New Bracket Polynomials Associated with the General Gough-Stewart Parallel Robot Singularities

Federico Thomas

***Abstract*— It is well known that the singularities of a Gough-Stewart platform arise when the determinant of the Plücker coordinates of the robot leg lines vanish. The direct expansion of this determinant in terms of the configuration of the moving platform leads to an intimidating algebraic expression which is difficult to organize in a manner that facilitates extracting geometric conditions for singularities to occur. The use of Grassmann-Cayley algebra has permitted expressing this determinant as a *bracket polynomial* which is easier to manipulate symbolically. Each monomial in this polynomial is the product of three *brackets*, 4×4 determinants involving the homogeneous coordinates of four leg attachments. In this paper, we show how to derive, using elementary linear algebra arguments, bracket polynomials where all brackets can be interpreted as reciprocal products between lines. Contrarily to what one might expect, these new bracket polynomials are simpler in general than those previously obtained using Grassmann-Cayley algebra.**

I. INTRODUCTION

A Gough-Stewart parallel robot incorporates six prismatic actuators, or *legs*, all of them connected simultaneously to a fixed *base* and a *moving platform* through spherical joints, or *attachments* (Fig. 1). It triggered the research on parallel manipulators and continues to be the center of many researches because, despite its simple geometry, its analysis translates into challenging mathematical problems. One of these problems is to characterize the configurations in which the moving platform becomes uncontrollable, that is, the singularities. In a singularity, there is a change in the robot's rigidity in certain directions. Therefore, the identification and avoidance of singularities are issues of practical importance. Although the singularity geometric identification problem has attracted a significant volume of research, it is still a far from simple task in the general case. It is generally restricted to relatively simple particular cases in which some extra constraints in the location of the leg attachments are introduced.

The algebraic characterization of the singularities of a Gough-Stewart parallel robot reduces to finding the robot configurations in which the determinant of the Plücker coordinates of the six leg lines vanish [1]. Then, a singularity implies a linear dependence between these vectors [2]. Alternatively, the geometric characterization approach tries to identify the geometric constraints that these lines must satisfy for these linear dependencies to occur [3], [4], [5], [6], and to translate them into geometric constructions to identify all the singularities. Since the nineteenth century both approaches, analytic and synthetic, have competed to provide elegant solutions to geometric problems [7].

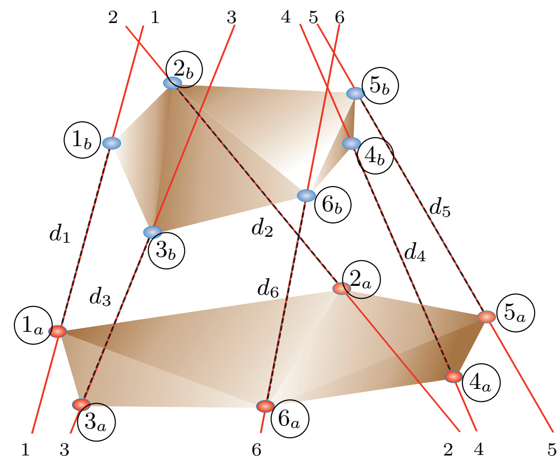

Fig. 1. A general Gough-Stewart parallel robot consists of six extensible legs (in dotted black) connecting a base (in orange) to a moving platform (in blue) through spherical joints. The leg line $i = 1,\ldots,6$ (in red) is determined by the location of the spherical joint attachments centers in the base and the platform, i_a and i_b, respectively. The length of leg i is $d_i = \|i_b - i_a\|$.

A synthetic approach to solve a geometry problem leads to geometric constructions without the use of coordinates or formulae. In general, this has the advantage of producing more general algorithms than those resulting from the analytic approach. In our case, the primitives of the synthetic geometry associated with the Gough-Stewart platform singularity problem are six lines, each of them defined by two points, and the tetrahedra defined by any subset of four of these points. In the characterization of parallel robot singularities, the synthetic approach has been dominated by the use of Grassmann-Cayley algebra. This paper is essentially devoted to present new alternative tools and results —to those already obtained using Grassmann-Cayley algebra— in which derivation only elementary linear algebra arguments have been used.

This paper is organized as follows. Section II briefly summarizes some basic facts on the singularities of the general Gough-Stewart parallel robot and the different bracket polynomials that have been previously derived using Grassmann-Cayley algebra. Section III presents the required operations between two lines that are needed in the derivation of the new bracket polynomials presented in Section IV. These new polynomials rely on the concept of *focal points* whose geometric interpretation is provided in Section V. Finally, we conclude in Section VI.

