A Spectral Decomposition Approach to the Accurate Conversion of 4D Rotation Matrices to Double Quaternions

Conference Paper · September 2021

2 authors:

Soheil Sarabandi

Federico Thomas
Spanish National Research Council

A SPECTRAL DECOMPOSITION APPROACH TO THE ACCURATE CONVERSION OF 4D ROTATION MATRICES TO DOUBLE QUATERNIONS

S. SARABANDI and F. THOMAS

Abstract. The problem of approximating dual quaternions by double quaternions emerges when trying to approximate 3D displacements by 4D rotations to simplify some problems arising in Robotics and Computer Graphics. This has triggered a renewed interest in 4D rotations. While 3D rotations can be represented using ordinary quaternions, 4D rotations require the use of double quaternions. Analogously to the 3D case, the mapping from double quaternions to rotation matrices cannot be smoothly inverted because it is a 2-to-1 mapping. This induces numerical problems near singularities that are exacerbated when the elements of the rotation matrices are noisy.

This paper focuses on the inversion of the mentioned mapping, including the important case in which the rotation matrices are contaminated by noise, and presents a new spectral decomposition approach which compares favorably with Rosen-Elfrinkhof method both in terms of time and accuracy.

1. Introduction

In mechanics and geometry, the 4D rotation group, often denoted $SO(4)$, is the group of all rotations about the origin of four-dimensional Euclidean space $\mathbb{R}^4$ under the operation of composition. The group $SO(4)$ is usually identified with the group of 4×4 orthogonal proper matrices under matrix multiplication. Thus, although each element of $SO(4)$ is identified with a 4×4 real matrix, its entries are determined by only six independent parameters due to the orthogonality condition [14, Sec. 2.7],

After a proper change in the orientation of the reference frame, an arbitrary 4D rotation can be expressed as [3, Ch. 3] [6, Ch. 6]:

$$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}. \tag{1.1}$$

2010 *MSC:* primary 34K06, 34K25; secondary 39.

Keywords: 4D Rotation matrices, double quaternions, spectral methods.

This work was partially supported by the Spanish Ministry of Economy and Competitiveness through the projects DPI2017-88282-P and MDM-2016-0656.

Therefore, a 4D rotation is defined by two mutually orthogonal planes of rotation with different rotation angles, each of the planes being fixed in the sense that points in each plane stay within the planes [2]. The group of 4D rotations can be expressed as the direct product of two planar rotations. That is, $SO(4) = SO(2)\times SO(2)$.

Now, observe that matrix (1.1) can be factored into the commutative product of

$$\begin{pmatrix} \cos\beta_1 & -\sin\beta_1 & 0 & 0 \\ \sin\beta_1 & \cos\beta_1 & 0 & 0 \\ 0 & 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix} \tag{1.2}$$

and

$$\begin{pmatrix} \cos\beta_2 & -\sin\beta_2 & 0 & 0 \\ \sin\beta_2 & \cos\beta_2 & 0 & 0 \\ 0 & 0 & \cos\beta_2 & \sin\beta_2 \\ 0 & 0 & -\sin\beta_2 & \cos\beta_2 \end{pmatrix}, \tag{1.3}$$

where $\beta_1 = \frac{\alpha_1+\alpha_2}{2}$ and $\beta_2 = \frac{\alpha_1-\alpha_2}{2}$. The rotation matrices above are called right- and left-isoclinic rotation matrices, respectively. They correspond to rotations in which the rotated angles in both invariant rotation planes have the same or opposite signs, respectively.

In the general case, that in which the 4D rotation matrix, say $\mathbf{R}$, is not expressed in a reference frame is in general orientation, the factorization into the commutative products of left- and right-isoclinic rotations (known as Cayley's factorization) can be expressed as [1]:

$$\mathbf{R} = \mathbf{R}^L\mathbf{R}^R = \mathbf{R}^R\mathbf{R}^L \tag{1.4}$$

where

$$\mathbf{R}^L = l_0\mathbf{I} + l_1\mathbf{A}_1 + l_2\mathbf{A}_2 + l_3\mathbf{A}_3 = \begin{pmatrix} l_0 & -l_3 & l_2 & -l_1 \\ l_3 & l_0 & -l_1 & -l_2 \\ -l_2 & l_1 & l_0 & -l_3 \\ l_1 & l_2 & l_3 & l_0 \end{pmatrix} \tag{1.5}$$

and

$$\mathbf{R}^R = r_0\mathbf{I} + r_1\mathbf{B}_1 + r_2\mathbf{B}_2 + r_3\mathbf{B}_3 = \begin{pmatrix} r_0 & -r_3 & r_2 & r_1 \\ r_3 & r_0 & -r_1 & r_2 \\ -r_2 & r_1 & r_0 & r_3 \\ -r_1 & -r_2 & -r_3 & r_0 \end{pmatrix} \tag{1.6}$$

where $\mathbf{I}$ stands for the 4×4 identity matrix and

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \mathbf{A}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ \mathbf{B}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

