Linear Quadratic Zonotopic Control of Switched Systems: Application to Autonomous Vehicle Path-Tracking

Shuang Zhang, Sara Ifqir and Vicenç Puig

***Abstract*—This letter proposes a zonotopic approach for the state feedback control problem of a class of uncertain switched systems subject to unknown but bounded disturbances and measurement noises. The proposed approach is the zonotopic analogous case of the switched Linear Quadratic Gaussian (LQG) control, in which the feedback loop is closed using the optimal estimates of a Switched Zonotopic Kalman Filter (SZKF) leading to a Switched Linear Quadratic Zonotopic (SLQZ) control scheme. In this context, first, a SZKF with offline filter gains design is proposed so that the unmeasurable system states can be estimated. Then, to tackle the synthesis of the SZKF and the state feedback controller, separation principle is proved so that the computation of the optimal controller and estimator can be done separately by finding the solutions to a finite set of Linear Matrix Inequalities (LMIs). At last, a reference path tracking controller of the vehicle lateral dynamics is designed to demonstrate the validity and performance of the proposed method.**

***Index Terms*—Switched system, optimal control, zonotope, autonomous vehicles, path tracking.**

I. Introduction

Switched systems are an important class of hybrid systems and have wide practical applications in many fields, as e.g., networked control systems, power systems, automotive and aircraft control, among others. Therefore, important results have been achieved dealing with stability [1], stabilization and controllability [2], observability and estimation [3]. The goal of the letter is to investigate a state feedback/estimation design strategy for discrete-time switched systems affected by additive uncertainties, such as process disturbances or measurement noises.

Among the existing robust state estimation approaches, one category to characterize the uncertainties relies on the assumption of their probability distributions. However, it may fail in many practical situations [4]. Hence, in the remaining of the letter, set-based approaches [5] are considered, in which the modelling uncertainties are assumed to be unknown but bounded by a priori known bound, requiring no assumption about the probability distributions. In the literature, several approaches have been proposed to address the state bounding problem for switched systems, describing the bound by several types of sets, such as intervals [6], [7], ellipsoids [8], and zonotopes [9].

Regarding the research on set-based control, an interval observer-based feedback control framework was first proposed in [10] for solving the problem of output stabilization of a class of nonlinear and Linear Parameter Varying (LPV) systems subject to uncertainties. After that, it was extended to nonlinear switched systems [11] and switched LPV systems [12]. Meanwhile, this interval observer-based control scheme has been applied to several practical scenarios, as e.g. multi-DOF micropositioner [13] and solid oxide fuel cells stacks [14]. However, the condition of designing an interval observer could be very restrictive, as the designed observer gain is required to ensure not only that the observer system is robustly stable, but also that the estimation error system matrix is Metzler [6]. This motivates the focus of this letter on zonotopes, which provide efficient computation, a much more compact representation and effectively mitigate wrapping effect. To this end, the zonotopic observer/estimator has been widely applied to state estimation [4], [15] and fault diagnosis [5], [16].

As we know, Linear Quadratic Gaussian (LQG) control, a synthesis of a linear-quadratic regulator (LQR) and a Kalman Filter (KF), can provide an optimal solution in terms of minimizing a quadratic cost function that includes both state errors and control efforts. Inspired by the aforementioned discussions, the aim of this letter is to solve state feedback control problems of uncertain switched systems by designing an LQG control scheme using zonotopic approaches. In particular, [17] proposed a linear-quadratic regulator (LQR) controller that operates based on the estimates of a Zonotopic Kalman Filter (ZKF) for Linear Time Invariant (LTI) system. This letter extends these results to switched systems proving the separation principle in the design of controller and observer. To the best of our knowledge, there is no literature on zonotopic LQG control for switched systems. This motivates the work of the present letter.

To address the state feedback control problem using zonotopes, the Switched Zonotopic Kalman Filter (SZKF) is first

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developed for discrete-time switched LTI systems, in which the switched filter gain is computed offline by solving Linear Matrix Inequality (LMI) optimization problems instead of calculated online using the explicit solution of [4]. Then, as a zonotopic counterpart of the Switched Linear Quadratic Gaussian (SLQG) scheme, the Switched Linear Quadratic Zonotopic (SLQZ) control scheme is first proposed, which describes uncertainties by zonotopic sets instead of Gaussian probability distributions. The SLQZ control scheme operates a Switched Linear Quadratic Regulator (SLQR) controller on the estimates of the SZKF to achieve optimal state feedback control performance. Moreover, the separation principle of SLQZ control is proved, which allows to calculate the optimal switched controller gain and filter gain separately. The proposed SLQZ control scheme is applied to vehicle reference path-tracking control. Effectiveness and applicability of the proposed scheme is demonstrated through simulation by using real collected data. Finally, it should be underlined that the use of the zonotopic state feedback technique is a novelty in the context of vehicle dynamics control.

