathcal{H}}}(\cdot)$ satisfies that $\epsilon_1\mathbf{z}^\top\mathrm{D}_{\mathbf{x}}\tilde{\boldsymbol{\mathcal{H}}}(\mathbf{x})\mathbf{z} \leq -\mu\mathbf{z}^\top\mathbf{z}$, for some $\mu\in\mathbb{R}_{>0}$, all $\mathbf{x}\in\Delta$, and all $\mathbf{z}\in\mathbb{R}^n$, then Assumption 2 is immediately satisfied. We highlight that smoothing PDMs are useful to smooth short-term fluctuations and to isolate long-term trends in the decision-making process of the society [4]. As illustration, a smoothing PDM has been applied in [6] to consider delays in congestion games over vehicle traffic networks, e.g., to account for the time lag with which the drivers receive and process congestion information.

Example 2: Consider any system with state $\mathbf{q}\in\mathbb{R}^d$, input $\mathbf{x}\in\mathbb{R}^n$, and dynamics given by $\dot{\mathbf{q}}(t) = \tilde{\mathbf{M}}\mathbf{q}(t)+\tilde{\boldsymbol{\Phi}}(\mathbf{x}(t))$, where $\tilde{\mathbf{M}}\in\mathbb{R}^{d\times d}$ is Hurwitz, and $\tilde{\boldsymbol{\Phi}}:\mathbb{R}^n_{\geq 0}\to\mathbb{R}^d$ is continuously differentiable and Lipschitz continuous. We highlight that any continuous-time asymptotically stable linear system has this form. Besides, note that $\tilde{\boldsymbol{\Phi}}(\cdot)$ might represent a smooth (nonlinear) input saturation function, e.g., an hyperbolic tangent or a sigmoid function, which are often considered in the context of control systems. Now, suppose that the control objective regarding such a system is to drive the state-input pair to a point $(\mathbf{q}^*,\mathbf{x}^*)\in\arg\max_{(\mathbf{q},\mathbf{x})\in\mathbb{R}^d\times\Delta} J(\mathbf{q},\mathbf{x})$, where the utility $J:\mathbb{R}^d\times\mathbb{R}^n_{\geq 0}\to\mathbb{R}$ is twice-continuously differentiable, $L_{\boldsymbol{\mathcal{F}}}$-smooth in its first argument, and μ-strongly concave in its second argument. By setting $d_1 = d$, $d_2 = 0$, $\mathbf{q}_1(t)\triangleq\mathbf{q}(t)$, $\mathbf{A} = \tilde{\mathbf{M}}$, $\boldsymbol{\Phi}(\cdot) = \tilde{\boldsymbol{\Phi}}(\cdot)$, and $\boldsymbol{\mathcal{F}}(\mathbf{x}(t),\mathbf{q}_1(t)) = \nabla_{\mathbf{x}}J(\mathbf{q}(t),\mathbf{x}(t))$, it follows that the PDM in (5) is able to capture the dynamics and control objective of the considered system. In addition, if $\tilde{\mathbf{M}}+\tilde{\mathbf{M}}^\top$ is negative definite, then all the conditions in Assumption 2 hold. Also, recall that any linear system with all simple, real, and strictly negative eigenvalues, can be rewritten in modal canonical form characterized by a real diagonal matrix $\tilde{\mathbf{M}}$ for which the negative definiteness of $\tilde{\mathbf{M}}+\tilde{\mathbf{M}}^\top$ immediately holds. Hence, the considered PDM is flexible enough to capture several dynamical systems, which may account for additional internal dynamics in the society.

In Section IV, we provide yet a third example of dynamics that can be captured by the PDM in (5), which are relevant for the problem of distributed NE seeking in population games. Now, however, we proceed to characterize the boundedness and δ-antipassivity of the PDM in (5).

Lemma 3: Let Assumption 2-ii) hold. Then, the PDM in (5) is bounded in the sense of Definition 2.

Proof: Observe that the dynamics of the PDM in (5) can be rewritten as

$$\dot{\mathbf{q}}(t) = \frac{1}{\tau}\tilde{\mathbf{A}}\mathbf{q}(t)+\frac{1}{\tau}\begin{bmatrix}\boldsymbol{\Phi}(\mathbf{x}(t))\\ \mathbf{0}_{d_2}\end{bmatrix}, \tag{6}$$

with $\tilde{\mathbf{A}}$ taken as in Assumption 2-ii). Since $\tilde{\mathbf{A}}$ is Hurwitz, it follows that the PDM in (5) is input-to-state stable [33, Lemma 4.6]. Thus, under any input $\mathbf{x}(t)\in\Delta$, the state of the PDM in (5) is bounded, for all $t\geq 0$. ■

Lemma 4: Let Assumption 2 hold and consider the matrix

$$\mathbf{M}(\mathbf{q}_1,\mathbf{x}) = \begin{bmatrix} -\frac{1}{\tau}\mathbf{A} & -\frac{1}{\tau}\mathrm{D}_{\mathbf{x}}\boldsymbol{\Phi}(\mathbf{x}) \\ -\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1) & -\mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)\end{bmatrix}. \tag{7}$$

Moreover, suppose that the following inequalities hold[3]:

$$\frac{2L_{\boldsymbol{\Phi}}L_{\boldsymbol{\mathcal{F}}}}{|\lambda_{\max}(\mathbf{A}+\mathbf{A}^\top)|} \leq \mu \tag{8a}$$

$$\frac{2L_{\boldsymbol{\Phi}}^2}{\mu|\lambda_{\max}(\mathbf{A}+\mathbf{A}^\top)|} \leq \tau \leq \frac{\mu\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right|}{2L_{\boldsymbol{\mathcal{F}}}^2}. \tag{8b}$$

Then, $\mathbf{z}^\top\mathbf{M}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$, for all $\mathbf{q}_1\in\mathbb{R}^{d_1}$, all $\mathbf{x}\in\Delta$, and all $\mathbf{z}\in\mathbb{R}^{d_1+n}$.

Proof: First, note that for (8b) to hold it is necessary that

$$\frac{2L_{\boldsymbol{\Phi}}^2}{\mu|\lambda_{\max}(\mathbf{A}+\mathbf{A}^\top)|} \leq \frac{\mu\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right|}{2L_{\boldsymbol{\mathcal{F}}}^2}.$$

Clearly, such a condition is satisfied only if (8a) holds. Thus, (8a) is a necessary condition for (8b) to hold. With this fact in mind, we now proceed to show that if (8b) holds, then $\mathbf{z}^\top\mathbf{M}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$, for all $\mathbf{q}_1\in\mathbb{R}^{d_1}$, $\mathbf{x}\in\Delta$, and $\mathbf{z}\in\mathbb{R}^{d_1+n}$. Throughout this proof, let $\mathbf{q}_1$, $\mathbf{x}$, and $\mathbf{z}$ be arbitrary.

Now, observe that

$$\begin{aligned}\mathbf{z}^\top\mathbf{M}(\mathbf{q}_1,\mathbf{x})\mathbf{z} &= \frac{1}{2}\mathbf{z}^\top\left(\mathbf{M}(\mathbf{q}_1,\mathbf{x})+\mathbf{M}(\mathbf{q}_1,\mathbf{x})^\top\right)\mathbf{z}\\ &= \mathbf{z}^\top\left(\mathbf{M}_{\boldsymbol{\Phi}}(\mathbf{q}_1,\mathbf{x})+\mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1,\mathbf{x})\right)\mathbf{z},\end{aligned}$$

where

$$\mathbf{M}_{\boldsymbol{\Phi}}(\mathbf{q}_1,\mathbf{x}) = \begin{bmatrix} -\frac{1}{4\tau}\left(\mathbf{A}+\mathbf{A}^\top\right) & -\frac{1}{2\tau}\mathrm{D}_{\mathbf{x}}\boldsymbol{\Phi}(\mathbf{x}) \\ -\frac{1}{2\tau}\mathrm{D}_{\mathbf{x}}\boldsymbol{\Phi}(\mathbf{x})^\top & -\frac{1}{4}\mathbf{S}(\mathbf{x},\mathbf{q}_1)\end{bmatrix}$$

$$\mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1,\mathbf{x}) = \begin{bmatrix} -\frac{1}{4\tau}\left(\mathbf{A}+\mathbf{A}^\top\right) & -\frac{1}{2}\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)^\top \\ -\frac{1}{2}\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1) & -\frac{1}{4}\mathbf{S}(\mathbf{x},\mathbf{q}_1)\end{bmatrix},$$

with $\mathbf{S}(\mathbf{x},\mathbf{q}_1) = \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)+\mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x},\mathbf{q}_1)^\top$. Hence, to show that $\mathbf{z}^\top\mathbf{M}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$, it suffices to show that both $\mathbf{z}^\top\mathbf{M}_{\boldsymbol{\Phi}}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$ and $\mathbf{z}^\top\mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$.

Let us consider $\mathbf{z}^\top\mathbf{M}_{\boldsymbol{\Phi}}(\mathbf{q}_1,\mathbf{x})\mathbf{z}\geq 0$ first. By Assumption 2-i) and the Schur complement characterization of definite matrices [34, Prop. 8.2.4], it follows that $\mathbf{M}_{\boldsymbol{\Phi}}(\mathbf{q}_1,\mathbf{x})\succeq 0$ if and only if

$$-\frac{1}{4}\mathbf{S}(\mathbf{x},\mathbf{q}_1)+\frac{1}{\tau}\mathrm{D}_{\mathbf{x}}\boldsymbol{\Phi}(\mathbf{x})^\top\left(\mathbf{A}+\mathbf{A}^\top\right)^{-1}\mathrm{D}_{\mathbf{x}}\boldsymbol{\Phi}(\mathbf{x})\succeq 0.$$

[3]If τ is a tunable parameter and (8a) holds, then setting $\tau = L_{\boldsymbol{\Phi}}/L_{\boldsymbol{\mathcal{F}}}$ is sufficient to satisfy (8b). To see this fact, simply replace the equality value of (8a) into both sides of (8b).