In other words, left- and right-isoclinic rotations are completely determined by the vectors

$$\mathbf{l} = (l_0\, l_1\, l_2\, l_3) \tag{1.7}$$

and

$$\mathbf{r} = (r_0\, r_1\, r_2\, r_3), \tag{1.8}$$

respectively. Since (1.5) and (1.6) are rotation matrices, their rows and columns are unit vectors. As a consequence,

$$\mathbf{l}^T \mathbf{l} = 1 \tag{1.9}$$

and

$$\mathbf{r}^T \mathbf{r} = 1. \tag{1.10}$$

Therefore, $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ and $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ can be seen, respectively, as bases for left- and right-isoclinic rotations.

Now, it can be verified that

$$\mathbf{A}_1^2 = \mathbf{A}_2^2 = \mathbf{A}_3^2 = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 = -\mathbf{I}, \tag{1.11}$$

and

$$\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{B}_3^2 = \mathbf{B}_1 \mathbf{B}_2 \mathbf{B}_3 = -\mathbf{I}, \tag{1.12}$$

where we can recognize the quaternion definition. Moreover, it can be verified that

$$\mathbf{A}_i \mathbf{B}_j = \mathbf{B}_j \mathbf{A}_i. \tag{1.13}$$

which is actually a consequence of the commutativity of left- and right-isoclinic rotations.

It can be concluded that $\mathbf{R}_i^L$ and $\mathbf{R}_i^R$ can be seen either as 4×4 rotation matrices or, when expressed as in (1.5) and (1.6), as unit quaternions [17]. Therefore, a 4D rotation can be represented by a double quaternion expressible in vector form as $(\mathbf{l}, \mathbf{r})$.

While the passage from $(\mathbf{l}, \mathbf{r})$ to $\mathbf{R}$ is trivial, the way round is somewhat tricky. Indeed, by observing (1.4), it is concluded that $(\mathbf{l}, \mathbf{r})$ and $(-\mathbf{l}, -\mathbf{r})$ lead to the same 4D rotation matrix. Since this is a 2-to-1 map, it cannot be smoothly inverted.

The computation of the double quaternion corresponding to a 4D rotation matrix has important applications in robotics (for recent references, see [18], where this is used to solve the hand-eye calibration problem, and [15], where it is used to solve the pointcloud registration problem).

The first practical method to solve the 4D rotation matrix to double quaternion conversion is usually attributed to Rosen [10]. Unfortunately, it does not include any optimality criterion thus leading to some inconveniences when handling noisy rotation matrices (see [11] for a broader view of existing approaches). In this paper, we present a new method, based on two consecutive spectral decompositions.

This paper is organized as follows. We start in Section 2 with a description of Rosen-Elfrinkhof method. Using this method, each element of the double quaternion can be interpreted as the result of a squared mean root. Then, in Section 3, we propose a new method based on spectral decompositions. A performance comparison of the two described methods is presented in Section 4. We conclude in Section 5 with a summary of the main contributions and points deserving further attention.

2. Rosen-Elfrinkhof method

The development of the first effective procedure for computing Cayley's factorization is attributed in [8] to van Elfrinkhof [16]. Since this work, written in Dutch, remained unnoticed, other sources (see, for example, [5]) attribute to Rosen, a close collaborator of Einstein, the first procedure to obtain it [10]. The methods of Elfrinkhof and Rosen are equivalent. They are based on clever manipulation of the 16 algebraic scalar equations resulting from solving the matrix equation given in (1.4).