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Federico Thomas is with the Institut de Robòtica i Informàtica Industrial (CSIC-UPC), 08028 Barcelona, Spain. f.thomas@csic.es

II. GOUGH-STEWART PARALLEL ROBOT SINGULARITIES AND BRACKET POLYNOMIALS

Fig. 1 presents part of the notation used throughout this paper. To simplify the presentation, we incur in some abuses of notation. For example, no notational difference is used between scalars and vectors, or between points and their coordinates, but we have been careful to make the nature of each symbol unambiguous in its context. Although the usual practice when using Grassmann-Cayley algebra is to refer to the twelve legs attachments as $a, b, c, ..., l$, the information on which two points define a given leg is not explicit when using this notation. This decreases readability when working with reciprocal products. Thus, we have opted to denote leg lines by $i = 1, \ldots 6$, and the attachments of leg i to the base and the moving platform by i_a and i_b, respectively.

From the algebraic point of view, the singularities of a Gough-Stewart parallel robot are those robot configuration in which the matrix that maps the twist of the platform into the velocities of the actuators is singular. Using the principle or virtual work, this is equivalent to say that the singularities are those configurations in which the matrix that maps the wrench exerted by the moving platform into the forces exerted by the linear actuators in the extensible legs is singular. It directly follows that one matrix is the transpose of the other. Since the version of this analysis based on the robot statics is much easier [8], it is the one summarized next.

Since the legs of a Gough-Stewart parallel robot are connected to the base and the moving platform through spherical joints, only forces along the leg lines can be transmitted between the base and the platform. A force of magnitude f_i applied along leg i leads to the following force and torque in the moving platform expressed in the base reference frame:

$$\sigma_i = \frac{f_i}{d_i}(i_b - i_a),$$

and

$$\tau_i = i_a \times \sigma_i = \frac{f_i}{d_i}\left[i_a \times (i_b - i_a)\right] = \frac{f_i}{d_i}\left(i_a \times i_b\right),$$

respectively. Therefore, the total transmitted force and torque (wrench) is

$$W = \sum_{i=1}^{6} \begin{pmatrix} \sigma_i \\ \tau_i \end{pmatrix} = \underbrace{\begin{pmatrix} 1_b - 1_a & \ldots & 6_b - 6_a \\ 1_b \times 1_a & \ldots & 6_b \times 6_a \end{pmatrix}}_{\triangleq K} \begin{pmatrix} \frac{1}{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{d_6} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_6 \end{pmatrix}.$$

Now, assuming that all leg lengths are not null, if we want to know how a given wrench applied to the moving platform is distributed into six forces along the leg lines, we have to invert K. When this is not possible, the robot might collapse for some particular applied wrenches.

The columns of K are the Plücker coordinates [1] of the six leg lines. Then, a singularity implies a linear dependence between these vectors [2], a situation that occurs if these lines satisfy particular geometric constraints [3], [4]. The result of the leg lines becoming linearly dependent is that the matrix K drops rank. Then, the most straightforward way to identify the singular configurations is to analyze when det(K) vanishes. However, it is not obvious from the determinant of K alone as to which specific condition causes the drop in rank, despite all the different representations and strategies that have been used to achieve the explicit expression of det(K) [9], [10], [11], [12].

A way around the complexities of working with the analytic expansion of the determinant of K consists in using Grassman-Cayley algebra (see, for example, [13], [14]). Using this algebra, the determinant of K can be rewritten as a *bracket polynomial* formed by adding monomials of three multiplying brackets each. A bracket is a 4×4 determinant whose elements are the homogeneous coordinates of four leg attachments. Therefore, the singularity condition reduces to a bracket polynomial equated to zero. Since each bracket can be interpreted, up to an irrelevant constant factor, as the oriented volume of the tetrahedron formed by four attachments, the singularity condition thus obtained is known as the *pure condition* to stress the fact that it is intrinsically associated to the robot (that is, independent of the chosen reference frame). Hence, brackets may also be thought of as coordinate-free symbolic expressions. In what follows, the bracket of p_1, p_2, p_3 and p_4 will be denoted by $[p_1, p_2, p_3, p_4]$.

The pure condition is unique up to *syzygies*, a term borrowed from the literature on classic invariant theory, which refer to the three-term Grassmann-Plücker relations. In practice, this means that there are different equivalent pure conditions that vary in their number of terms and involved brackets. This certainly complicates things.

In 1983, White obtained the first pure condition for the Gough-Stewart parallel robot [15]. The obtained bracket polynomial had 96 adding monomials. White suggested that this was the minimum number of possible monomials for this polynomial. Nevertheless, in 1990, McMillan obtained a simpler polynomial in his thesis [16]. This bracket polynomial had 24 bracket monomials. Later on, in 2002, Downing *et al.* came up with even a simpler polynomial containing only 16 monomials [13]. Nevertheless, Ben-Horin and Shoham found more useful McMillan's than Downing *et al.*'s polynomial because the former was simpler to manipulate as all brackets appearing in it were in a particular lexicographic order known as straightened form [14]. For comparison purposes, these three bracket polynomials are included in the appendix.

