Finding the Common Tangents to Four Spheres via Dimensionality Reduction

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Abstract One of the advantages of formulating position analysis problems in terms of distances is that the dimension of the problem can be reduced by projecting the problem onto a subspace. Although, in general, this operation does not provide a significant advantage, when parallelism or alignment constraints must be enforced, a proper projection results in an important simplification. This is the case when computing the common tangents to four spheres in $\mathbb{R}^3$. In this paper, we first show how this problem can be formulated in terms of just five points in $\mathbb{R}^2$ thanks to projection, and then this is applied to solve the forward kinematics of the 4-S$\underline{\text{P}}$C parallel robot.

1 Introduction

The problem of finding the common tangent lines to four spheres is an elementary geometric problem —at least in its wording— that apparently was first posed in 1990 by Larman [1], and later discussed by Karger [2] and Verschelde [3]. Macdonald *et al.* [4] proved that four equal-radius spheres in $\mathbb{R}^3$ can have at most 12 common tangents, and that this bound is tight. They paid particular attention to the case in which the spheres are centered at the vertices of a regular tetrahedron. If the spheres overlap pairwise, but no three have a common point, then there will be exactly 12 common real tangents. Almost contemporaneously, Megyesi [5] showed that this result remains true if the spheres have coplanar centers, but that there can only be eight common real tangents if the spheres have the same radii. The general case was established soon after by Sottile and Theobald in [6]. They proved that $2n-2$ general spheres in $\mathbb{R}^n$ $(n \geq 3)$ have $3 \cdot 2^{n-1}$ complex common tangent lines, and there are $2n-2$ such spheres with all common tangent lines real.

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Porta *et al.* presented a coordinate-free formulation specially tailored to be solved using a branch-and-prune method [7]. Herein, we use a similar formulation with the important simplification of reducing the problem to that of embedding five points, whose pairwise distances depend on three variables, on a plane. This simplification is accomplished by projecting the problem onto a plane perpendicular to the sought tangents. The projection and backprojections of problems formulated in terms of distances was first introduced in [8], extended in [9], and it has been recently applied to deal with revolute axis parallelism in 6R serial robots [10].

The described problem arises in several computer graphics and computational geometry applications, including visibility computations with moving viewpoints [11], computing the smallest enclosing cylinders of point sets [12], and placement problems in geometric modeling [13]. We show here that it also arises when solving the forward kinematics of the 4-S$\underline{\text{P}}$C parallel robot which is of particular interest as a pointing device.

This paper is structured as follows. While Section 2 explains the standard approach, Section 3 presents the new coordinate-free approach consisting in reducing the dimension of the problem by projection. Section 4 applies the obtained results to the forward kinematics of the tetrapod 4-SPC. Finally, Section 5 summarizes the main contributions and enumerates points deserving further attention.

2 The standard approach

Let us assume that we have four spheres of radii $r_1,\ldots,r_4$, with their centers located at $P_1,\ldots,P_4$, as shown in Fig. 1. The vector position of P_i in the world reference frame will be denoted by $\mathbf{p}_i$. The problem of finding the common tangent lines to these four spheres can be fully expressed in terms of four lines orthogonally traversing the tangent line $\mathscr{L}$. This line will be represented by its nearest point to the origin (with location vector $\mathbf{q}$) and a unit vector $\mathbf{u}$ along it. Therefore,

$$\mathbf{u}\cdot\mathbf{u}=1, \quad (1) \qquad\qquad \mathbf{q}\cdot\mathbf{u}=0. \quad (2)$$

Then, for each sphere we have that

$$(\mathbf{p}_i-\mathbf{q})\cdot(\mathbf{p}_i-\mathbf{q})-(\mathbf{u}\cdot(\mathbf{p}_i-\mathbf{q}))^2-r_i^2=\mathbf{p}_i\cdot\mathbf{p}_i-2\,\mathbf{p}_i\cdot\mathbf{q}+\mathbf{q}\cdot\mathbf{q}-(\mathbf{u}\cdot\mathbf{p}_i)^2-r_i^2=0. \quad (3)$$