Let us first define the matrix of products as:

$$\mathbf{P} = \mathbf{l}\mathbf{r}^T = \begin{pmatrix} l_0 r_0 & l_0 r_1 & l_0 r_2 & l_0 r_3 \\ l_1 r_0 & l_1 r_1 & l_1 r_2 & l_1 r_3 \\ l_2 r_0 & l_2 r_1 & l_2 r_2 & l_2 r_3 \\ l_3 r_0 & l_3 r_1 & l_3 r_2 & l_3 r_3 \end{pmatrix}, \tag{2.1}$$

and the matrix

$$\mathbf{K} = \frac{1}{4} \begin{pmatrix} r_{11}+r_{22}+r_{33}+r_{44} & -r_{41}+r_{32}-r_{23}+r_{14} & -r_{31}-r_{42}+r_{13}+r_{24} & r_{21}-r_{12}-r_{43}+r_{34} \\ r_{41}+r_{32}-r_{23}-r_{14} & r_{11}-r_{22}-r_{33}+r_{44} & r_{21}+r_{12}+r_{43}+r_{34} & r_{31}-r_{42}+r_{13}-r_{24} \\ -r_{31}+r_{42}+r_{13}-r_{24} & r_{21}+r_{12}-r_{43}-r_{34} & -r_{11}+r_{22}-r_{33}+r_{44} & r_{41}+r_{32}+r_{23}+r_{14} \\ r_{21}-r_{12}+r_{43}-r_{34} & r_{31}+r_{42}+r_{13}+r_{24} & -r_{41}+r_{32}+r_{23}-r_{14} & -r_{11}-r_{22}+r_{33}+r_{44} \end{pmatrix}. \tag{2.2}$$

The merit of Elfrinkhof and Rosen was to realize that equation (1.4) can be reformulated as:

$$\mathbf{l}\mathbf{r}^T = \mathbf{P} = \mathbf{K}. \tag{2.3}$$

From which it follows that the norm of row i (column j) of $\mathbf{K}$ is $\|l_{i-1}\|$ ($\|r_{j-1}\|$). In other words, the absolute values of the $\mathbf{l}$ and $\mathbf{r}$ components can be obtained by computing the norms of the rows and columns of $\mathbf{K}$, respectively. To assign a consistent set of signs to them, we can take any positive entry in $\mathbf{K}$, say for example the element (i, j). Then, according to (2.1), l_{i-1} and r_{j-1} are both positive or negative. If we assume that they are both positive, then we have that:

$$\operatorname{sign}(\mathbf{l}_{l-1}) = \operatorname{sign}(p_{l,i}),\ l \in \{1,2,3,4\}\backslash i, \tag{2.4}$$

and

$$\operatorname{sign}(\mathbf{r}_{m-1}) = \operatorname{sign}(p_{j,m}),\ m \in \{1,2,3,4\}\backslash j. \tag{2.5}$$

The other set of consistent signs is obtained by assuming that $\hat{\mathbf{l}}_{k-1}$ and $\hat{\mathbf{r}}_{l-1}$ are both negative, thus accounting for the double covering of the space of rotations.

3. Spectral decomposition-based method

The spectral decomposition of 4D rotations was introduced in [9], where its practical consequence was not analyzed. This decomposition results from observing that the set of matrices $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ form an orthogonal basis in the sense of Hilbert-Schmidt for the real Hilbert space of 4×4 real orthonormal matrices representing left-isoclinic rotations. Then, (1.5) can be seen as a spectral decomposition. If we left-multiply it by each of the elements of the set $\{\mathbf{I}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, to obtain the different projection coefficients, we have that

$$l_0\mathbf{I} = -\mathbf{R}^L + l_1\mathbf{A}_1 + l_2\mathbf{A}_2 + l_3\mathbf{A}_3, \tag{3.1}$$

$$l_1\mathbf{I} = -\mathbf{A}_1\mathbf{R}^L - l_0\mathbf{A}_1 + l_2\mathbf{A}_3 - l_3\mathbf{A}_2, \tag{3.2}$$

$$l_2\mathbf{I} = -\mathbf{A}_2\mathbf{R}^L - l_0\mathbf{A}_2 - l_1\mathbf{A}_3 + l_3\mathbf{A}_1, \tag{3.3}$$

$$l_3\mathbf{I} = -\mathbf{A}_3\mathbf{R}^L - l_0\mathbf{A}_3 + l_1\mathbf{A}_2 - l_2\mathbf{A}_1. \tag{3.4}$$