We can say that the complexity of a bracket polynomial is not only given by the number of its monomials, but also by the total number of different brackets involved. It is important to observe that there are three kinds of brackets: 3-1 brackets (those involving three attachments in the base and one in the platform), 2-2 brackets (those involving two attachments in the base and two in the platform), and 1-3 brackets (those involving one attachment in the base and three in the platform). Calculating the corresponding combinations, we conclude that there are 120, 225, and 120 possible different 3-1, 2-2, and 1-3 brackets, respectively.

An important subset of 2-2 brackets is the set of brackets involving the attachments of two legs. There are only 15 such brackets which will be called *reciprocal brackets* for the reason that will be clear in the next section.

Each monomial in a bracket polynomial is the product of three brackets. All twelve attachments are present in every monomial and occur only once in each. Therefore, if a monomial is the product of a a-a' bracket, a b-b' bracket and a c-c' bracket, then $a+b+c = a'+b'+c' = 6$. In other words, each monomial in a bracket polynomial is the product of either three 2-2 brackets, or the product of one 1-3 bracket, one 3-1 bracket, and one 2-2 bracket. Now, one important question arises: would it be possible to derive a bracket polynomial involving only 2-2 brackets? In Section IV, we show that this is indeed possible. Before presenting these results, we need to review some facts on the reciprocal product of two lines and related concepts.

III. OPERATIONS WITH TWO LINES

A. Reciprocal product

The reciprocal product of lines i and j is defined as

$$i \otimes j \triangleq (i_b - i_a)\cdot(j_b \times j_a) + (j_b - j_a)\cdot(i_b \times i_a), \quad (1)$$

which, in matrix form, can be expressed as

$$i \otimes j = \begin{pmatrix} i_b - i_a \\ i_b \times i_a \end{pmatrix}^T \underbrace{\begin{pmatrix} 0_{3\times3} & I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{pmatrix}}_{\triangleq M} \begin{pmatrix} j_b - j_a \\ j_b \times j_a \end{pmatrix}. \quad (2)$$

Now, observe that, expressing the two triple products in (1) in determinant form, this product can be rewritten as

$$\begin{aligned} i \otimes j &= \begin{vmatrix} i_b - i_a & j_b & j_a \end{vmatrix} + \begin{vmatrix} j_b - j_a & i_b & i_a \end{vmatrix} \\ &= \begin{vmatrix} i_b - i_a & j_b - j_a & j_a \end{vmatrix} + \begin{vmatrix} j_b - j_a & i_b - i_a & i_a \end{vmatrix} \\ &= \begin{vmatrix} i_a & i_b - i_a & j_a & j_b - j_a \\ 1 & 0 & 1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} i_a & i_b & j_a & j_b \\ 1 & 1 & 1 & 1 \end{vmatrix} = [i_a, i_b, j_a, j_b]. \end{aligned} \quad (3)$$

In other words, the reciprocal product of two lines can be seen as a bracket. That is, as one sixth the oriented volume of the tetrahedron defined by the points defining the two lines. It is thus invariant to any Euclidean isometry.

In 1869, Cayley proved in [1] that the reciprocal product of two lines can also be expressed as

$$i \otimes j = d_i\, d_j\, \delta_{ij}\, \sin\varphi_{ij}, \quad (4)$$

where $d_k = \|k_b - k_a\|$, δ_{ij} is the shortest distance between the two lines, and φ_{ij} is the angle formed by both lines when projected along their common normal with the orientation resulting from the application of the right-hand rule.

Also in 1869, Drach proved in [17, p. 132] the further alternative expression

$$i \otimes j = d_i\, d_j\, \eta_{ij}\, \cos\vartheta_{ij}, \quad (5)$$

where $\eta_{ij} = \|i_a - j_a\|$ and ϑ_{ij} is the angle between the normal to the plane defined by the three points i_a, i_b and j_a and line j.

Fig. 2 graphically presents the above three alternative geometric interpretations of the reciprocal product between two lines. These alternative forms are highly relevant when combined with the new bracket polynomials presented in the next section to come up with new geometric interpretations of singularities.

B. Reverse product

We define the reverse product as the standard reciprocal product in which the second line is reflected over the origin [18, §7.2]. That is, the reverse product between lines i and j is defined as

$$\begin{aligned} i \odot j &\triangleq \begin{pmatrix} i_b - i_a \\ i_b \times i_a \end{pmatrix}^T \begin{pmatrix} j_b \times j_a \\ j_a - j_b \end{pmatrix} \\ &= \begin{pmatrix} i_b - i_a \\ i_b \times i_a \end{pmatrix}^T \underbrace{\begin{pmatrix} 0_{3\times3} & I_{3\times3} \\ -I_{3\times3} & 0_{3\times3} \end{pmatrix}}_{\triangleq N} \begin{pmatrix} j_b - j_a \\ j_b \times j_a \end{pmatrix}. \end{aligned} \quad (6)$$

As above, this product can be rewritten as

$$\begin{aligned} i \odot j &= \begin{vmatrix} i_a & i_b & -j_a & -j_b \\ 1 & 1 & 1 & 1 \end{vmatrix} = [i_a, i_b, -j_a, -j_b] \\ &= -\begin{vmatrix} -i_a & -i_b & j_a & j_b \\ 1 & 1 & 1 & 1 \end{vmatrix} = -[-i_a, -i_b, j_a, j_b]. \end{aligned}$$

Therefore, while the reciprocal product is commutative operation ($i \otimes j = j \otimes i$), the reverse product is anticommutative ($i \odot j = -j \odot i$).