If we assume, without loss of generality, that $\mathbf{p}_1=(0,0,0)^T$, then

$$\mathbf{q}\cdot\mathbf{q}=r_1^2, \quad (4) \qquad\qquad \mathbf{p}_i\cdot\mathbf{p}_i-2\,\mathbf{p}_i\cdot\mathbf{q}-(\mathbf{u}\cdot\mathbf{p}_i)^2-r_i^2+r_1^2=0, \quad (5)$$

for $i=2,3,4$. From (5), we can express $\mathbf{q}$ in terms of $\mathbf{u}$ as

$$\mathbf{q}=\mathbf{M}^{-1}\begin{pmatrix}\mathbf{p}_2\cdot\mathbf{p}_2-(\mathbf{u}\cdot\mathbf{p}_2)^2-r_2^2+r_1^2\\ \mathbf{p}_3\cdot\mathbf{p}_3-(\mathbf{u}\cdot\mathbf{p}_3)^2-r_3^2+r_1^2\\ \mathbf{p}_4\cdot\mathbf{p}_4-(\mathbf{u}\cdot\mathbf{p}_4)^2-r_4^2+r_1^2\end{pmatrix}, \quad (6)$$

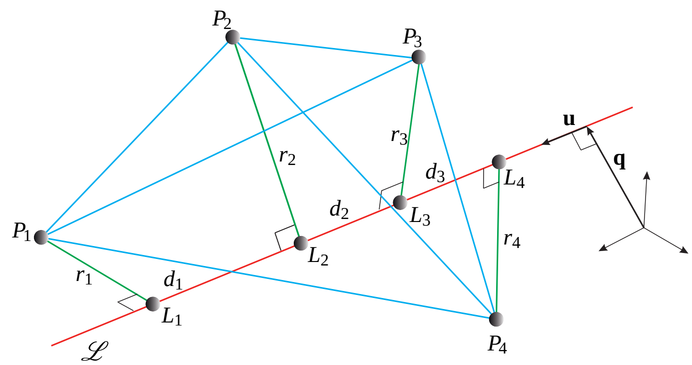


Fig. 1 Four spheres are centered at $P_1,\ldots,P_4$. The pairwise distances between these points are constant (shown in blue). The line $\mathscr{L}$ (in red) is a common tangent to the four spheres of radii $r_1,\ldots,r_4$. The segments in green connect the centers of the spheres and the points of contact between the line and the corresponding sphere which are denoted by $L_1,\ldots,L_4$.

with $\mathbf{M} = 2\left(\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4\right)^\top$. Finally, equation (1) and the equations resulting from substituting (6) in (2) and (4) form a system of three equations of degree 2, 3, and 4 in the components of $\mathbf{u}$. This system yields up to 24 real solutions that corresponds to 12 tangent lines, since each tangent is obtained with the two possible orientations of its director vector $\mathbf{u}$. Thus, two important disadvantages of this approach are that the three resulting equations involve the three variables, and that the obtained solutions are actually duplicated because of the used representation for the tangent line. These two issues are addressed in the next section.

3 A coordinate-free approach

Points P_1, P_2, P_3 and P_4 define a tetrahedron in $\mathbb{R}^3$ whose edge lengths will be denoted by $d_{i,j} = \overline{P_iP_j}$, and $s_{i,j} = d_{ij}^2$ for $i, j = 1,\ldots 4, i < j$. These edges appear in blue in Fig. 1. In the same figure, the segments in green connect the centers of the spheres and the points of contact between the line and the corresponding sphere which are denoted by $L_1,\ldots,L_4$. Let us also denote $d_i = \overline{L_iL_{i+1}}$, for $i = 1,2,3$.

Coordinate-free formulations using distance geometry not only involve distances but also orientations of simplices. These orientations are defined with respect to one simplex whose orientation is arbitrarily chosen so that all other orientations are given with respect to it. Since, in our case, we only have a simplex in $\mathbb{R}^3$ (the one defined by $\{P_1,P_2,P_3,P_4\}$), its orientation is irrelevant. Nevertheless, we have three simplices in $\mathbb{R}^1$ (those defined by $\overline{L_1,L_2}$, $\overline{L_2,L_3}$, and $\overline{L_3,L_4}$). In what follows, we will assign positive orientation to the simplex $\overline{L_2,L_3}$, and the other two simplices will have either negative, or positive, orientations whether they overlap, or not, $\overline{L_2,L_3}$.