Equivalently, $\mathbf{z}^\top \mathbf{M}_{\mathbf{\Phi}}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$ if and only if

$$\frac{1}{\tau}\mathbf{z}^\top \mathbf{T}_1(\mathbf{x})\mathbf{z} \geq \frac{1}{4}\mathbf{z}^\top \mathbf{S}(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z}, \tag{9}$$

with $\mathbf{T}_1(\mathbf{x}) = \mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})^\top \left(\mathbf{A}+\mathbf{A}^\top\right)^{-1} \mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$. Now, let $\mathbf{T}_2(\mathbf{x}) = \mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})^\top \mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$ and notice that

$$\begin{aligned}
\frac{1}{\tau}\mathbf{z}^\top \mathbf{T}_1(\mathbf{x})\mathbf{z} &\geq \frac{1}{\tau}\lambda_{\min}\left(\left(\mathbf{A}+\mathbf{A}^\top\right)^{-1}\right)\mathbf{z}^\top \mathbf{T}_2(\mathbf{x})\mathbf{z} \\
&= \frac{1}{\tau}\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)}\mathbf{z}^\top \mathbf{T}_2(\mathbf{x})\mathbf{z} \\
&\geq \frac{1}{\tau}\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)}\left\|\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})\right\|_2^2 \mathbf{z}^\top \mathbf{z} \\
&\geq \frac{1}{\tau}\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)} L_{\mathbf{\Phi}}^2 \mathbf{z}^\top \mathbf{z},
\end{aligned}$$

where the last two inequalities follow from the fact that $\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right) < 0$, in conjunction with the fact that $\lambda_{\max}(\mathbf{T}_2(\mathbf{x})) = \|\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})\|_2^2 \leq L_{\mathbf{\Phi}}^2$ (here, the latter inequality follows from the mean value theorem together with the definition of $L_{\mathbf{\Phi}}$-Lipschitz continuity). On the other hand, observe that

$$\frac{1}{2}\mathbf{z}^\top \mathbf{S}(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z} = \mathbf{z}^\top \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\mathbf{z}.$$

Thus, from Assumption 2-v) it holds that

$$-\frac{1}{2}\mu\mathbf{z}^\top \mathbf{z} \geq \frac{1}{4}\mathbf{z}^\top \mathbf{S}(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z}.$$

Hence, to satisfy the inequality in (9), it suffices to ensure that $L_{\mathbf{\Phi}}^2 / \left(\tau\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right) \geq -\mu/2$, which, by the fact that $\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right) < 0$, is equivalent to

$$\frac{2L_{\mathbf{\Phi}}^2}{\tau\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right|} \leq \mu. \tag{10}$$

Therefore, if $\tau \geq 2L_{\mathbf{\Phi}}^2 / \left(\mu\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right|\right)$, then the inequalities in (9)-(10) hold, and $\mathbf{z}^\top \mathbf{M}_{\mathbf{\Phi}}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$.

We now consider $\mathbf{z}^\top \mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$. Similar as before, by Assumption 2-i) and the Schur complement characterization of definite matrices, it follows that $\mathbf{z}^\top \mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$ if and only if

$$\tau\mathbf{z}^\top \mathbf{T}_3(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z} \geq \frac{1}{4}\mathbf{z}^\top \mathbf{S}(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z}, \tag{11}$$

with $\mathbf{T}_3(\mathbf{x}, \mathbf{q}_1) = \mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\left(\mathbf{A}+\mathbf{A}^\top\right)^{-1}\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top$. By setting $\mathbf{T}_4(\mathbf{x}, \mathbf{q}_1) = \mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top$, it follows that

$$\begin{aligned}
\tau\mathbf{z}^\top \mathbf{T}_3(\mathbf{x}, \mathbf{q}_1)\,\mathbf{z} &\geq \tau\lambda_{\min}\left(\left(\mathbf{A}+\mathbf{A}^\top\right)^{-1}\right)\mathbf{z}^\top \mathbf{T}_4(\mathbf{x}, \mathbf{q}_1)\mathbf{z} \\
&= \tau\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)}\mathbf{z}^\top \mathbf{T}_4(\mathbf{x}, \mathbf{q}_1)\mathbf{z} \\
&\geq \tau\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)}\left\|\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\right\|_2^2 \mathbf{z}^\top \mathbf{z} \\
&\geq \tau\frac{1}{\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)} L_{\boldsymbol{\mathcal{F}}}^2 \mathbf{z}^\top \mathbf{z},
\end{aligned}$$

where we have used the facts that $\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right) < 0$, and $\lambda_{\max}(\mathbf{T}_4(\mathbf{x})) = \|\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\|_2^2 \leq L_{\boldsymbol{\mathcal{F}}}^2$. Thus, following a similar analysis as before, to satisfy the inequality in (11) it suffices to ensure that

$$\frac{2\tau L_{\boldsymbol{\mathcal{F}}}^2}{\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right|} \leq \mu. \tag{12}$$

Consequently, if $\tau \leq \mu\left|\lambda_{\max}\left(\mathbf{A}+\mathbf{A}^\top\right)\right| / \left(2L_{\boldsymbol{\mathcal{F}}}^2\right)$, then the inequalities in (11)-(12) hold, and $\mathbf{z}^\top \mathbf{M}_{\boldsymbol{\mathcal{F}}}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$. ■

Lemma 5: Consider the matrix $\mathbf{M}(\mathbf{q}_1, \mathbf{x})$ in (7). Let Assumptions 2-i) and 2-v) hold, with 2-v) relaxed with $\mu = 0$, and suppose that $\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top = -(1/\tau)\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$, for all $\mathbf{x} \in \Delta$ and all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$. Then, $\mathbf{z}^\top \mathbf{M}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$, for all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$, all $\mathbf{x} \in \Delta$, and all $\mathbf{z} \in \mathbb{R}^{d_1+n}$.

Proof: Given that $\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top = -(1/\tau)\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$, for all $\mathbf{x} \in \Delta$ and all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$, it holds that

$$\begin{aligned}
\mathbf{z}^\top \mathbf{M}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} &= \frac{1}{2}\mathbf{z}^\top\left(\mathbf{M}(\mathbf{q}_1, \mathbf{x}) + \mathbf{M}(\mathbf{q}_1, \mathbf{x})^\top\right)\mathbf{z} \\
&= -\frac{1}{2}\mathbf{z}^\top \begin{bmatrix} \frac{1}{\tau}\left(\mathbf{A}+\mathbf{A}^\top\right) & \mathbf{0}_{d_1\times n} \\ \mathbf{0}_{n\times d_1} & \mathbf{S}(\mathbf{x}, \mathbf{q}_1) \end{bmatrix}\mathbf{z} \\
&\geq 0,
\end{aligned}$$

for all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$, all $\mathbf{x} \in \Delta$, and all $\mathbf{z} \in \mathbb{R}^{d_1+n}$. Here, $\mathbf{S}(\mathbf{x}, \mathbf{q}_1) = \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1) + \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top$, and the last inequality follows from Assumption 2-i) and the relaxed Assumption 2-v). ■

Due to Lemma 3, we conclude that an EDM-PDM subject to the PDM in (5) satisfying Assumption 2-ii) is admissible in the sense of Definition 3, which implies that solutions to the initial value problem of such an EDM-PDM exist, are unique, and are within the input spaces of the EDM and PDM (c.f., Remark 2). On the other hand, based on Lemmas 4 and 5, we now provide our main result regarding the δ-antipassivity of the PDM in (5).

Theorem 1: Consider the PDM in (5) and suppose that at least one of the following two cases holds:

i) Assumption 2 and the inequalities in (8) hold.
ii) Assumptions 2-i), 2-ii), and 2-v) hold, with 2-v) relaxed with $\mu = 0$, and $\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top = -(1/\tau)\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$, for all $\mathbf{x} \in \Delta$ and all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$.

Then, the considered PDM is bounded and δ-antipassive.

Proof: Throughout this proof, let $\mathbf{M}(\mathbf{q}_1, \mathbf{x})$ be the matrix given in (7). Moreover, note that from Lemma 3 it holds that the considered PDM is bounded, and from Lemmas 4 and 5 it follows that $\mathbf{z}^\top \mathbf{M}(\mathbf{q}_1, \mathbf{x})\,\mathbf{z} \geq 0$, for all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$, all $\mathbf{x} \in \Delta$, and all $\mathbf{z} \in \mathbb{R}^{d_1+n}$.

Given the PDM in (5), it follows that

$$\dot{\mathbf{p}}(t) = \mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}(t), \mathbf{q}_1(t))\,\dot{\mathbf{q}}_1(t) + \mathrm{D}_{\mathbf{x}}\boldsymbol{\mathcal{F}}(\mathbf{x}(t), \mathbf{q}_1(t))\,\dot{\mathbf{x}}(t). \tag{13}$$

Moreover, consider the (valid) δ-antistorage function given by

$$Q(\mathbf{q}, \mathbf{x}) = \frac{1}{2\tau^2}\left\|\mathbf{A}\mathbf{q}_1 - \mathbf{B}\mathbf{q}_2 + \mathbf{\Phi}(\mathbf{x})\right\|_2^2 + \frac{1}{2\tau^2}\left\|\mathbf{B}^\top\mathbf{q}_1\right\|_2^2,$$

such that

$$\nabla_{\mathbf{q}_1} Q\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \frac{1}{\tau}\mathbf{A}^\top \dot{\mathbf{q}}_1(t) + \frac{1}{\tau}\mathbf{B}\dot{\mathbf{q}}_2(t)$$
$$\nabla_{\mathbf{q}_2} Q\left(\mathbf{q}(t), \mathbf{x}(t)\right) = -\frac{1}{\tau}\mathbf{B}^\top \dot{\mathbf{q}}_1(t)$$
$$\nabla_{\mathbf{x}} Q\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \frac{1}{\tau}\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}\left(\mathbf{x}(t)\right)^\top \dot{\mathbf{q}}_1(t).$$

Consequently,

$$\dot{Q}(t) = \frac{1}{\tau}\dot{\mathbf{q}}_1(t)^\top \mathbf{A}\dot{\mathbf{q}}_1(t) + \frac{1}{\tau}\dot{\mathbf{q}}_1(t)^\top \mathrm{D}_{\mathbf{x}}\mathbf{\Phi}\left(\mathbf{x}(t)\right)\dot{\mathbf{x}}(t). \quad (14)$$