C. Generalized reverse product and focal points

The reverse product can be extended to incorporate a reflection across an arbitrary point, say f, instead of the origin. We will refer to f as a *focal point*. Since the result of reflecting another arbitrary point, say v, across f is $2f - v$, we can generalize the definition of reverse product to the reverse product with respect to point p as follows

$$i \odot_p j \triangleq \begin{vmatrix} i_a & i_b & 2f - j_a & 2f - j_b \\ 1 & 1 & 1 & 1 \end{vmatrix}. \quad (7)$$

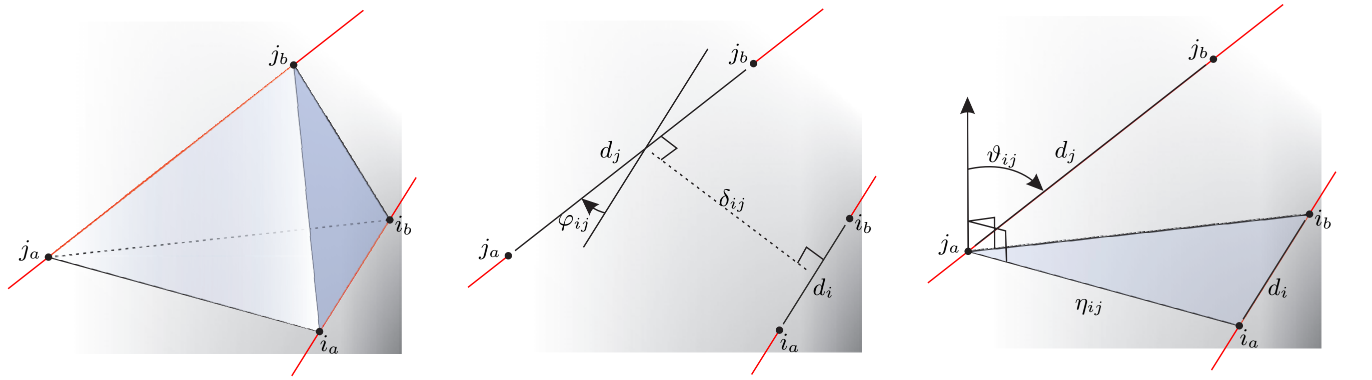
Now, an interesting problem arises: where have we to locate f so that $i \odot_f j = 0$? To solve this problem, first observe that

$$i \odot_f j = i \odot j + 4 \begin{vmatrix} i_a & i_b & f & \frac{j_a - j_b}{2} \\ 1 & 1 & 0 & 1 \end{vmatrix}. \quad (8)$$

Then, if f is chosen to be a point in the plane whose equation is

$$-\begin{vmatrix} i_{ay} & i_{by} & \frac{j_{ay}-j_{by}}{2} \\ i_{az} & i_{bz} & \frac{j_{az}-j_{bz}}{2} \\ 1 & 1 & 1 \end{vmatrix} x + \begin{vmatrix} i_{ax} & i_{bx} & \frac{j_{ax}-j_{bx}}{2} \\ i_{az} & i_{bz} & \frac{j_{az}-j_{bz}}{2} \\ 1 & 1 & 1 \end{vmatrix} y - \begin{vmatrix} i_{ax} & i_{bx} & \frac{j_{ax}-j_{bx}}{2} \\ i_{ay} & i_{by} & \frac{j_{ay}-j_{by}}{2} \\ 1 & 1 & 1 \end{vmatrix} z = \frac{i \odot j}{4},$$

we have that $i \odot_f j = 0$. Therefore, if we have three lines, say 1, 2 and 3, in general position, the value of f that makes $1 \odot_f 2 = 0$, $1 \odot_f 3 = 0$, and $2 \odot_f 3 = 0$, is uniquely determined as the intersection of three planes. This will be useful in the next section.



$i \otimes j = [i_a, i_b, j_a, j_b]$ $\quad$ $i \otimes j = d_i\, d_j\, \delta_{ij}\, \sin\varphi_{ij}$ $\quad$ $i \otimes j = d_i\, d_j\, \eta_{ij}\, \cos\vartheta_{ij}$

Fig. 2. Three geometric interpretations of the algebraic definition of the reciprocal product between two lines, say i and j (depicted in red), due to Sylvester (left), Cayley (center), and Drach (right).