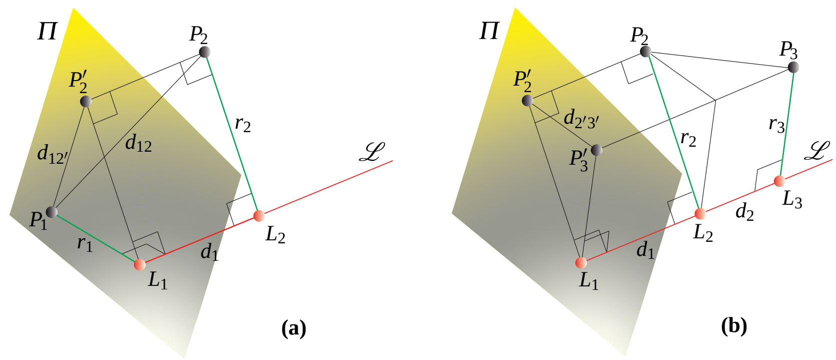


Fig. 2 Π is the normal plane to $\mathscr{L}$ that contains P_1. Then, L_1 is the point of intersection of Π and $\mathscr{L}$. If P'_2 is the projection of P_2 onto Π, then $d_{1,2'} = \sqrt{d_{1,2}^2 - d_1^2}$ **(a)**. Analogously, if P'_3 is the projection of P_3 onto Π, then $d_{1,3'} = \sqrt{d_{1,3}^2 - (d_1 + d_2)^2}$ **(b)**.

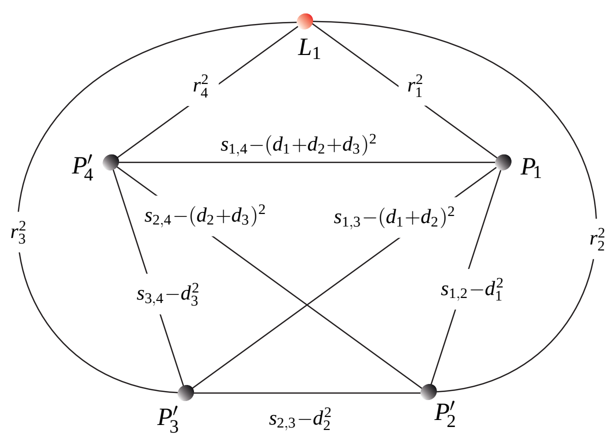


Fig. 3 Planar distance graph involving five points obtained by projecting the eight points in Fig. 1 onto the plane perpendicular to $\mathscr{L}$. The value attached to each edge stands for the square distance between the corresponding points.

Using the distance-based formulation detailed in this section, the problem will be stated as follows: given the distances between the sphere centers and a set of values for $r_1, \ldots, r_4$, the problem consists in obtaining d_1, d_2, and d_3. For the reason given above, we will assign positive sign to d_2, while d_1 and d_3 will be positive or negative. We will next see how this sign convention avoids the duplication of solutions intrinsic to the standard formulation.

As shown in the previous section, the standard formulation requires considering eight points. Nevertheless, observe that we can project the problem onto a plane orthogonal to $\mathscr{L}$ so that the problem is reduced to consider five points in $\mathbb{R}^2$ instead of eight points in $\mathbb{R}^3$. Indeed, if P_2' denotes the projection of P_2 onto the plane orthogonal to $\mathscr{L}$ that contains P_1 (see Fig. 2a), then $\overline{L_1P_2'}^2 = r_2^2$ and $\overline{P_1P_2'}^2 = s_{1,2} - d_1^2$. Likewise, if P_3' denotes the projection of P_3 onto the same plane (see Fig. 2b), $\overline{L_1P_3'}^2 = r_3^2$, $\overline{P_1P_3'}^2 = s_{1,3} - (d_1 + d_2)^2$ and $\overline{P_2'P_3'}^2 = s_{2,3} - d_2^2$. A similar result is obtained for the projection of P_4. Then, after computing all distances between the projected points, we can depict the planar distance graph in Fig. 3.

Now, since all subsets of four points in Fig. 3 are coplanar, their pairwise distances are not independent. Their dependency can be easily formulated using the theory of Cayley-Menger determinants [14]. For example, the pairwise distances between the points in the two sets $\{L_1, P_1, P_2', P_3'\}$ and $\{L_1, P_2', P_3', P_4'\}$ must satisfy the equations

$$f_1(d_1,d_2): \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_1^2 & r_2^2 & r_3^2 \\ 1 & r_1^2 & 0 & s_{1,2}-d_1^2 & s_{1,3}-(d_1+d_2)^2 \\ 1 & r_2^2 & s_{1,2}-d_1^2 & 0 & s_{2,3}-d_2^2 \\ 1 & r_3^2 & s_{1,3}-(d_1+d_2)^2 & s_{2,3}-d_2^2 & 0 \end{vmatrix} = 0, \quad (7)$$

$$f_2(d_2,d_3): \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_2^2 & r_3^2 & r_4^2 \\ 1 & r_2^2 & 0 & s_{2,3}-d_2^2 & s_{2,4}-(d_2+d_3)^2 \\ 1 & r_3^2 & s_{2,3}-d_2^2 & 0 & s_{3,4}-d_3^2 \\ 1 & r_4^2 & s_{2,4}-(d_2+d_3)^2 & s_{3,4}-d_3^2 & 0 \end{vmatrix} = 0. \quad (8)$$