Finally, consider the function

$$\nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \begin{bmatrix} \dot{\mathbf{q}}_1(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}^\top \mathbf{M}\left(\mathbf{q}_1(t), \mathbf{x}(t)\right) \begin{bmatrix} \dot{\mathbf{q}}_1(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}. \quad (15)$$

Since, $\mathbf{z}^\top \mathbf{M}\left(\mathbf{q}_1, \mathbf{x}\right)\mathbf{z} \geq 0$, for all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$, all $\mathbf{x} \in \Delta$, and all $\mathbf{z} \in \mathbb{R}^{d_1+n}$, it follows that $\nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) \geq 0$ for all times. Furthermore, from (7) and (13)-(15) it holds that

$$-\nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) = \dot{\mathbf{x}}(t)^\top \dot{\mathbf{p}}(t) + \dot{Q}(t),$$

which implies that the δ-antipassivity inequality in (4b) is satisfied (in this case with equality). ■

Theorem 1 provides sufficient conditions to guarantee the δ-antipassivity of the PDM in (5). Moreover, such sufficient conditions can be checked by examining the parameters' values of the PDM in (5). In particular, Case i) of Theorem 1 requires for the output map of the PDM in (5) to be sufficiently strongly contractive in the sense of Assumption 2-v) and the inequalities in (8). In contrast, Case ii) of Theorem 1 removes such a strong contractivity requirement and replaces it by mere contractivity, but presumes a particular matching condition between the Jacobians of $\boldsymbol{\mathcal{F}}(\mathbf{x}, \cdot)$ and $\mathbf{\Phi}(\cdot)$. Nevertheless, such a matching condition can be easily satisfied in certain especial cases. For instance, consider the PDM in (5) subject to $\mathbf{\Phi}(\mathbf{x}) = \mathbf{C}\mathbf{x}$, with $\mathbf{C} \in \mathbb{R}^{d_1 \times n}$, and $\boldsymbol{\mathcal{F}}\left(\mathbf{x}, \mathbf{q}_1\right) = \tilde{\boldsymbol{\mathcal{F}}}(\mathbf{x}) - \tau\mathbf{C}^\top \mathbf{q}_1$, with $\tilde{\boldsymbol{\mathcal{F}}} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ satisfying $\mathbf{z}^\top \mathrm{D}_{\mathbf{x}}\tilde{\boldsymbol{\mathcal{F}}}(\mathbf{x})\mathbf{z} \leq 0$, for all $\mathbf{x} \in \Delta$ and all $\mathbf{z} \in \mathbb{R}^n$. We highlight that such PDMs are relevant for the class of aggregative population games reported in [23], which we reconsider and generalize in Section VI.

Although Theorem 1 allows us to assert the δ-antipassivity of the PDM in (5), note that the auxiliary function $\nu(\cdot, \cdot)$ in (15), employed in the proof of Theorem 1, is not informative in the sense of Definition 5. For instance, observe that such a $\nu(\cdot, \cdot)$ might be zero even if $\dot{\mathbf{q}}_2(t) \neq \mathbf{0}_{d_2}$. Consequently, the result of Proposition 1 cannot be directly invoked when the PDM in (5) is considered (at least not with the given $\nu(\cdot, \cdot)$). To overcome this issue, we provide a result alternative to Proposition 1 for the case where the PDM in (5) is considered.

Corollary 1: Consider an EDM-PDM comprised of a δ-passive EDM with informative $\zeta(\cdot, \cdot)$, and the PDM in (5). Moreover, suppose that the set of equilibria $\mathcal{E}$ of such an EDM-PDM is nonempty and compact. In addition, let at least one of the following two cases hold:

i) Assumption 2 and the inequalities in (8) hold.

ii) Assumptions 2-i), 2-ii), and 2-v) hold, with 2-v) relaxed with $\mu = 0$, and $\mathrm{D}_{\mathbf{q}_1}\boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)^\top = -(1/\tau)\mathrm{D}_{\mathbf{x}}\mathbf{\Phi}(\mathbf{x})$, for all $\mathbf{x} \in \Delta$ and all $\mathbf{q}_1 \in \mathbb{R}^{d_1}$.

Then, the set $\mathcal{E}$ is asymptotically stable under the considered EDM-PDM.

Proof: From Lemma 3, it follows that the considered EDM-PDM is admissible in the sense of Definition 3 (c.f., Remark 2). In addition, from Theorem 1, it follows that the PDM in (5) is δ-antipassive with the (non-informative) $\nu(\cdot, \cdot)$ given in (15). Hence, consider the function $V\left(\mathbf{x}(t), \mathbf{q}(t)\right) = S\left(\mathbf{x}(t), \mathbf{p}(t)\right) + Q\left(\mathbf{q}(t), \mathbf{x}(t)\right)$, where $\mathbf{p}(t) = \boldsymbol{\mathcal{F}}\left(\mathbf{x}(t), \mathbf{q}_1(t)\right)$. From (3a) and (4a), it follows that $V(\cdot, \cdot)$ is a valid Lyapunov function candidate. Moreover, from (3b) and (4b) we have that, for all $t \geq 0$,

$$\begin{aligned} \dot{V}(t) &= \dot{S}(t) + \dot{Q}(t) \\ &\leq -\zeta\left(\mathbf{x}(t), \mathbf{p}(t)\right) - \nu\left(\mathbf{q}(t), \mathbf{x}(t)\right) \\ &\leq 0, \end{aligned}$$

where the last inequality follows from the non-negativity of $\zeta(\cdot, \cdot)$ and $\nu(\cdot, \cdot)$. Thus, we conclude that $\mathcal{E}$ is stable in the sense of Lyapunov [30, Corollary 4.7].

From the non-negativity of $\zeta(\cdot, \cdot)$ and $\nu(\cdot, \cdot)$, it holds that $\dot{V}(t) = 0$ only if $\zeta(\mathbf{x}(t), \mathbf{p}(t)) = \nu(\mathbf{q}(t), \mathbf{x}(t)) = 0$. Besides, given that $\zeta(\cdot, \cdot)$ is informative, we conclude that $\zeta\left(\mathbf{x}(t), \mathbf{p}(t)\right) = 0 \Leftrightarrow \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \mathbf{p}(t)\right) = \mathbf{0}_n$. Hence, $\dot{\mathbf{x}}(t) = \mathbf{0}_n$ is a necessary condition for $\dot{V}(t) = 0$ (recall that $\dot{\mathbf{x}}(t) = \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \mathbf{p}(t)\right)$ by definition). On the other hand, from the definition of $\nu(\cdot, \cdot)$ in (15), the definition of $\mathbf{M}(\cdot, \cdot)$ in (7), and Assumption 2-i), it follows that if $\dot{\mathbf{x}}(t) = \mathbf{0}_n$, then $\nu(\mathbf{q}(t), \mathbf{x}(t)) = 0$ if and only if $\dot{\mathbf{q}}_1(t) = \mathbf{0}_{d_1}$. Therefore, we can further conclude that

$$\dot{V}(t) = 0 \Leftrightarrow [\dot{\mathbf{x}}(t) = \mathbf{0}_n \text{ and } \dot{\mathbf{q}}_1(t) = \mathbf{0}_{d_1}]$$

Equivalently, due to (5), $\dot{V}(t) = 0 \Leftrightarrow (\mathbf{x}(t), \mathbf{q}(t)) \in \mathcal{R}$, with

$$\mathcal{R} = \left\{ (\mathbf{x}, \mathbf{q}) \in \Delta \times \mathbb{R}^d : \begin{array}{c} \boldsymbol{\mathcal{V}}\left(\mathbf{x}, \boldsymbol{\mathcal{F}}(\mathbf{x}, \mathbf{q}_1)\right) = \mathbf{0}_n \\ \mathbf{A}\mathbf{q}_1 - \mathbf{B}\mathbf{q}_2 + \mathbf{\Phi}\left(\mathbf{x}\right) = \mathbf{0}_{d_1} \end{array} \right\}.$$

Now, let $\mathcal{I} \subseteq \mathcal{R}$ be the largest invariant set of the EDM-PDM within $\mathcal{R}$. By LaSalle's Theorem [30, Theorem 3.3], it follows that $(\mathbf{x}(t), \mathbf{q}(t)) \to \mathcal{I}$ as $t \to \infty$, i.e., $\mathcal{I}$ is attractive under the EDM-PDM. Also, by Definition 3 it holds that $\mathcal{E} \subseteq \mathcal{I}$. Hence, if $\mathcal{E} = \mathcal{I}$, then $\mathcal{E}$ is asymptotically stable and the proof is complete. We now proceed to show that $\mathcal{E} = \mathcal{I}$ indeed.

Suppose that $\mathcal{E} \subset \mathcal{I}$ and let $\mathcal{T} = \mathcal{I} \setminus \mathcal{E}$. Thus, $\mathcal{T} \neq \emptyset$. Besides, let $\tilde{t} \in \mathbb{R}_{\geq 0}$ be an arbitrary time instant (which might even be the initial time $\tilde{t} = 0$). Given that $\mathcal{T} \subset \mathcal{I} \subseteq \mathcal{R}$ and $\mathcal{E} \cap \mathcal{T} = \emptyset$, it follows from the definition of $\mathcal{R}$ and (5b) that

$$\left(\mathbf{x}\left(\tilde{t}\right), \mathbf{q}\left(\tilde{t}\right)\right) \in \mathcal{T} \Rightarrow \left[\dot{\mathbf{q}}_2\left(t\right) = \frac{1}{\tau}\mathbf{B}^\top \mathbf{q}_1\left(\tilde{t}\right) \neq \mathbf{0}_{d_2}, \forall t \geq \tilde{t}\right].$$

Therefore, $\left(\mathbf{x}\left(\tilde{t}\right), \mathbf{q}\left(\tilde{t}\right)\right) \in \mathcal{T}$ implies that $\|\mathbf{q}_2(t)\|_2 \to \infty$ as $t \to \infty$. Clearly, since $\tilde{t}$ is arbitrary, such an implication contradicts the boundedness of the PDM given by Lemma 3. Consequently, we conclude that $\mathcal{T}$ must be an empty set and thus $\mathcal{E} = \mathcal{I}$. This observation completes the proof. ■

Corollary 1 provides sufficient conditions to certify the asymptotic stability of the set of equilibria of EDM-PDMs when the PDM in (5) is considered. As we show in Section V, such a result allows us to guarantee the effectiveness of our proposed approach for solving the distributed NE seeking problem of the forthcoming section.