IV. NEW BRACKET POLYNOMIALS

A. *New polynomial in terms of reciprocal products*

It is straightforward to observe that

$$\underset{123456}{\overset{123456}{\bigotimes}} \triangleq \begin{vmatrix} 1\otimes 1 & 1\otimes 2 & \dots & 1\otimes 6 \\ 2\otimes 1 & 2\otimes 2 & \dots & 1\otimes 6 \\ \vdots & \vdots & & \vdots \\ 6\otimes 1 & 6\otimes 2 & \dots & 6\otimes 6 \end{vmatrix}$$
$$= \det(K^T M K) = \det(K^T)\det(M)\det(K) = -\det^2(K). \tag{9}$$

As early as in 1861, Sylvester already showed that, when the determinant of recirpocal products in (9) vanishes, the six involved lines are in *involution*, a term used by him and Cayley to refer to our singularity condition [19]. Sylvester also knew that this determinant was the squared of the determinant of K [20], a fact rediscovered in [21].

The determinant (9) can be expanded using the generalized Laplace's expansion formula with respect to the 3×3 minors of its first three rows and their complementary 3×3 minors in the other three rows. This idea was almost simultaneously explored in [9] and [22] for the direct expanssion of $\det(K)$. In [22], half of the terms are missing which might induce some confusions. The application of this expansion to $\det^2(K)$ reads as follows

$$\det^2(K) =$$
$$\underset{123}{\overset{123}{\bigotimes}}\underset{456}{\overset{456}{\bigotimes}} - \underset{123}{\overset{456}{\bigotimes}}\underset{456}{\overset{123}{\bigotimes}} - \underset{123}{\overset{124}{\bigotimes}}\underset{456}{\overset{356}{\bigotimes}} + \underset{123}{\overset{356}{\bigotimes}}\underset{456}{\overset{124}{\bigotimes}} + \underset{123}{\overset{125}{\bigotimes}}\underset{456}{\overset{346}{\bigotimes}}$$
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The most interesting feature of (9) is that it is completely expressed in terms of reciprocal products. This allows us to explore other forms of the singularity conditions using Cayley's and Drach's formulas for these products. For example, using Cayley's formula, we have, after extracting non-null common row and column factors, that

$$\underset{123}{\overset{456}{\bigotimes}} = D \begin{vmatrix} \delta_{14}\sin\varphi_{14} & \delta_{15}\sin\varphi_{15} & \delta_{16}\sin\varphi_{16} \\ \delta_{24}\sin\varphi_{24} & \delta_{25}\sin\varphi_{25} & \delta_{26}\sin\varphi_{26} \\ \delta_{34}\sin\varphi_{34} & \delta_{35}\sin\varphi_{35} & \delta_{36}\sin\varphi_{36} \end{vmatrix}. \tag{11}$$

where $D = d_1 d_2 d_3 d_4 d_5 d_6$. Here, both δ_{ij} and φ_{ij} vary with the pose of the moving platform. D also varies, but, since it is common to all factors in (10), it can be dropped. Alternatively, using Drach's formula, we have that

$$\underset{123}{\overset{456}{\bigotimes}} = D \begin{vmatrix} \eta_{14}\cos\vartheta_{14} & \eta_{15}\cos\vartheta_{15} & \eta_{16}\cos\vartheta_{16} \\ \eta_{24}\cos\vartheta_{24} & \eta_{25}\cos\vartheta_{25} & \eta_{26}\cos\vartheta_{26} \\ \eta_{34}\cos\vartheta_{34} & \eta_{35}\cos\vartheta_{35} & \eta_{36}\cos\vartheta_{36} \end{vmatrix}. \tag{12}$$

Since η_{ij} is the distance between i_a and j_a, it is independent of the moving platform location. In this case, only ϑ_{ij} varies. Moreover, if we exchange the role of the base and the moving platform, we can obtain an equivalent expression involving the distances between the attachments in the moving platform instead. Both, (11) and (12), will be useful in the next subsection.

The main problem with expression (10) is that it actually is the square of the pure condition which, in general, does not factorize in terms of reciprocal brackets. Apparently, we can only express the square of the pure condition in terms of reciprocal brackets, not the pure condition itself. A way around this difficulty is presented in the next section thanks to the use of focal points.