Then, by iterative substituting and rearranging terms in (3.1)-(3.4), we conclude that the coefficients of the spectral decomposition (1.5) can be expressed as:

$$l_0\mathbf{I} = -\frac{1}{4}\left(-\mathbf{R}^L + \mathbf{A}_1\mathbf{R}^L\mathbf{A}_1 + \mathbf{A}_2\mathbf{R}^L\mathbf{A}_2 + \mathbf{A}_3\mathbf{R}^L\mathbf{A}_3\right), \tag{3.5}$$

$$l_1\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_1 + \mathbf{A}_1\mathbf{R}^L + \mathbf{A}_3\mathbf{R}^L\mathbf{A}_2 - \mathbf{A}_2\mathbf{R}^L\mathbf{A}_3\right), \tag{3.6}$$

$$l_2\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_2 + \mathbf{A}_2\mathbf{R}^L + \mathbf{A}_1\mathbf{R}^L\mathbf{A}_3 - \mathbf{A}_3\mathbf{R}^L\mathbf{A}_1\right), \tag{3.7}$$

$$l_3\mathbf{I} = -\frac{1}{4}\left(\mathbf{R}^L\mathbf{A}_3 + \mathbf{A}_3\mathbf{R}^L + \mathbf{A}_2\mathbf{R}^L\mathbf{A}_1 - \mathbf{A}_1\mathbf{R}^L\mathbf{A}_2\right). \tag{3.8}$$

Likewise, we can consider the set of matrices $\{\mathbf{I}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3\}$ as an orthogonal basis for right-isoclinic rotations. Then, the coefficients in (1.6) could also be obtained as above.

Now, we can define the following matrix linear operators for arbitrary 4D rotation matrices:

$$\begin{aligned}
\mathcal{L}_0(\mathbf{R}) &= -\frac{1}{4}\left(-\mathbf{R} + \mathbf{A}_1\mathbf{R}\mathbf{A}_1 + \mathbf{A}_2\mathbf{R}\mathbf{A}_2 + \mathbf{A}_3\mathbf{R}\mathbf{A}_3\right),\\
\mathcal{L}_1(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_1 + \mathbf{A}_1\mathbf{R} + \mathbf{A}_3\mathbf{R}\mathbf{A}_2 - \mathbf{A}_2\mathbf{R}\mathbf{A}_3\right),\\
\mathcal{L}_2(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_2 + \mathbf{A}_2\mathbf{R} + \mathbf{A}_1\mathbf{R}\mathbf{A}_3 - \mathbf{A}_3\mathbf{R}\mathbf{A}_1\right),\\
\mathcal{L}_3(\mathbf{R}) &= -\frac{1}{4}\left(\mathbf{R}\mathbf{A}_3 + \mathbf{A}_3\mathbf{R} + \mathbf{A}_2\mathbf{R}\mathbf{A}_1 - \mathbf{A}_1\mathbf{R}\mathbf{A}_2\right).
\end{aligned} \tag{3.9}$$

According to (3.5)-(3.8), $\mathcal{L}_i(\mathbf{R}^L) = l_i\mathbf{I}$, $i = 0, \dots, 3$. Then, using the commutativity property of left- and right-isoclinic rotations, it is straightforward to prove that

$$\mathcal{L}_i(\mathbf{R}) = \mathcal{L}_i(\mathbf{R}^L\mathbf{R}^R) = \mathcal{L}_i(\mathbf{R}^L)\mathcal{L}_i(\mathbf{R}^R) = l_i\mathbf{R}^R. \tag{3.10}$$

By expanding and simplifying the right-hand side of (3.5), and identifying the result with the entries of $\mathbf{R}^R$ in (1.6), we obtain the following expression:

$$l_0\mathbf{r}^T = \frac{1}{4}\big(\; r_{11}+r_{22}+r_{33}+r_{44} \quad r_{12}-r_{21}-r_{34}+r_{43} \quad r_{13}+r_{24}-r_{31}-r_{42} \quad r_{14}-r_{23}+r_{32}-r_{41} \;\big) \tag{3.11}$$

Repeating the same procedure for (3.6)-(3.8), and compiling all the results in a single matrix expression, we obtain expression (2.3). Thus, the fundamental formula used in Rosen-Elfrinkhof method can now be seen as the result of a spectral decomposition, not as the result of a felicitous brainwave.