IV. Distributed NE Seeking in Population Games

Recall that, according to the NE seeking problem of Definition 7, the goal is to design a PDM that renders the set $\mathrm{NE}(\mathbf{f})$ asymptotically stable under the corresponding EDM-PDM. Moreover, we seek to accomplish such a goal under a partial-decision information scheme. Namely, it is assumed that each population $k \in \mathcal{P}$ has an associated higher level entity, here referred to as the payoff provider of population k, which provides the payoff vector $\mathbf{p}^k(t)$ to the agents of population k at time t. In addition, the payoff provider of population k has direct access only to the strategic distribution of population k ($\mathbf{x}^k$), and it is not able to directly measure the strategic distribution of any other population $\ell \in \mathcal{P} \setminus \{k\}$. However, the payoff providers are allowed to communicate with each other through a possibly non-complete network. Consequently, by defining the PDM as the ensemble of all payoff providers, the goal is to design each payoff provider (synthesized as a continuous-time dynamical system), so that the resulting PDM solves the aforementioned distributed NE seeking problem.

Throughout, we impose the following assumptions on the fitness functions of the population game G.

Standing Assumption 1: For all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, the continuously differentiable and Lipschitz continuous map $f_i^k(\cdot)$ is of the aggregative form $f_i^k(\mathbf{x}) = g_i^k\left(\mathbf{x}^k, \boldsymbol{\sigma}(\mathbf{x})\right)$, where $g_i^k : \mathbb{R}_{\geq 0}^{n^k} \times \mathbb{R}^r \to \mathbb{R}$ and $\boldsymbol{\sigma}(\mathbf{x}) = (1/N)\sum_{\ell\in\mathcal{P}} \boldsymbol{\phi}^\ell\left(\mathbf{x}^\ell\right)$, with $r \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{\phi}^\ell : \mathbb{R}_{\geq 0}^{n^\ell} \to \mathbb{R}^r$, for all $\ell \in \mathcal{P}$. Moreover, $\left\|g_i^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right) - g_i^k\left(\mathbf{x}^k, \tilde{\mathbf{q}}_1^k\right)\right\|_2 \leq L_{g_i^k}\left\|\mathbf{q}_1^k - \tilde{\mathbf{q}}_1^k\right\|_2$, for some $L_{g_i^k} \in \mathbb{R}_{>0}$, for all $\mathbf{x}^k \in \Delta^k$, and for all $\mathbf{q}_1^k, \tilde{\mathbf{q}}_1^k \in \mathbb{R}^r$; and $\left\|\boldsymbol{\phi}^k\left(\mathbf{x}^k\right) - \boldsymbol{\phi}^k\left(\tilde{\mathbf{x}}^k\right)\right\|_2 \leq L_{\boldsymbol{\phi}^k}\left\|\mathbf{x}^k - \tilde{\mathbf{x}}^k\right\|_2$, for some $L_{\boldsymbol{\phi}^k} \in \mathbb{R}_{>0}$, and for all $\mathbf{x}^k, \tilde{\mathbf{x}}^k \in \mathbb{R}_{\geq 0}^{n^k}$.

Under Standing Assumption 1, it follows that the fitness functions depend on the strategic distribution of the entire society through the aggregate term $\boldsymbol{\sigma}(\mathbf{x})$. Thus, the fitness function of a given strategy at a given population is, in general, affected by the strategic distribution of the whole society. It is important to keep this observation in mind as we seek to design an NE seeking method under partial-decision information.

Now, to define the communication between the multiple payoff providers, we let $\mathcal{G}_c = (\mathcal{P}, \mathcal{L}_c, \mathbf{W})$ be the directed graph (digraph) associated to the communication network. Here, $\mathcal{P}$ is the set of nodes corresponding to the payoff providers (which are indexed according to their associated populations), $\mathcal{L}_c \subseteq \mathcal{P} \times \mathcal{P}$ is the set of links of possible communication, and $\mathbf{W} \in \mathbb{R}_{\geq 0}^{N\times N}$ is the weighted adjacency matrix that captures the topology of the digraph. Throughout, we say that $(\ell, k) \in \mathcal{L}_c$ if and only if node k can receive information from node ℓ, and for simplicity we adopt the convention that $(k,k) \notin \mathcal{L}_c$. Moreover, $w_{k\ell} > 0$ for all $(\ell, k) \in \mathcal{L}_c$, and $w_{k\ell} = 0$ otherwise. Here, $w_{k\ell}$ denotes the (k,ℓ)-th element of $\mathbf{W}$. Furthermore, we let $\mathcal{N}_{\text{in}}^k = \{\ell \in \mathcal{P} : w_{k\ell} > 0\}$ denote the set of in-neighbors of node k, for all $k \in \mathcal{P}$, and we let $\mathbf{L} = \operatorname{diag}(\mathbf{W}\mathbf{1}_N) - \mathbf{W}$ be the Laplacian matrix associated to $\mathcal{G}_c$. In addition, we impose the following conditions on $\mathcal{G}_c$.

Standing Assumption 2: The digraph $\mathcal{G}_c$ is strongly connected and weight-balanced.

Remark 3: Following [35], the strong connectivity in Standing Assumption 2 implies that there exists a directed path from any node to any other node in $\mathcal{G}_c$. On the other hand, the weigh-balanced condition means that $\sum_{\ell\in\mathcal{P}} w_{k\ell} = \sum_{\ell\in\mathcal{P}} w_{\ell k}$, for all $k \in \mathcal{P}$, i.e., the sum of in-weights matches the sum of out-weights at every node, and thus $\mathbf{1}_N^\top \mathbf{L} = \mathbf{0}_N^\top$. Overall, Standing Assumption 2 implies that $\operatorname{rank}(\mathbf{L}) = N - 1$ and $\lambda_{\min}\left(\mathbf{L} + \mathbf{L}^\top\right) = 0$. Finally, it is worth to highlight that a weight-balanced digraph can be constructed in a distributed manner from a strongly connected digraph [36], [37].

Having defined the information and communication related constraints to be considered, we now proceed to formulate the continuous-time dynamics that characterize each payoff provider. Namely, following the ideas in [24] and [32] on the so-called proportional integral consensus algorithm[4], for each population $k \in \mathcal{P}$ we formulate the corresponding payoff provider as a dynamical system of the form

$$\dot{\mathbf{q}}_1^k(t) = \frac{1}{\tau}\Bigg(-\mathbf{q}_1^k(t) - \sum_{\ell\in\mathcal{P}} w_{k\ell}\left(\mathbf{q}_1^k(t) - \mathbf{q}_1^\ell(t)\right) - \sum_{\ell\in\mathcal{P}} w_{\ell k}\left(\mathbf{q}_2^k(t) - \mathbf{q}_2^\ell(t)\right) + \boldsymbol{\phi}^k\left(\mathbf{x}^k(t)\right)\Bigg) \tag{16a}$$

$$\dot{\mathbf{q}}_2^k(t) = \frac{1}{\tau}\sum_{\ell\in\mathcal{P}} w_{k\ell}\left(\mathbf{q}_1^k(t) - \mathbf{q}_1^\ell(t)\right) \tag{16b}$$

$$\mathbf{p}^k(t) = \boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k(t), \mathbf{q}_1^k(t)\right), \tag{16c}$$

with $\mathbf{q}_1^k(0), \mathbf{q}_2^k(0) \in \mathbb{R}^r$, $\tau \in \mathbb{R}_{>0}$, and $\boldsymbol{\mathcal{F}}^k : \mathbb{R}_{\geq 0}^{n^k} \times \mathbb{R}^r \to \mathbb{R}^{n^k}$ given by $\boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right) = \operatorname{col}\left(g_1^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right), \cdots, g_{n^k}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right)\right)$. Namely, the intuition behind (16) is as follows. Under the dynamics in (16a)-(16b), the state variable $\mathbf{q}_1^k$ comprises an estimate of the aggregate term $\boldsymbol{\sigma}(\mathbf{x})$ held by the payoff provider of population k. If at time t such an estimate is correct, i.e., $\mathbf{q}_1^k(t) = \boldsymbol{\sigma}(\mathbf{x}(t))$, then it holds that

$$\begin{aligned}\mathbf{p}^k(t) &= \boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k(t), \boldsymbol{\sigma}(\mathbf{x}(t))\right)\\ &= \operatorname{col}\left(g_1^k\left(\mathbf{x}^k(t), \boldsymbol{\sigma}(\mathbf{x}(t))\right), \cdots, g_{n^k}^k\left(\mathbf{x}^k(t), \boldsymbol{\sigma}(\mathbf{x}(t))\right)\right)\\ &= \mathbf{f}^k(\mathbf{x}(t)).\end{aligned}$$

That is, under a correct estimate of the aggregate term, the dynamics in (16) align the payoff vector of population k to

[4] As discussed in [24], in contrast to the merely proportional consensus algorithm, the proportional integral consensus method does not require a correct initialization of the estimators to eliminate steady-state errors under constant inputs. This property is due to (16b) and the term $\sum_{\ell\in\mathcal{P}} w_{\ell k}\left(\mathbf{q}_2^k(t) - \mathbf{q}_2^\ell(t)\right)$ in (16a) (see the proof of Lemma 6).

the fitness vector of population k. Moreover, the dynamics in (16) can be computed in a distributed fashion under the communication ruled by $\mathcal{G}_c$. For the sake of clarity, in Fig. 2 we depict the overall considered framework.