B. *New polynomial in terms of reverse products*

As in the previous subsection, we observe that

$$\underset{123456}{\overset{123456}{\odot}} \triangleq \begin{vmatrix} 1\odot 1 & 1\odot 2 & \dots & 1\odot 6 \\ 2\odot 1 & 2\odot 2 & \dots & 1\odot 6 \\ \vdots & \vdots & & \vdots \\ 6\odot 1 & 6\odot 2 & \dots & 6\odot 6 \end{vmatrix}$$
$$= \det(K^T N K) = \det(K^T)\det(N)\det(K) = \det^2(K). \tag{13}$$

Since the reverse product is anticommutative, the above determinant is skew-symmetric. What is important for us is that the determinant of a $n \times n$ skew-symmetric matrix, with n an even number, is the square of a polynomial of degree $n/2$, called the Pfaffian, a term coined by Cayley in [23]. In our case, by symbolically computing the determinant in (13) and factorizing the result, it can be verified that its Pfaffian can be expressed as

$$\begin{aligned}\pm\det(K) = &(1\odot 2)\,[(3\odot 4)(5\odot 6) - (3\odot 5)(4\odot 6) + (3\odot 6)(4\odot 5)]\\ &-(1\odot 3)\,[(2\odot 4)(5\odot 6) - (2\odot 5)(4\odot 6) + (2\odot 6)(4\odot 5)]\\ &+(1\odot 4)\,[(2\odot 3)(5\odot 6) - (2\odot 5)(3\odot 6) + (2\odot 6)(3\odot 5)]\\ &-(1\odot 5)\,[(2\odot 3)(4\odot 6) - (2\odot 4)(3\odot 6) + (2\odot 6)(3\odot 4)]\\ &+(1\odot 6)\,[(2\odot 3)(4\odot 5) - (2\odot 4)(3\odot 5) + (2\odot 5)(3\odot 4)]. \quad (14)\end{aligned}$$

This sign ambiguity is irrelevant because we are just interested in the roots of $\det(K)$. Moreover, since the root locus of $\det(K)$ is independent of the chosen reference frame, $\odot$ can be substituted with $\odot_f$ without altering the result (this can be verified using a computer algebra system). This is an important observation because we can chose f so that (14) is simplified as explained in Section III. For example, let us suppose that 1, 2 and 3 are in general position. Then, we can find f such that $1\odot_f 2 = 0$, $1\odot_f 3 = 0$, and $2\odot_f 3 = 0$. In this particular case, (14) can be rewritten in this very compact form

$$\pm\det(K) = \begin{vmatrix} 1\odot_f 4 & 1\odot_f 4 & 1\odot_f 4\\ 2\odot_f 5 & 2\odot_f 5 & 2\odot_f 5\\ 3\odot_f 6 & 3\odot_f 6 & 3\odot_f 6\end{vmatrix}. \quad (15)$$

As a consequence, in terms of reciprocal products, we have that

$$\pm\det(K) = \bigotimes_{123}^{4'5'6'} = \begin{vmatrix} 1\otimes 4' & 1\otimes 4' & 1\otimes 4'\\ 2\otimes 5' & 2\otimes 5' & 2\otimes 5'\\ 3\otimes 6' & 3\otimes 6' & 3\otimes 6'\end{vmatrix}. \quad (16)$$

This expression still involves six leg lines; namely, those denoted by 1, 2, and 3 in the original Gough-Stewart parallel robot and 4', 5', and 6' which result from the point reflection of leg lines 4, 5, and 6 across f (the focal point determined by leg lines 1, 2, and 3). It is easy to see that 4', 5', and 6' are parallel to 4, 5, and 6, respectively. Point f, whose geometric interpretation is given in the next section, is thus essential in our formulation. Since there are 20 different combinations of three lines out of six, there are up to 20 different focal points. This is useful when looking for the most convenient expression for a given particular robot.

It is also important to observe that (16) can be reformulated in terms of angles directly using either (11) or (12). Also observe that $\varphi_{14'} = \varphi_{14}$ and $\vartheta_{14'} = \vartheta_{14}$ because 4 and 4' are parallel, and analogously for all other angles.

The expansion of (16) leads to a bracket polynomial with only six monomials, each of them involving the six lines. It is a bracket polynomial with only nine different reciprocal brackets which represents an important simplification with respect to the previously obtained bracket polynomials. Table I compiles the number of monomials and the different kinds of brackets appearing in the five bracket polynomials considered in this paper.

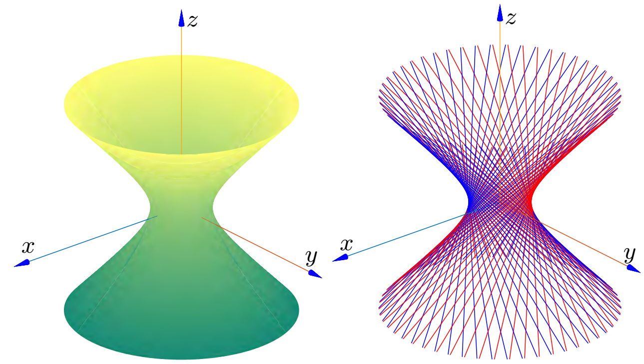

Fig. 3. Three lines in space in general position determine a one-sheeted hyperboloid (left), a doubled-ruled surface that contains two reguli (right).