Now, observe that the singular value decomposition (sometimes also called spectral decomposition) of $\mathbf{P} = \mathbf{l}\mathbf{r}^T$ can be expressed as

$$\mathbf{P} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \underbrace{\begin{pmatrix}\mathbf{l} & \mathbf{C}^{\mathbf{l}}\end{pmatrix}}_{\mathbf{U}} \underbrace{\operatorname{diag}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}}_{\boldsymbol{\Sigma}} \underbrace{\begin{pmatrix}\mathbf{r} & \mathbf{C}^{\mathbf{r}}\end{pmatrix}^T}_{\mathbf{V}^T} \tag{3.12}$$

where $\mathbf{C}^{\mathbf{l}}$ and $\mathbf{C}^{\mathbf{r}}$ are $4{\times}3$ matrices where their three columns are unit orthogonal vectors spanning the orthogonal complement of $\mathbf{l}$ and $\mathbf{r}$, respectively. These two matrices are not unique, but fully determined by $\mathbf{l}$ and $\mathbf{r}$. Their exact expressions are actually irrelevant in our case.

In the case in which the rotation matrix is not noisy, clearly the spectrum of $\mathbf{K}$ is the set $\{1,0,0,0\}$ because $\mathbf{K} = \mathbf{P}$. Nevertheless, if the rotation matrix is noisy, this is not necessarily true. Therefore, in this latter case, the equality $\mathbf{K} = \mathbf{P}$ does not hold. Clearly, the spectral decomposition of $\mathbf{K}$, when it is computed from a noisy rotation matrix, can be expressed as:

$$\begin{pmatrix}\hat{\mathbf{l}} & \hat{\mathbf{C}}^{\mathbf{l}}\end{pmatrix} \operatorname{diag}\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4\end{pmatrix} \begin{pmatrix}\hat{\mathbf{r}} & \hat{\mathbf{C}}^{\mathbf{r}}\end{pmatrix} \tag{3.13}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4$, σ_1 being *close* to 1 and σ_2, σ_3 and σ_4, to 0.

By comparing the spectral decompositions (3.12) and (3.13), we observe that the equality $\mathbf{P} = \mathbf{K}$ can be satisfied if the spectrum of $\mathbf{K}$ is normalized to $\{1,0,0,0\}$. Observe how this operation cancels the effect of the orthogonal complements of $\hat{\mathbf{l}}$ and $\hat{\mathbf{r}}$ on the result.

Finally, we can summarize the new method in two simple steps: fist compute $\mathbf{K}$ from the input rotation matrix, $\mathbf{R}$, according to its definition in (2.2), and then compute the singular value decomposition of $\mathbf{K}$ to obtain (3.13). The double quaternion corresponding to $\mathbf{R}$ is then simply given by $(\hat{\mathbf{l}}, \hat{\mathbf{r}})$.

4. Performance analysis

The two described methods have been implemented in MATLAB®, running on an Intel® Core™i7 with 16 GB of RAM. All comparisons have been performed using single-precision floating-point numbers according to IEEE Standard 754.

The next presented comparison is based on a statistical analysis. To this end, we first need to generate random double quaternions. Since this is equivalent to generate random points uniformly distributed in $\mathbb{S}^3$, we can use the algorithm described in [7] (this problem has also recently been treated in [4]). For each generated double quaternion, we can generate a 4D rotation matrix using (1.4), and then recover the original double quaternions using the described methods. The committed error is evaluated as

$$\varepsilon = \left\|\mathbf{l} - \hat{\mathbf{l}}\right\| + \|\mathbf{r} - \hat{\mathbf{r}}\|, \tag{4.1}$$

where $(\mathbf{l}, \mathbf{r})$ and $(\hat{\mathbf{l}}, \hat{\mathbf{r}})$ represent the original and the recovered double quaternion, respectively. In general, this is not a good way to compute the distance between two 4D rotations. Nevertheless, since in our case the error is assumed to be small, the length of the vector connecting both orientations in $\mathbb{S}^3$ is going to coincide with the value of the angle formed by them as seen from the center of $\mathbb{S}^3$.

Table 1. Time and error performances for the computation of double quaternions from 4D noiseless rotation matrices using the two explained methods. When adding an increasing level of noise to the input rotation matrices, error figures evolve quite differently for both methods (see Fig. 1).