Remark 4: Notice that to compute (16a)-(16b), the payoff provider of each population $k \in \mathcal{P}$ should in general know the global parameter τ. In addition, the payoff provider of each population k must know the k-th row of the adjacency matrix $\mathbf{W}$ (which contains its own weights on its neighbors' data), as well as the k-th column of $\mathbf{W}$ (which contains its neighbors' weights on its own data) [24]. Throughout, we consider such informational requirements as granted.

Based on (16), it follows that the ensemble of all payoff providers' dynamics can be compactly written as

$$\dot{\mathbf{q}}_1(t) = \frac{1}{\tau}\bigg(- (\mathbf{I}_{Nr} + \mathbf{L} \otimes \mathbf{I}_r)\, \mathbf{q}_1(t) - \left(\mathbf{L}^\top \otimes \mathbf{I}_r\right) \mathbf{q}_2(t) + \boldsymbol{\Phi}\left(\mathbf{x}(t)\right)\bigg) \tag{17a}$$

$$\dot{\mathbf{q}}_2(t) = \frac{1}{\tau}\left(\mathbf{L} \otimes \mathbf{I}_r\right) \mathbf{q}_1(t) \tag{17b}$$

$$\mathbf{p}(t) = \boldsymbol{\mathcal{F}}\left(\mathbf{x}(t), \mathbf{q}_1(t)\right), \tag{17c}$$

with $\mathbf{q}_1(0), \mathbf{q}_2(0) \in \mathbb{R}^{Nr}$, $\mathbf{q}_1(t) = \mathrm{col}\left(\mathbf{q}_1^1(t), \cdots, \mathbf{q}_1^N(t)\right)$, $\mathbf{q}_2(t) = \mathrm{col}\left(\mathbf{q}_2^1(t), \cdots, \mathbf{q}_2^N(t)\right)$,

$$\boldsymbol{\Phi}\left(\mathbf{x}(t)\right) = \mathrm{col}\left(\boldsymbol{\phi}^1\left(\mathbf{x}^1(t)\right), \cdots, \boldsymbol{\phi}^N\left(\mathbf{x}^N(t)\right)\right),$$

and

$$\boldsymbol{\mathcal{F}}\left(\mathbf{x}(t), \mathbf{q}_1(t)\right) = \begin{bmatrix} \boldsymbol{\mathcal{F}}^1\left(\mathbf{x}^1(t), \mathbf{q}_1^1(t)\right) \\ \vdots \\ \boldsymbol{\mathcal{F}}^N\left(\mathbf{x}^N(t), \mathbf{q}_1^N(t)\right) \end{bmatrix}.$$

Clearly, Standing Assumption 1 implies that $\boldsymbol{\Phi}(\cdot)$ is $L_{\boldsymbol{\Phi}}$-Lipschitz continuous under the Euclidean norm with $L_{\boldsymbol{\Phi}} = \max_{k\in\mathcal{P}} L_{\boldsymbol{\phi}^k}$, and that $\boldsymbol{\mathcal{F}}(\mathbf{x}, \cdot)$ is $L_{\boldsymbol{\mathcal{F}}}$-Lipschitz continuous under the Euclidean norm with $L_{\boldsymbol{\mathcal{F}}} = \max_{k\in\mathcal{P}} \max_{i\in\mathcal{S}^k} L_{g_i^k}$. Hence, under Standing Assumption 1, the dynamics in (17) can be viewed as a PDM in the sense of Definition 2, with $\mathbf{q}(t) = \mathrm{col}\left(\mathbf{q}_1(t), \mathbf{q}_2(t)\right)$ and $d = 2Nr$. Furthermore, to exploit the results in Section III, we impose the next requirement.

Standing Assumption 3: For all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, the functions $g_i^k(\cdot, \mathbf{q}_1^k)$ are such that the resulting function $\boldsymbol{\mathcal{F}}(\cdot, \mathbf{q}_1)$ is μ-strongly contractive in the sense that there exists a $\mu \in \mathbb{R}_{>0}$ such that $\mathbf{z}^\top \mathrm{D}_{\mathbf{x}} \boldsymbol{\mathcal{F}}\left(\mathbf{x}, \mathbf{q}_1\right) \mathbf{z} \leq -\mu \mathbf{z}^\top \mathbf{z}$, for all $\mathbf{x} \in \Delta$, all $\mathbf{q}_1 \in \mathbb{R}^{Nr}$, and all $\mathbf{z} \in \mathbb{R}^n$.

It is worth to highlight that for a given population game G and EDM, the convergence to an NE under the conventional memoryless oracle-like payoff provider of [2] does not guarantee the convergence to an NE under the proposed PDM in (17). More precisely, the fact that the EDM-PDM with the oracle-like PDM characterized by $\mathbf{p}(t) = \mathbf{f}\left(\mathbf{x}(t)\right)$ converges to an NE of a game G does not immediately imply, in general, that the EDM-PDM with the PDM in (17) converges to an NE of the same game G. We support this claim through the following academic example.

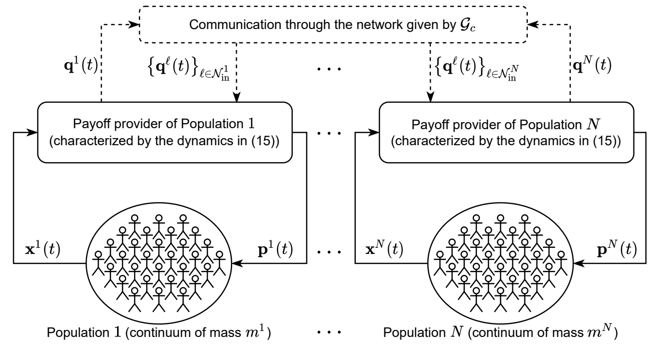

Fig. 2. Considered framework. Here, $\mathbf{q}^k(t) = \mathrm{col}\left(\mathbf{q}_1^k(t), \mathbf{q}_2^k(t)\right)$, for all $k \in \mathcal{P}$. Besides, observe that the payoff providers receive the state vectors of their in-neighbors.

Example 3: Consider a population game G with $N = 10$, $m^k = 1$, for all $k \in \mathcal{P}$, and characterized by fitness vectors of the form[5]

$$\mathbf{f}^k\left(\mathbf{x}\right) = -\mu \mathbf{x}^k + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \boldsymbol{\sigma}\left(\mathbf{x}\right), \quad \forall k \in \mathcal{P},$$

with $\mu \in \mathbb{R}_{>0}$, and $\boldsymbol{\sigma}(\mathbf{x}) = (1/N)\sum_{\ell\in\mathcal{P}} \mathbf{x}^\ell$. Moreover, consider an EDM characterized by the Smith dynamics in (1), and consider two different PDMs: i) the conventional oracle-like PDM given by $\mathbf{p}(t) = \mathbf{f}\left(\mathbf{x}(t)\right)$; and ii) the PDM in (17) with $\tau = L_{\boldsymbol{\Phi}}/L_{\boldsymbol{\mathcal{F}}} \approx 1/1.73$ and with $\mathcal{G}_c$ taken as a directed cycle with unitary weights. Namely, the PDM in i) corresponds to a centralized payoff provider with full-decision information, as in [2], while the PDM in ii) represents the proposed distributed approach subject to partial-decision information. In Fig. 3, we depict the numerical simulations of the corresponding EDM-PDMs considering two different values of μ. We remark that whilst the oracle-like approach converges to the NE for both values of μ, the distributed approach only converges to the NE for the first value of $\mu = 1.73$. Therefore, the convergence of the conventional (centralized) dynamics is not enough to guarantee the convergence under the proposed distributed scheme.

Example 3 highlights the importance of characterizing sufficient conditions on the underlying population game to assert the convergence of the proposed distributed approach to an NE. As such, we now proceed to formally characterize such sufficient conditions.

V. Convergence Analysis

In this section, we analyze the distributed NE seeking approach proposed in Section IV under the light of the results of Section III. Throughout, the reader should keep in mind Standing Assumptions 1-3.

To start the discussion, we first show that the PDM in (17) satisfies Assumption 1.

[5]The considered fitness vectors represent an aggregative Rock-Paper-Scissors [2] multi-population game with added μ-strong contractivity.

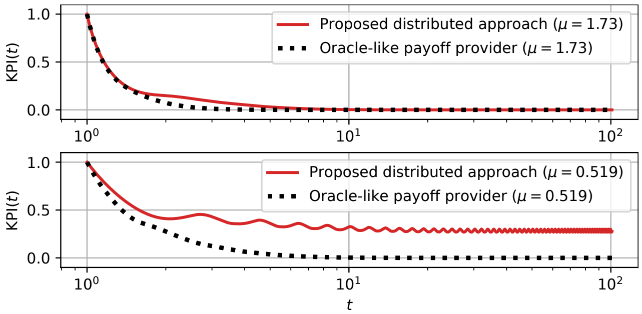

Fig. 3. Simulation results for Example 3. Without loss of generality, in all cases we let $\mathbf{x}^k(0) = \mathrm{col}(1, 0, 0)$, and $\mathbf{q}_1^k(0) = \mathbf{q}_2^k(0) = \mathbf{0}_{Nr}$, for all $k \in \mathcal{P}$. Moreover, the key performance index (KPI) is taken as $\mathrm{KPI}(t) = \|\mathbf{x}(t) - \mathbf{x}^*\|_2 / \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^* = (1/3)\mathbf{1}_n$ is the unique NE of G. Hence, the convergence of $\mathrm{KPI}(t)$ to 0 implies the convergence of $\mathbf{x}(t)$ to the NE of the population game G.

Lemma 6: Assumption 1 holds under the PDM in (17) with any $\tau > 0$.