These two equations are important because they involve only two variables, which is advantageous when performing eliminations. Unfortunately, the equations derived for the other three sets of four points involve the three variables. For example, for the sets $\{L_1, P_1', P_3', P_4'\}$ and $\{L_1, P_1', P_2', P_4'\}$, we obtain the equations

$$f_3(d_1,d_2,d_3): \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_1^2 & r_3^2 & r_4^2 \\ 1 & r_1^2 & 0 & s_{1,3}-(d_1+d_2)^2 & s_{1,4}-(d_1+d_2+d_3)^2 \\ 1 & r_3^2 & s_{1,3}-(d_1+d_2)^2 & 0 & s_{3,4}-d_3^2 \\ 1 & r_4^2 & s_{1,4}-(d_1+d_2+d_3)^2 & s_{3,4}-d_3^2 & 0 \end{vmatrix} = 0, \quad (9)$$

$$f_4(d_1,d_2,d_3): \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_1^2 & r_2^2 & r_4^2 \\ 1 & r_1^2 & 0 & s_{1,2}-d_2^2 & s_{1,4}-(d_1+d_2+d_3)^2 \\ 1 & r_2^2 & s_{1,2}-d_1^2 & 0 & s_{2,4}-(d_2+d_3)^2 \\ 1 & r_4^2 & s_{1,4}-(d_1+d_2+d_3)^2 & s_{2,4}-(d_2+d_3)^2 & 0 \end{vmatrix} = 0. \quad (10)$$

Equations (7), (8), (9), and (10) form a redundant system of equations, but it is advantageous to use them all in the elimination process schematized in Fig. 4. As a result of this process, we obtain a 12-order polynomial in d_2^2. Since we decided to to take the simplex $\overline{L_2L_3}$ in $\mathbb{R}^1$ as a reference, the roots of this polynomial yield up to 12 real positive solutions for d_2. To obtain the corresponding value of d_3 for each value of d_2, we can substitute d_2 in $r_1(d_2,d_3) = 0$ and $f_2(d_2,d_3) = 0$ (see Fig. 4). Therefore, the common root of the resulting two polynomials in d_3 is the sought value. Analogously, to obtain the corresponding value for d_1, we can substitute d_1 in $r_2(d_1,d_2) = 0$ and $f_1(d_1,d_2) = 0$ to obtain their common root. Observe that the

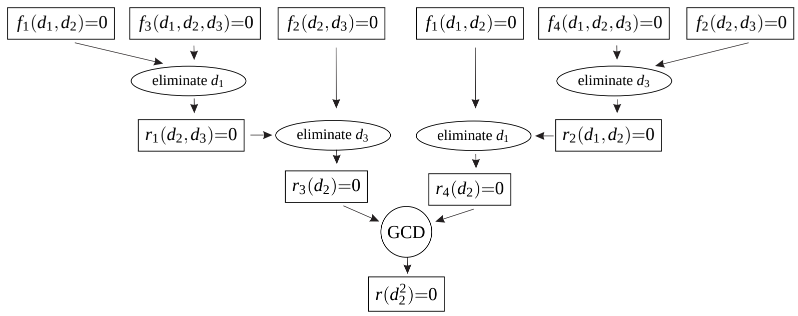


Fig. 4 As an alternative to multivariate elimination, we have exploited the redundancy of the derived set of equations to obtain a 12-degree polynomial equation in d_2^2 by applying four eliminations of a single variable and the computation of a greatest common divisor.

signs of d_1 and d_3 indicate the location of L_1 and L_4 with respect to L_2 and L_3, respectively.

As an example, let us consider the case in which we locate the sphere centers at the vertices of a regular tetrahedron. To this end, consider the cube whose vertices are located at $\frac{1}{\sqrt{8}}(\pm 1, \pm 1, \pm 1)$. The inscribed regular tetrahedron whose edges are alternate face diagonals of the cube have edge length 1. Then, if the squared radii are $r_1^2 = 0.2750$, $r_2^2 = 0.2500$, $r_3^2 = 0.2625$, and $r_4^2 = 0.2875$, the resulting polynomial equation in d_2^2 is

$$d^{12} - 4.333d^{11} + 8.099d^{10} - 8.628d^9 + 5.836d^8 - 2.641d^7 + 0.821d^6 - 0.177d^5$$
$$+ 2.618 \cdot 10^{-2} d^4 - 2.617 \cdot 10^{-4} d^3 + 1.68 \cdot 10^{-4} d^2 - 6.245 \cdot 10^{-6} d + 1.02 \cdot 10^{-7} = 0,$$