Method	Average time (μs)	Worst-case error $\times 10^{-7}$	Average error $\times 10^{-8}$	Standard deviation $\times 10^{-8}$
Rosen-Elfrinkhof method	11.1	1.63	5.03	5.80
New method	10.5	6.12	14.72	16.19

The time and error performances of the two described methods for 10^5 random 4D rotations are compiled in Table 1. The second column gives the average time required by each method. It is interesting to observe that, although the proposed method requires the computation of a singular value decomposition, it is a bit faster than Rosen-Elfrinkhof method, probably because it does not require the computation of square roots. The other three columns give the worst-case errors, the average errors, and the error standard deviations, respectively. Since the worst-case errors are lower than 10^{-6} in both cases, both methods can be considered as appropriate for most practical applications. It remains to be seen what happens under the presence of noise.

Now, to evaluate the performance of both methods under the presence of noise, we can perturb the randomly generated 4D rotations with uncorrelated uniformly distributed noise in the internal $[-\sigma, \sigma]$ and repeat the same procedure described above. The plots obtained for values of σ ranging in the interval $[0, 0.5]$ appear in Fig. 1. From these plots, it can be concluded that, when the considered rotation matrices are contaminated by numerical or experimental noise, Rosen-Elfrinkhof method should be avoided. In these cases, the presented spectral decomposition-based method performs much better.

5. Conclusion

Spectral theory is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. We have applied two different spectral decompositions to come up with a method for the computation of the double quaternion corresponding to a given, possibly noisy, rotation matrix. This new method has been shown to be superior in terms of computational time,

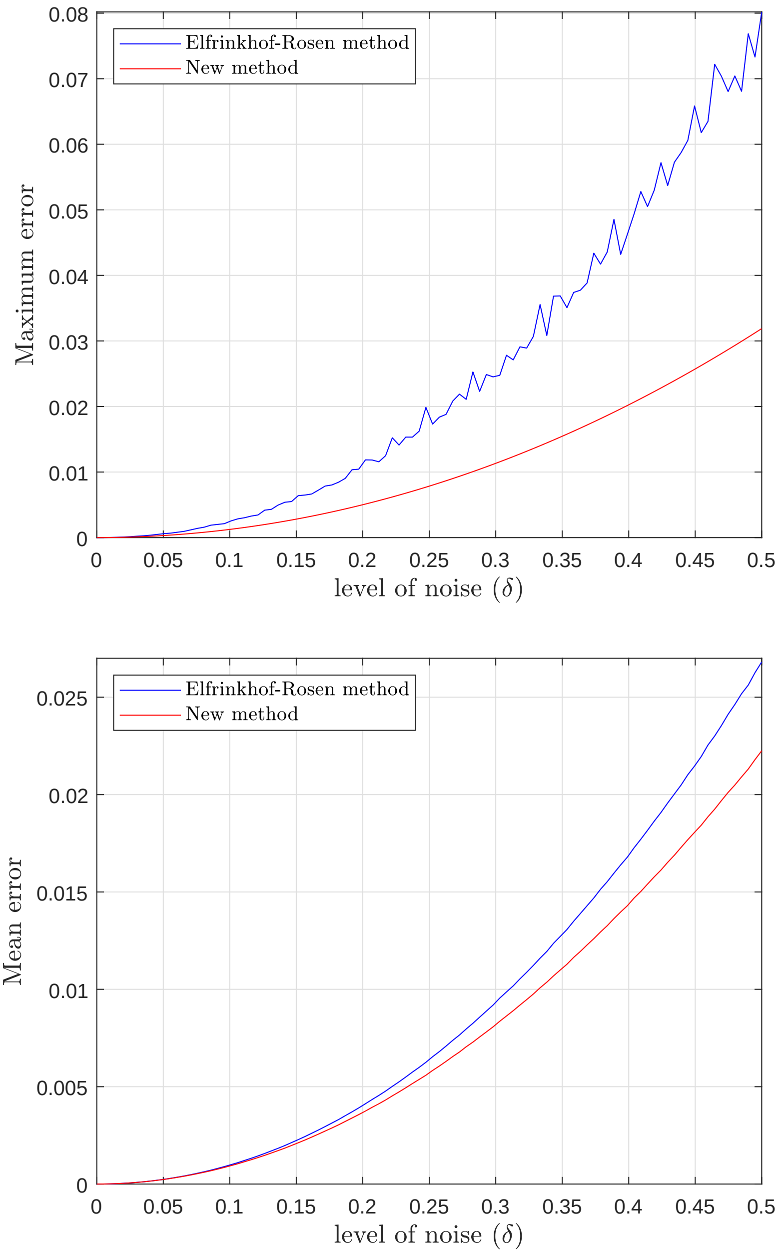

Figure 1. Maximum (top) and mean (bottom) error as a function of the level of noise δ added to the elements of $\mathbf{R}$.

and less sensitive to noise with respect to Rosen-Elfrinkhof method, the standard method of choice until the present.