Proof: Due to Standing Assumption 2, it holds that

$$\begin{aligned}\sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{k\ell}\left(\mathbf{q}_1^k - \mathbf{q}_1^\ell\right) &= \sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{k\ell}\mathbf{q}_1^k - \sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{k\ell}\mathbf{q}_1^\ell \\ &= \sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{k\ell}\mathbf{q}_1^k - \sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{\ell k}\mathbf{q}_1^k \\ &= \sum_{k\in\mathcal{P}} \mathbf{q}_1^k \left(\sum_{\ell\in\mathcal{P}} w_{k\ell} - \sum_{\ell\in\mathcal{P}} w_{\ell k}\right) \\ &= \mathbf{0}_{Nr}.\end{aligned}$$

Similarly, $\sum_{k\in\mathcal{P}}\sum_{\ell\in\mathcal{P}} w_{\ell k}\left(\mathbf{q}_2^k - \mathbf{q}_2^\ell\right) = \mathbf{0}_{Nr}$. Therefore, from (16a) it follows that, for all $t \geq 0$,

$$\sum_{k\in\mathcal{P}} \dot{\mathbf{q}}_1^k(t) = -\frac{1}{\tau}\sum_{k\in\mathcal{P}} \mathbf{q}_1^k(t) + \frac{1}{\tau}\sum_{k\in\mathcal{P}} \boldsymbol{\phi}^k\left(\mathbf{x}^k(t)\right).$$

Hence, $\dot{\mathbf{q}}_1(t) = \mathbf{0}_{Nr} \Rightarrow \sum_{k\in\mathcal{P}} \mathbf{q}_1^k(t) = \sum_{k\in\mathcal{P}} \boldsymbol{\phi}^k\left(\mathbf{x}^k(t)\right)$. On the other hand, from Standing Assumption 2 and (16b), it holds that $\dot{\mathbf{q}}_2(t) = \mathbf{0}_{Nr}$ if and only if $\mathbf{q}_1^k(t) = \mathbf{q}_1^\ell(t)$ for all $k, \ell \in \mathcal{P}$. Consequently,

$$\begin{aligned}\dot{\mathbf{q}}(t) = \mathbf{0}_{2Nr} \Rightarrow \mathbf{q}_1^k(t) &= \frac{1}{N}\sum_{\ell\in\mathcal{P}} \boldsymbol{\phi}^\ell\left(\mathbf{x}^\ell(t)\right), \ \forall k \in \mathcal{P} \\ \Rightarrow \mathbf{q}_1^k(t) &= \boldsymbol{\sigma}\left(\mathbf{x}(t)\right), \ \forall k \in \mathcal{P} \\ \Rightarrow \mathbf{p}^k(t) &= \mathbf{f}^k\left(\mathbf{x}(t)\right), \ \forall k \in \mathcal{P} \\ \Rightarrow \mathbf{p}(t) &= \mathbf{f}\left(\mathbf{x}(t)\right),\end{aligned}$$

which completes the proof. ■

Lemma 6 allows us to invoke the results of Lemma 2 for any EDM-PDM comprised of a Nash stationary EDM and the PDM in (17). This fact is summarized in Lemma 7.

Lemma 7: Consider an EDM-PDM comprised of a Nash stationary EDM and the PDM in (17) with any $\tau > 0$. The corresponding set of equilibria $\mathcal{E}$ is nonempty and satisfies that $(\mathbf{x}^*, \mathbf{q}^*) \in \mathcal{E} \Rightarrow \mathbf{x}^* \in \mathrm{NE}(\mathbf{f})$.

Proof: The implication ($\Rightarrow$) follows from Lemmas 2 and 6. To prove that $\mathcal{E}$ is nonempty, it suffices to see that $\mathrm{NE}(\mathbf{f})$ is nonempty and compact (by Lemma 1), and to note that $\mathbf{x}^* \in \mathrm{NE}(\mathbf{f})$, $\mathbf{q}_1^* = \mathbf{1}_N \otimes \boldsymbol{\sigma}(\mathbf{x}^*)$, and $\mathbf{q}_2^* \in \mathrm{span}(\mathbf{1}_{Nr})$, comprise an equilibrium of the considered EDM-PDM. ■

We now proceed to show that the PDM in (17) also satisfies Assumption 2. In order to do so, we reformulate the PDM in (17) in an equivalent reduced-order form as follows. Let $\mathbf{v} = \left(1/\sqrt{N}\right)\mathbf{1}_N$ and let $\mathbf{U} = \left[\tilde{\mathbf{U}}, \mathbf{v}\right] \in \mathbb{R}^{N\times N}$ be an orthonormal matrix. Based on $\mathbf{U}$, consider the change of variable $\mathbf{q}_2(t) = (\mathbf{U}\otimes\mathbf{I}_r)\begin{bmatrix}\tilde{\mathbf{q}}_2(t)\\ \hat{\mathbf{q}}_2(t)\end{bmatrix}$, with $\tilde{\mathbf{q}}_2(t) \in \mathbb{R}^{Nr-r}$ and $\hat{\mathbf{q}}_2(t) \in \mathbb{R}^r$. Under such a change of variable, we have

$$\begin{aligned}\left(\mathbf{L}^\top\otimes\mathbf{I}_r\right)\mathbf{q}_2(t) &= \left(\mathbf{L}^\top\otimes\mathbf{I}_r\right)\left(\mathbf{U}\otimes\mathbf{I}_r\right)\begin{bmatrix}\tilde{\mathbf{q}}_2(t)\\ \hat{\mathbf{q}}_2(t)\end{bmatrix} \\ &= \left(\left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_r\right)\tilde{\mathbf{q}}_2(t).\end{aligned}$$

Here, the second equality follows from the properties of the Kronecker product and the fact that $\mathbf{L}^\top\mathbf{v} = \mathbf{0}_N$ (c.f., Remark 3). On the other hand, notice that

$$\begin{aligned}\begin{bmatrix}\dot{\tilde{\mathbf{q}}}_2(t)\\ \dot{\hat{\mathbf{q}}}_2(t)\end{bmatrix} &= \left(\mathbf{U}\otimes\mathbf{I}_r\right)^{-1}\dot{\mathbf{q}}_2(t) \\ &= \left(\mathbf{U}^\top\otimes\mathbf{I}_r\right)\dot{\mathbf{q}}_2(t) \quad \text{[since } \mathbf{U} \text{ is orthonormal]} \\ &= \frac{1}{\tau}\left(\mathbf{U}^\top\otimes\mathbf{I}_r\right)\left(\mathbf{L}\otimes\mathbf{I}_r\right)\mathbf{q}_1(t) \quad \text{[using (17b)]} \\ &= \begin{bmatrix}\frac{1}{\tau}\left(\left(\tilde{\mathbf{U}}^\top\mathbf{L}\right)\otimes\mathbf{I}_r\right)\mathbf{q}_1(t)\\ \mathbf{0}_r\end{bmatrix},\end{aligned}$$

Therefore, under the considered change of variable, the dynamics in (17a)-(17b) can be equivalently written as

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{q}}_1(t)\\ \dot{\tilde{\mathbf{q}}}_2(t)\\ \dot{\hat{\mathbf{q}}}_2(t)\end{bmatrix} &= \frac{1}{\tau}\begin{bmatrix}\mathbf{A} & -\mathbf{B} & \mathbf{0}\\ \mathbf{B}^\top & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{q}_1(t)\\ \tilde{\mathbf{q}}_2(t)\\ \hat{\mathbf{q}}_2(t)\end{bmatrix} \\ &\quad + \frac{1}{\tau}\begin{bmatrix}\boldsymbol{\Phi}\left(\mathbf{x}(t)\right)\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix},\end{aligned}$$

with $\mathbf{0}$ having appropriate dimensions in all cases, and

$$\begin{aligned}\mathbf{A} &= -\left(\mathbf{I}_{Nr} + \mathbf{L}\otimes\mathbf{I}_r\right)\\ \mathbf{B} &= \left(\mathbf{L}^\top\tilde{\mathbf{U}}\right)\otimes\mathbf{I}_r.\end{aligned}$$

In particular, observe that $\hat{\mathbf{q}}_2$ is an uncontrollable state variable that does not interact with the rest of the states. Hence, without loss of generality, we can ignore $\hat{\mathbf{q}}_2$ and reformulate the PDM in (17) as the reduced-order PDM given by

$$\dot{\mathbf{q}}_1(t) = \frac{1}{\tau}\left(\mathbf{A}\mathbf{q}_1(t) - \mathbf{B}\tilde{\mathbf{q}}_2(t) + \boldsymbol{\Phi}\left(\mathbf{x}(t)\right)\right) \tag{18a}$$

$$\dot{\tilde{\mathbf{q}}}_2(t) = \frac{1}{\tau}\mathbf{B}^\top\mathbf{q}_1(t) \tag{18b}$$

$$\mathbf{p}(t) = \boldsymbol{\mathcal{F}}\left(\mathbf{x}(t), \mathbf{q}_1(t)\right). \tag{18c}$$

Remark 5: Given that the state variable $\hat{\mathbf{q}}_2$ does not interact with $\mathbf{q}_1$ and $\tilde{\mathbf{q}}_2$, it follows that, under matching initial conditions, the PDMs in (17) and (18) have the same input-output behaviors. More precisely, if the $\tilde{\mathbf{q}}_2$ and $\mathbf{q}_2$ initial conditions satisfy that $\tilde{\mathbf{q}}_2(0) = \tilde{\mathbf{U}}^\top\mathbf{q}_2(0)$, and both PDMs are given the exact same input $\mathbf{x}(t)$ for all $t \geq 0$, then both PDMs produce exactly the same output $\mathbf{p}(t)$, for all $t \geq 0$. Consequently, to study the δ-antipassivity of the PDM in (17), it suffices to analyze the reduced-order PDM in (18).

Clearly, the reduced-order PDM in (18) has the same form as the PDM in (5), studied in Section III. Moreover, from Standing Assumption 2, the matrix $\mathbf{A} + \mathbf{A}^\top = -\left(2\mathbf{I}_{Nr} + \left(\mathbf{L} + \mathbf{L}^\top\right) \otimes \mathbf{I}_r\right)$ is negative definite, and the matrix $\mathbf{B}$ is full column rank (c.f., Remark 3). Thus, from [32, Lemma 2.2] we conclude that the matrix $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} \end{bmatrix}$ is Hurwitz. Consequently, the PDM in (18) satisfies Assumptions 2-i) and 2-ii). Furthermore, from Standing Assumption 1 it follows that the PDM in (18) also satisfies Assumptions 2-iii) and 2-iv). Finally, the PDM in (18) satisfies Assumption 2-v) by means of Standing Assumption 3. Therefore, by Remark 5, the PDM in (17) satisfies Assumption 2.