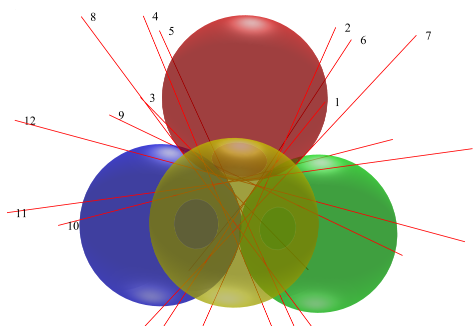
where $d = d_2^2$. This polynomial has 12 real solutions. Each value of d yields one value for d_2 due to our assumption for its sign. Then, following the procedure sketched above, we can obtain the corresponding values for d_1 and d_3. The 12 resulting solutions are shown in Fig. 5.

4 The forward kinematics of the 4-SPS robot

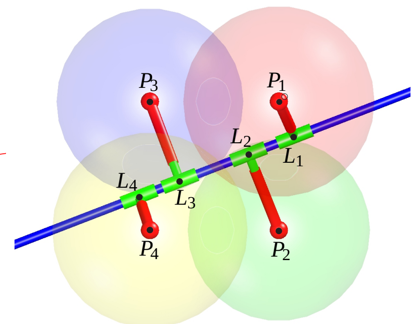
There are many industrial tasks —laser-engraving, spray-based painting, or water-jet cutting— that require controlling the orientation of an end-effector to be perpendicular to a 3D free-from surface along a given trajectory no matter its axial orientation. This kind of tasks can be automated using, for example, pentapods consisting of five SPU legs where the centers of the universal joints are aligned [15,16]. Nevertheless, when the translation along the axis of the end-effector becomes irrelevant, as is the case in most pointing devices, the pentapod can be substituted with a tetrapod. One possible implementation for such a device consists in using a 4-SPP parallel robot where the non-actuated prismatic joint axes are aligned. The location

sol.	1	2	3	4	5	6	7	8	9	10	11	12
d_1	0.3745	0.6194	-0.6725	-0.9533	-0.6653	0.2707	0.2999	-0.9543	-0.3729	-0.3299	-0.3703	-0.6166
d_2	0.3054	0.3194	0.3362	0.3528	0.4038	0.4042	0.6399	0.6399	0.6628	0.6651	0.9677	0.9679
d_3	-0.9263	-0.6624	0.6043	0.2997	-0.6533	0.2395	-0.2980	-0.2984	-0.9277	0.2762	-0.3461	-0.6024

(a)



(b)



(c)

Fig. 5 Example with 12 common tangents to four spheres of radii $r_1^2 = 0.2750$ (red), $r_2^2 = 0.2500$ (green), $r_3^2 = 0.2625$ (blue), and $r_4^2 = 0.2875$ (yellow) centered at the vertices of a regular tetrahedron of unitary edge length. The obtained numerical solutions in **(a)** are depicted in **(b)**. If this is applied to solve the forward kinematics of a 4-S$\underline{P}$C parallel manipulator with the same dimensions whose end-effector attachments are ordered L_1, L_2, L_3 and L_4 only one solution is valid **(c)**.

of this axis is what is actually controlled. Since it is tangent to the four spheres centered at the spherical joints, with radius equal to the length of the corresponding legs, solving the forward kinematics of this parallel robot is partially equivalent to finding all tangent lines to four spheres. This equivalence is not absolute because the order of the attachments on end-effector cannot be altered without disassembling the robot itself. In other words, this order must be preserved. For example, following with the same example in the previous section, let us assume that the order of the attachments in the end-effector is L_1, L_2, L_3 and L_4. Then, d_1 and d_3 must be positive. Observe that, in the example of the previous section, there is only only solution satisfying these conditions. The corresponding robot configuration appears in Fig. 5(c).

5 Conclusion

Despite the apparent simplicity of obtaining all tangents to four spheres, it seems that there are still some questions associated with it that remain unanswered. For example, the question whether it is possible for four disjoint unit spheres to have 12 common tangents remains still open [17, p. 113]. This is a point that we are currently trying to elucidate using the approach presented in this paper.

The proposed distance-based approach has allowed us to gain a new insight into the forward kinematics of a tetrapod used as a pointing device. The use of a tetrapod instead of a pentapod not only reduces the number of actuators and the risk of leg collisions, but also the number of forward kinematic solutions. If the ordering of the moving platform attachments is also taken into account, we have shown how this number might be reduced to just one.

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