All the performed operations in the derivation of the new method can be seen as orthogonal projections onto the desired solution subspaces. Thus, the obtained solution minimize an Euclidean distance. Based on this fact, we conjecture that the proposed method is equivalent to obtain the nearest proper rotation matrix (in Frobenius norm) to the input noisy rotation matrix (see [12] for different approaches to solve this problem in 3D, and [13] for the extension of these approaches to 4D), and then to obtain the double quaternion corresponding to the resulting matrix using Rosen-Elfrinkhof method. Preliminary experiments support this conjecture. From our point of view, this point deserves further inquiries.

References

[1] A. Cayley, *Recherches ultérieures sur les déterminants gauches*, The Collected Mathematical Papers of Arthur Cayley, article 137, (1891), pp. 202–215.

[2] F. Cole, *On rotations in space of four dimensions*, American Journal of Mathematics, 12 (1890), pp. 191–210.

[3] P. Du Val, *Homographies, Quaternions and Rotations*, Clarendon Press: Oxford University Press, 1964.

[4] M. Erdoğdu and M. Özdemir, *Simple, double and isoclinic rotations with a viable algorithm*, Mathematical Sciences and Applications E-Notes, 8 (2020), pp. 11–24.

[5] F. Hitchcock, *Analysis of rotations in Euclidean four-space by sedenions*, Journal of Mathematics and Physics, 9 (1930), pp. 188–193.

[6] C. Hsiung and G. Mao, *Linear Algebra*, Allied Publishers, 1998.

[7] G. Marsaglia, *Choosing a point from the surface of a sphere*, The Annals of Mathematical Statistics, 43 (1972), pp. 645–646.

[8] J. Mebius, *Applications of Quaternions to Dynamical Simulation, Computer Graphics and Biomechanics*, PhD thesis, Delft University of Technology, 1994.

[9] A. Perez-Gracia and F. Thomas, *On Cayley's factorization of 4D rotations and applications*, Advances in Applied Clifford Algebras, 27 (2017), pp. 523–538.

[10] N. Rosen, *Note on the general Lorentz transformation*, Journal of Mathematics and Physics, 9 (1930), pp. 181–187.

[11] S. Sarabandi, *Solving the Nearest Rotation Matrix Problem in Three and Four Dimensions with Applications in Robotics*, PhD thesis, Technical University of Catalonia, 2021.

[12] S. Sarabandi, A. Shabani, J. Porta, and F. Thomas, *On closed-form formulas for the 3D nearest rotation matrix problem*, IEEE Transactions on Robotics, 36 (2020), pp. 1333–1339.

[13] S. Sarabandi and F. Thomas, *The 4D nearest rotation matrix problem*, 8th Conference on Applied Geometric Algebras in Computer Science and Engineering, (2021).

[14] J. Stillwell, *Naive Lie Theory*, Springer, 2000.

[15] F. Thomas and S. Sarabandi, *Approximating displacements in R^3 by rotations in R^4 and its application to pointcloud registration*, submitted for publication.

[16] L. van Elfrinkhof, *Eene eigenschap van de orthogonale substitutie van de vierde orde*, Handelingen van het zesde Nederlandsch Natuur- en Geneeskundig Congres (Acts of the sixth Dutch nature and medical congress), (1897), pp. 237–240.

[17] J. Weiner and G. Wilkens, *Quaternions and rotations in E^4*, Mathematical Association of America, 12 (2005), pp. 69–76.

[18] J. Wu, Y. Sun, M. Wang, and M. Liu, *Hand-eye calibration: 4D Procrustes analysis approach*, IEEE Transactions on Instrumentation and Measurement, 69 (2020), pp. 2966–2981.

S. Sarabandi, Institut de Robòtica i Informàtica Industrial (CSIC-UPC), Llorens Artigas 4-6, 08028 Barcelona, Spain,
e-mail: ssarabandi@iri.upc.edu

F. Thomas, Institut de Robòtica i Informàtica Industrial (CSIC-UPC), Llorens Artigas 4-6, 08028 Barcelona, Spain,
e-mail: f.thomas@csic.es