Based on these observations, and exploiting the results in Section III, we now formally state some sufficient conditions to assert the asymptotic stability of the set of Nash equilibria of population games under any EDM-PDM comprised of a Nash stationary EDM and our proposed PDM in (17).

Corollary 2: Consider an EDM-PDM comprised of a Nash stationary δ-passive EDM with informative $\zeta(\cdot,\cdot)$ and the PDM in (17). Moreover, let at least one of the following two cases hold:

i) The parameters $L_{\mathbf{\Phi}}$, $L_{\boldsymbol{\mathcal{F}}}$, μ, and τ satisfy that

$$L_{\mathbf{\Phi}} = \max\nolimits_{k\in\mathcal{P}} L_{\boldsymbol{\phi}^k} \tag{19a}$$

$$L_{\boldsymbol{\mathcal{F}}} = \max_{k\in\mathcal{P}} \max_{i\in\mathcal{S}^k} L_{g_i^k} \tag{19b}$$

$$L_{\mathbf{\Phi}} L_{\boldsymbol{\mathcal{F}}} \leq \mu \tag{19c}$$

$$\frac{L_{\mathbf{\Phi}}^2}{\mu} \leq \tau \leq \frac{\mu}{L_{\boldsymbol{\mathcal{F}}}^2}. \tag{19d}$$

ii) Standing Assumption 3 holds with the relaxation $\mu = 0$, and $\mathrm{D}_{\mathbf{q}_1^k} \boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right)^\top = -(1/\tau)\mathrm{D}_{\mathbf{x}^k} \boldsymbol{\phi}^k\left(\mathbf{x}^k\right)$, for all $\mathbf{x}^k \in \Delta^k$, all $\mathbf{q}_1^k \in \mathbb{R}^r$, and all $k \in \mathcal{P}$.

Then, the set of equilibria $\mathcal{E}$ is asymptotically stable and, for all $(\mathbf{x}^*, \mathbf{q}^*) \in \mathcal{E}$, it holds that $\mathbf{x}^* \in \mathrm{NE}(\mathbf{f})$, i.e., $\mathbf{x}^*$ is an NE of the game G.

Proof: As discussed in Remark 5, without loss of generality we can consider the reduced-order PDM in (18) instead of the one in (17). Now, from Lemma 7, and the fact that the matrix $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} \end{bmatrix}$ is Hurwitz, it follows that the set of equilibria $\mathcal{E}$ is nonempty and compact. Also, under the PDM in (18), it holds that $\left|\lambda_{\max}\left(\mathbf{A} + \mathbf{A}^\top\right)\right| = 2$, which implies that (8) reduces to (19c)-(19d). Consequently, the results follow from Corollary 1 and Lemma 7. ∎

Remark 6: Recall from Remark 4 that the payoff providers are assumed to know the value of τ. Also, note that if (19c) holds, then one can simply set $\tau = L_{\mathbf{\Phi}}/L_{\boldsymbol{\mathcal{F}}}$ to satisfy (19d). Thus, if τ is set as $\tau = L_{\mathbf{\Phi}}/L_{\boldsymbol{\mathcal{F}}}$, then the payoff providers should know both $L_{\mathbf{\Phi}}$ and $L_{\boldsymbol{\mathcal{F}}}$. Now, given that $L_{\mathbf{\Phi}} = \max_{k\in\mathcal{P}} L_{\boldsymbol{\phi}^k}$ and $L_{\boldsymbol{\mathcal{F}}} = \max_{k\in\mathcal{P}} \max_{i\in\mathcal{S}^k} L_{g_i^k}$, it turns out that $L_{\mathbf{\Phi}}$ and $L_{\boldsymbol{\mathcal{F}}}$ can be computed over $\mathcal{G}_c$ as follows. Define a variable $\tilde{L}_{\mathbf{\Phi}}^k$ with $\tilde{L}_{\mathbf{\Phi}}^k(0) = L_{\boldsymbol{\phi}^k}$, and update it iteratively as $\tilde{L}_{\mathbf{\Phi}}^k(\kappa+1) = \max\left\{\tilde{L}_{\mathbf{\Phi}}^k(\kappa), \tilde{L}_{\mathbf{\Phi}}^\ell(\kappa), \forall \ell \in \mathcal{N}_{\text{in}}^k\right\}$, with $\kappa \in \mathbb{Z}_{\geq 0}$. That way, one can compute $L_{\mathbf{\Phi}}$ within $N-1$ iterations. Clearly, a similar approach can be employed to compute $L_{\boldsymbol{\mathcal{F}}}$.

Based on Corollary 2, we conclude that the proposed approach is indeed suitable to solve the considered distributed NE seeking problem. For the sake of illustration, we now proceed to apply the overall developed theory to a practical numerical scenario.

VI. An Illustrative Application

To illustrate the application of our developed theory, we consider a multi-population congestion game. Note that congestion games, as relevant engineering problems [25], have been also considered in some of the previous works on population games [3], [6], and for that reason, we consider such an example game in this paper as well. Namely, consider a set of $N = 10$ populations and suppose that each population $k \in \mathcal{P}$ seeks to travel from an origin O^k to a destination D^k by using a set of available routes. Hence, the goal is for each population of agents to allocate themselves over their available routes while considering certain congestion costs for the routes. Moreover, the traveling routes are shared over the multiple populations, and so the decision-making task of the multiple populations is coupled. More precisely, while $O^k \neq O^\ell$ or $D^k \neq D^\ell$, for every $k, \ell \in \mathcal{P}$ with $k \neq \ell$, the routes available to travel from O^k to D^k might also be employed to travel from O^ℓ to D^ℓ, and so the congestion of a route depends on the allocation of all the populations that may employ that route. Without loss of generality, we assume that there is a total of $r = 7$ routes, and we let $\mathcal{S}^k$ represent the subset of routes that allow population k to travel from O^k to D^k (thus $\left|\mathcal{S}^k\right| \leq r$). Furthermore, we let $\mathbf{C}^k \in \mathbb{R}^{r\times n^k}$ be a matrix defining a bipartite graph between population k and the routes in $\mathcal{S}^k$. For instance, if the agents of population k were allowed to travel through routes 1, 7, and 14, then $\mathbf{C}^k = [\mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_{14}]$, where $\mathbf{e}_i \in \mathbb{R}^r$ denotes the i-th column vector of the $r \times r$ identity matrix. Therefore, the sum $\sum_{\ell\in\mathcal{P}} \mathbf{C}^\ell \mathbf{x}^\ell = \mathbf{C}\mathbf{x}$ summarizes how the entire society is allocated over all the routes (here, $\mathbf{C} \in \mathbb{R}^{r\times n}$ is constructed as $\mathbf{C} = \left[\mathbf{C}^1, \mathbf{C}^2, \ldots, \mathbf{C}^N\right]$).

Based on the allocation $\mathbf{C}\mathbf{x}$, it is assumed that the routes are subject to the congestion cost

$$\boldsymbol{\beta}(\mathbf{C}\mathbf{x}) = \mathrm{col}\left(\beta_1\left(\mathbf{e}_1^\top \mathbf{C}\mathbf{x}\right), \beta_2\left(\mathbf{e}_2^\top \mathbf{C}\mathbf{x}\right), \ldots, \beta_r\left(\mathbf{e}_r^\top \mathbf{C}\mathbf{x}\right)\right),$$

where $\beta_z : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, L_{β_z}-Lipschitz continuous (for some $L_{\beta_z} \in \mathbb{R}_{>0}$), and provides the congestion cost for route z, for all $z \in \{1, 2, \ldots, r\}$. Consequently, we let the fitness vector of every population k be given by

$$\mathbf{f}^k(\mathbf{x}) = -\mathbf{D}^k \mathbf{x}^k - \mathbf{C}^{k\top} \boldsymbol{\beta}(\mathbf{C}\mathbf{x}), \tag{20}$$

where $\mathbf{D}^k \in \mathbb{R}_{\geq 0}^{n^k\times n^k}$ is a diagonal matrix encoding the preferences of population k over their available routes.

Throughout, we denote the minimum diagonal entry of $\mathbf{D}^k$ as μ^k.

Remark 7: We highlight that, if interpreted as the gradients of a set of utility functions (one for each population), fitness vectors of the form in (20) appear in several practical applications even beyond congestion games. Some examples include demand response management [38], charging coordination of electric vehicles [39], and Cournot games [40, Section 7.1], among others. For further details on the interpretation of fitness vectors as pseudo-gradient mappings, we refer the reader to [26, Section 6].

Given that each population $k \in \mathcal{P}$ may have a different origin O^k, it is safe to assume that populations might be spatially distributed over some geographical region. Therefore, the conventional approach with a single centralized oracle-like payoff provider with full-decision information may not be viable for the problem under consideration. As such, we consider our distributed framework and introduce a dedicated payoff provider for each population. Moreover, without loss of generality, we let the communication graph $\mathcal{G}_c$ be a directed cycle with unitary weights. Hence, we let the payoff provider of each population k be characterized by the dynamics in (16) with

$$\begin{aligned}\boldsymbol{\phi}^k\left(\mathbf{x}^k\right) &= N\mathbf{C}^k\mathbf{x}^k \\ \boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right) &= -\mathbf{D}^k\mathbf{x}^k - \mathbf{C}^{k\top}\boldsymbol{\beta}\left(\mathbf{q}_1^k\right).\end{aligned}$$

Clearly, it follows that $\boldsymbol{\sigma}(\mathbf{x}) = (1/N)\sum_{\ell\in\mathcal{P}}\boldsymbol{\phi}^\ell\left(\mathbf{x}^\ell\right) = \mathbf{C}\mathbf{x}$, and so the considered fitness vectors in (20) are indeed of the aggregative form (c.f., Standing Assumption 1). Moreover, for all $k \in \mathcal{P}$ it holds that $\boldsymbol{\phi}^k(\cdot)$ is $L_{\boldsymbol{\phi}^k}$-Lipschitz continuous and $\boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \cdot\right)$ is $L_{\boldsymbol{\mathcal{F}}^k}$-Lipschitz continuous, with

$$\begin{aligned}L_{\boldsymbol{\phi}^k} &= N\left\|\mathbf{C}^k\right\|_2 \\ L_{\boldsymbol{\mathcal{F}}^k} &= \left\|\mathbf{C}^{k\top}\right\|_2\left(\max_{z\in\{1,2,\ldots,r\}} L_{\beta_z}\right).\end{aligned}$$

Here, we highlight that from our definition of $\mathbf{C}^k$ it follows that $\left\|\mathbf{C}^k\right\|_2 = \left\|\mathbf{C}^{k\top}\right\|_2 = 1$, for all $k \in \mathcal{P}$ (because it always holds that $\mathbf{C}^{k\top}\mathbf{C}^k = \mathbf{I}_{n^k}$, and thus the maximum singular value of $\mathbf{C}^k$ and $\mathbf{C}^{k\top}$ is 1). Therefore, Standing Assumptions 1 and 2 hold. In addition, if $\mu^k > 0$ for all $k \in \mathcal{P}$, then $\boldsymbol{\mathcal{F}}^k\left(\cdot, \mathbf{q}_1^k\right)$ is μ^k-strongly contractive for all k, and Standing Assumption 3 holds with $\mu = \min_{k\in\mathcal{P}}\mu^k$.