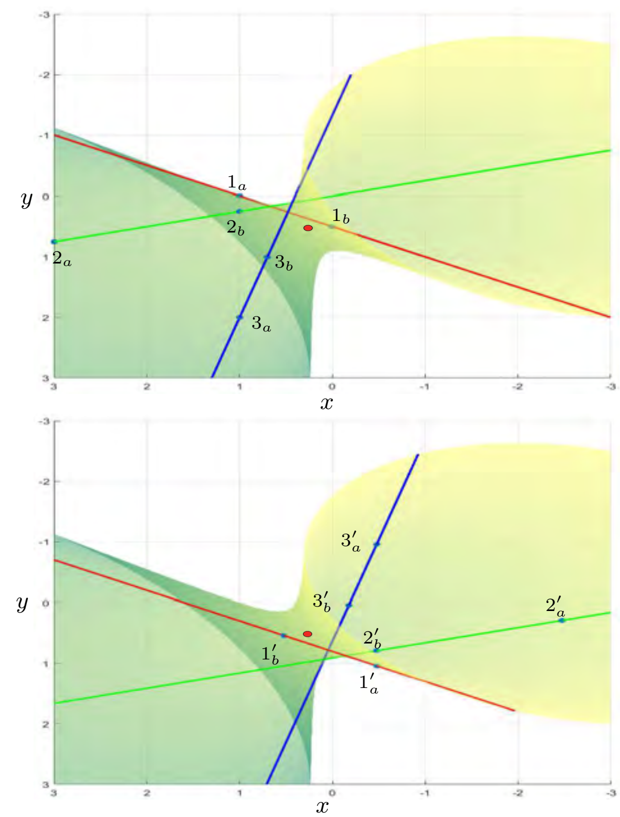

Fig. 4. Lines 1, 2, and 3 in space determine a one-sheeted hyperboloid (top), the projection of these lines across the center of the paraboloid (1', 2' and 3', respectively) also lie in the paraboloid (bottom). Both sets of lines are in the two different reguli contained in the paraboloid. The center of the paraboloid is actually the focal point determined by 1, 2, and 3. Observe that 1', 2', and 3' are parallel to 1, 2, and 3, respectively. In this drawing, the hiperboloid is projected onto the xy-plane.

V. Geometric Interpretation of the Focal Points

If we choose any three lines in space in general position, the set of lines meeting these three lines is a regulus. These three lines lie in an uniquely defined hyperboloid of one sheet [25, p. 15], a doubly ruled surface containing two reguli. One regulus contains the three lines, and the other, all the lines meeting these three lines (see Fig. 3).

TABLE I

NUMBER OF MONOMIALS AND DIFFERENT KINDS OF BRACKETS APPEARING IN THE FIVE CONSIDERED BRACKET POLYNOMIALS.

Bracket Polynomial	Number of monomials	Different brackets	1-3 brackets	3-1 brackets	2-2 brackets	Reciprocal brackets
White's [15]	96	56	14	14	28	0
McMillan's [16], [24], [14]	24	34	8	8	18	2
Downing *et al.*'s [13]	16	24	6	6	12	0
Bracket polynomial in (10)	130	15	0	0	15	15
Bracket polynomial in (16)	6	9	0	0	9	9

It is interesting to observe that the equation of the hyperboloid of one sheet defined by lines 1, 2, and 3 can be expressed in terms of brackets as:

$$[1_a\,1_b\,3_a\,p][2_a\,2_b\,3_b\,p] = [1_a\,1_b\,3_b\,p][2_a\,2_b\,3_a\,p], \qquad (17)$$

where $p = (x, y, z, 1)^T$. This equation was apparently first presented without proof in [26, p. 198] where it was delivered as an exercise (see [27] for its detailed derivation). The first expression for this hyperboloid was obtained by A. Cayley in [1]. It is a much more complicated expression than that in (17) as it is expressed in terms of the Plücker coordinates of the three lines.

Our problem is to obtain the value of f that satisfies the system of equations $1 \odot_f 2 = 0$, $1 \odot_f 3 = 0$, and $2 \odot_f 3 = 0$ (see Fig. 4). In other words line 2, after projected across f (let us call it $2'$), must meet line 1; and line 3, after projected across f (let us call it $3'$), must meet lines 1 and 2. As a consequence, $2'$ and $3'$ must be in the complementary regulus to the one containing 1, 2, and 3. Therefore, all lines (1, 2, 3, 2', and 3') lie in the same hyperboliod of one sheet. The point that, after projecting across it an arbitrary line in one regulus, is a line in the complementary regulus is the center of the hyperboloid containing both reguli. This provides a neat geometric interpretation of our focal points.

VI. CONCLUSION

From the use of Grassmann-Cayley algebra, we already knew that the singularity condition for the general Gough-Stewart platform could be expressed as the sum of products of three *brackets* (4×4 determinants involving the homogeneous of four leg attachments) equated to zero. However, using elementary linear algebra arguments, we have presented similar results with the important advantage that only those brackets that can be interpreted as reciprocal products between leg lines are needed.