For the sake of illustration, we now consider two scenarios with different congestion costs $\boldsymbol{\beta}(\cdot)$. For our numerical simulations we consider the Smith dynamics given in (1) and the BNN dynamics reported in [27], which are Nash stationary δ-passive EDMs with informative $\zeta(\cdot,\cdot)$ functions [3]. Besides, we let $m^k = 1$ and we randomly set $\mathcal{S}^k$ ensuring that $2 \leq \left|\mathcal{S}^k\right| \leq r$, for all $k \in \mathcal{P}$.

A. First Scenario: Affine Congestion Costs

For this scenario we consider affine congestion costs. Namely, for every route $z \in \{1, 2, \ldots, r\}$, we let $\beta_z(\omega) = \gamma_z\omega + \alpha_z$, with $\alpha_z \in \mathbb{R}$ and $\gamma_z \in \mathbb{R}_{>0}$. For simplicity[6], we let $\gamma_z = \gamma$, for all z, which implies that $\boldsymbol{\beta}\left(\mathbf{q}_1^k\right) = \gamma\mathbf{q}_1^k + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = \mathrm{col}\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right)$, and so

$$\boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right) = -\mathbf{D}^k\mathbf{x}^k - \gamma\mathbf{C}^{k\top}\mathbf{q}_1^k - \mathbf{C}^{k\top}\boldsymbol{\alpha}, \quad \forall k \in \mathcal{P}.$$

Thus, $\mathrm{D}_{\mathbf{q}_1^k}\boldsymbol{\mathcal{F}}^k\left(\mathbf{x}^k, \mathbf{q}_1^k\right)^\top = -\gamma\mathbf{C}^k = -(\gamma/N)\mathrm{D}_{\mathbf{x}^k}\boldsymbol{\phi}^k\left(\mathbf{x}^k\right)$, for all $\mathbf{x}^k \in \Delta^k$, all $\mathbf{q}_1^k \in \mathbb{R}^r$, and all $k \in \mathcal{P}$. Hence, Case ii) of Corollary 2 holds by setting $\tau = N/\gamma$. Consequently, by means of Corollary 2 we can assert the asymptotic stability of the set of Nash equilibria of the considered population game G under any Nash stationary δ-passive EDM with informative $\zeta(\cdot,\cdot)$, and the PDM in (17) with $\tau = N/\gamma$. Notice that such a result holds even if $\mu^k = 0$ for some $k \in \mathcal{P}$, i.e., even if $\boldsymbol{\mathcal{F}}(\cdot, \mathbf{q}_1)$ is only merely contractive. Figure 4 depicts some numerical simulations to validate our results.

B. Second Scenario: Huber Congestion Costs

As general practical applications might be better modeled by nonlinear (non-affine) congestion functions, for this second scenario we consider non-affine congestion costs. One way to achieve this goal, whist still maintaining global smoothness requirements, is by employing continuously differentiable piecewise functions. As an example, for this second scenario we consider congestion costs based on the Huber loss [41]. Namely, for every route $z \in \{1, 2, \ldots, r\}$, we let

$$\beta_z(\omega) = \begin{cases} \frac{\gamma_z}{2}\omega^2 + \alpha_z, & \text{if } |\omega| \leq 1, \\ \gamma_z\omega - \frac{\gamma_z}{2} + \alpha_z, & \text{otherwise,} \end{cases}$$

with $\alpha_z \in \mathbb{R}$ and $\gamma_z \in \mathbb{R}_{>0}$. Clearly, $\beta_z(\cdot)$ is continuously differentiable and L_{β_z}-Lipschitz continuous with $L_{\beta_z} = \gamma_z$. Under the considered framework and following the notation in Case i) of Corollary 2, it holds that $L_{\boldsymbol{\Phi}} = N$, and $L_{\boldsymbol{\mathcal{F}}} = \bar{\gamma} = \max_{z\in\{1,2,\ldots,r\}}\gamma_z$. Thus, if $\mu \geq N\bar{\gamma}$, then Corollary 2 can be invoked to assert the asymptotic stability of the (unique) NE of the considered population game G under any Nash stationary δ-passive EDM with informative $\zeta(\cdot,\cdot)$, and the PDM in (17) with $\tau = L_{\boldsymbol{\Phi}}/L_{\boldsymbol{\mathcal{F}}} = N/\bar{\gamma}$. Figure 5 depicts some numerical simulations to validate our results.

VII. Concluding Remarks

In this paper, we have formulated a framework for distributed Nash equilibrium (NE) seeking in aggregative population games subject to partial-decision information. By employing a δ-passivity-based perspective, we have deduced sufficient conditions to guarantee the asymptotic stability of the set of Nash equilibria of certain merely contractive and strongly contractive aggregative population games, and the provided results hold for several δ-passive Nash stationary dynamics.

[6] We use homogeneous values of γ_z to easily check Case ii) of Corollary 2. However, it is also possible to invoke Corollary 2 for heterogeneous values of γ_z. For such, we might simply redefine the matrices $\mathbf{C}^k$ as $\mathbf{C}^k \triangleq \boldsymbol{\Gamma}\tilde{\mathbf{C}}^k$, where $\boldsymbol{\Gamma} = \mathrm{diag}(\sqrt{\gamma_1}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_r})$ and $\tilde{\mathbf{C}}^k$ is the original $\mathbf{C}^k$ matrix.

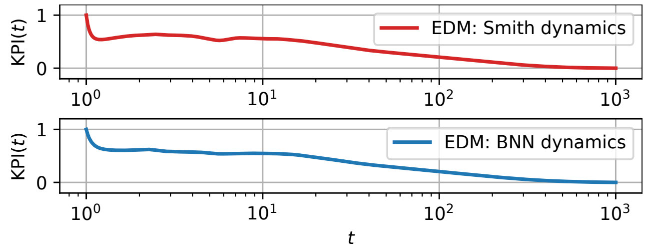

Fig. 4. Simulations for the first scenario under the proposed distributed NE seeking dynamics, for the Smith and BNN EDMs. Without loss of generality, we set $\gamma \sim \mathcal{U}[1,2]$, $\alpha_z \sim \mathcal{U}[-5,5]$, for all $z \in \{1,2,\ldots,r\}$, and we set $\mu^k \sim \mathcal{U}[1,2]$, for all $k \in \mathcal{P}\setminus\{\ell\}$, $\ell \in \mathcal{P}$, and $\mu^\ell = 0$ (to consider a merely contractive game). Besides, in both cases we let $\mathbf{x}^k(0) = (1/n^k)\mathbf{1}_{n^k}$, and $\mathbf{q}_1^k(0) = \mathbf{q}_2^k(0) = \mathbf{0}_{Nr}$, for all $k \in \mathcal{P}$. Moreover, the key performance index (KPI) is taken as $\text{KPI}(t) = \|\mathbf{x}(t) - \mathbf{x}^*\|_2 \,/\, \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^*$ is the achieved (non-unique) NE of the underlying population game G.

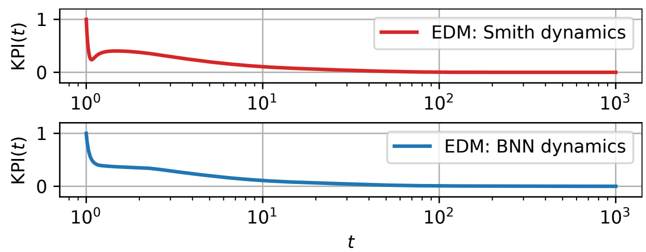

Fig. 5. Simulations for the second scenario under the proposed distributed NE seeking dynamics, for the Smith and BNN EDMs. Without loss of generality, we set $\gamma \sim \mathcal{U}[1,2]$ and $\alpha_z \sim \mathcal{U}[-5,5]$, for all $z \in \{1,2,\ldots,r\}$. Besides, we set $\mu^k \sim \mathcal{U}[2N, 2N+1]$, for all $k \in \mathcal{P}$, to ensure that $\mu \geq N\bar{\gamma}$. Besides, in both cases we let $\mathbf{x}^k(0) = (1/n^k)\mathbf{1}_{n^k}$, and $\mathbf{q}_1^k(0) = \mathbf{q}_2^k(0) = \mathbf{0}_{Nr}$, for all $k \in \mathcal{P}$. Moreover, the key performance index (KPI) is taken as $\text{KPI}(t) = \|\mathbf{x}(t) - \mathbf{x}^*\|_2 \,/\, \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^*$ is the achieved (unique) NE of the underlying population game G.

Beyond the context of distributed NE seeking dynamics, we have deduced sufficient conditions to certify the δ-antipassivity of a class of dynamic payoff mechanisms. Such a result is not only relevant for the distributed NE seeking problem studied in this paper, but also for future researches in the field of evolutionary game theory. As such, future work should further explore the applications of the developed theory to other evolutionary game theoretical problems.

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