Space limitations prevent us from including examples, but it is not difficult to imagine how all the presented new tools can be used to obtain new insights into the singularities of particular Gough-Stewart parallel robot architectures; that is, Gough-Stewart parallel robots in which some attachments, either on the base of the moving platform, coincide, are aligned, or are coplanar.

It can be said that the presented results are far reaching. For example, they can be easily adapted to characterize the singularities of 6R robots. In this case, the problem consists in identifying the linear dependencies between six lines representing the revolute joint axes with the important simplification that the reciprocal products between the lines corresponding to consecutive joint axes are constant.

APPENDIX

For comparison purposes, we include here the expressions for Whyte's, McMillan's, and Downing *et al.*'s bracket polynomials. Due to space limitations, we present them in terms of permutations with signum (see [24] for details on this notation).

- Whyte's bracket polynomial (1983) [15]

$$[\overset{\textcircled{7}}{1_a1_b}\overset{\textcircled{4}}{4_a}\overset{\textcircled{5}}{5_a}][\overset{\textcircled{7}}{2_a2_b}\overset{\textcircled{4}}{4_b}\overset{\textcircled{6}}{6_a}][\overset{\textcircled{7}}{3_a3_b}\overset{\textcircled{5}}{5_b}\overset{\textcircled{6}}{6_b}]$$
$$-[\overset{\textcircled{8}}{4_a4_b}\overset{\textcircled{1}}{1_a}\overset{\textcircled{2}}{2_a}][\overset{\textcircled{8}}{5_a5_b}\overset{\textcircled{1}}{1_b}\overset{\textcircled{3}}{3_a}][\overset{\textcircled{8}}{6_a6_b}\overset{\textcircled{2}}{2_b}\overset{\textcircled{3}}{3_b}],$$

where ①, ②, ③, ④, ⑤, ⑥ denote the permutations with signum of the 2-element sets $\{1_a, 1_b\}$, $\{2_a, 2_b\}$, $\{3_a, 3_b\}$, $\{4_a, 4_b\}$, $\{5_a, 5_b\}$, and $\{6_a, 6_b\}$, respectively, and ⑦ and ⑧, the permutations with signum of the 3-element sets $\{1_a1_b, 2_a2_b, 3_a3_b\}$ and $\{4_a4_b, 5_a5_b, 6_a6_b\}$, respectively. Then, the expansion of the above expression leads to $2{\cdot}2{\cdot}2{\cdot}6{+}2{\cdot}2{\cdot}2{\cdot}6 = 92$ monomials.

- McMillan's bracket polynomial (1990) [16]

$$[1_a1_b2_a2_b][3_a3_b\overset{\textcircled{1}}{4_a}\overset{\textcircled{2}}{5_a}][\overset{\textcircled{1}}{4_b}\overset{\textcircled{2}}{5_b}6_a6_b]$$
$$-[1_a1_b\overset{\textcircled{3}}{2_a}\overset{\textcircled{4}}{3_a}][\overset{\textcircled{3}}{2_b}\overset{\textcircled{4}}{3_b}4_a4_b][5_a5_b6_a6_b]$$
$$-[1_a1_b\overset{\textcircled{5}}{2_a}\overset{\textcircled{6}}{3_a}][\overset{\textcircled{5}}{2_b}4_a4_b\overset{\textcircled{7}}{5_a}][\overset{\textcircled{6}}{3_b}\overset{\textcircled{7}}{5_b}6_a6_b]$$
$$+[1_a1_b\overset{\textcircled{8}}{2_a}\overset{\textcircled{9}}{4_a}][\overset{\textcircled{8}}{2_b}3_a3_b\overset{\textcircled{10}}{5_a}][\overset{\textcircled{9}}{4_b}\overset{\textcircled{10}}{5_b}6_a6_b]$$

Observe that, in this case, all permutations involve only 2 elements. Therefore, the total number of monomials is $2{\cdot}2{+}2{\cdot}2{+}2{\cdot}2{\cdot}2{+}2{\cdot}2{\cdot}2 = 24$

- Downing *et al.*'s bracket polynomial (2002) [13]

$$[1_a1_b\overset{\textcircled{1}}{4_a}\overset{\textcircled{2}}{5_a}][2_a2_b\overset{\textcircled{1}}{4_b}\overset{\textcircled{3}}{6_a}][3_a3_b\overset{\textcircled{2}}{5_b}\overset{\textcircled{3}}{6_b}]$$
$$-[4_a4_b\overset{\textcircled{4}}{1_a}\overset{\textcircled{5}}{2_a}][5_a5_b\overset{\textcircled{4}}{1_b}\overset{\textcircled{6}}{3_a}][6_a6_b\overset{\textcircled{5}}{2_b}\overset{\textcircled{6}}{3_b}].$$

In this case, all permutations also involve only 2 elements. Therefore, the total number of monomials is $2{\cdot}2{\cdot}2{+}2{\cdot}2{\cdot}2 = 16$.

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