Notation: In the following, $\mathbb{R}^n$ denotes the set of n-dimensional real numbers and $\oplus$ denotes the Minkowski sum. The matrices are written using capital letter, e.g., A, the calligraphic notation is used for denoting sets, e.g., $\mathcal{X}$. For a vector $s \in \mathbb{R}^n$, $\|s\|$ denotes Euclidean vector norm, $\|s\|_P = \sqrt{s^T P s}$ denotes weighted Euclidean vector norm, where $P > 0$. A unitary interval is a vector denoted by $\boldsymbol{B} = [-1, 1]$. A unitary box in $\mathbb{R}^m$, denoted by $\boldsymbol{B}^m$, is a box composed of m unitary intervals. $\mathrm{Tr}[\cdot]$ denotes the trace of a square matrix. A^T denotes the transpose of A. An identity matrix of dimension n is denoted by I_n. A zonotope of order m in $\mathbb{R}^n$ is the translation by the center $c \in \mathbb{R}^n$ of the image of an unitary hypercube of dimension m in $\mathbb{R}^n$ under a linear transformation $R \in \mathbb{R}^{n\times m}$, the zonotope $\mathcal{X}$ is defined by: $\mathcal{X} = \langle c, R\rangle = c \oplus R\boldsymbol{B}^m = \{c + Rz : z \in \boldsymbol{B}^m\}$.

II. PROBLEM FORMULATION

In this work, the following class of stable (or stabilizable) uncertain discrete-time switched systems is considered

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + w_{\sigma(k)} \\ y(k) = C_{\sigma(k)}x(k) + v_{\sigma(k)} \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output. $\sigma(k) : \mathbb{R}^+ \rightarrow \mathcal{I} = \{1, 2, \ldots, I_\sigma\}$ is a known switching signal, satisfying the ADT switching scheme [18], assumed to be prior unknown but online available, where I denotes the number of subsystems. A sequence $k_1, k_2 \ldots k_l, k_{l+1}, \ldots, k_{N_\sigma(k_0,K)}$ is employed to represent the switching instants on the interval $[k_0, K)$, where $k_0 = 0$ denotes the initial time, k_l denotes the l^{th} switching instant and the active i^{th} subsystem $(\sigma(k) = i)$ when $k \in [k_l, k_{l+1})$. $A_{\sigma(k)} \in \mathbb{R}^{n_x\times n_x}$, $B_{\sigma(k)} \in \mathbb{R}^{n_x\times n_u}$ and $C_{\sigma(k)} \in \mathbb{R}^{n_y\times n_x}$ are state, input and output matrices. $w_{\sigma(k)} \in \mathbb{R}^{n_w}$ and $v_{\sigma(k)} \in \mathbb{R}^{n_v}$ are the process and measurement noises, respectively, assumed to be unknown but bounded by zonotopes, i.e., $w_{\sigma(k)} \in \mathcal{W}_{\sigma(k)} = \langle 0, E_{\sigma(k)}\rangle$, $v_{\sigma(k)} \in \mathcal{V}_{\sigma(k)} = \langle 0, F_{\sigma(k)}\rangle$, where $E_{\sigma(k)} \in \mathbb{R}^{n_x\times n_w}$ and $F_{\sigma(k)} \in \mathbb{R}^{n_y\times n_v}$ are known segment matrices of zonotope describing the worse bounds of the process and measurement noises.

In this letter, we address the optimal state feedback control problem of the uncertain switched system (1), in which the feedback loop is closed using the optimal estimates of a SZKF, as depicted by Fig. 1. Since the considered system (1) is affected by zonotopic uncertainties, a zonotopic counterpart of switched LQG control is proposed, i.e., Switched Linear Quadratic Zonotopic (SLQZ) control, a combination of a Switched Zonotopic Kalman Filter (SZKF) together with a Switched Linear Quadratic Regulator (SLQR).

In order to assess the system operation, the following performance criteria are introduced.

Definition 1: (F_W-Radius). Let $W \in \mathbb{R}^{n\times n}$ be a Symmetric Positive Definite (SPD) matrix: $W = W^T > 0$. The weighted Frobenius radius (F_W-radius) of the zonotope $\langle c, R\rangle \subset \mathbb{R}^n$ is the weighted Frobenius norm of R : $\|\langle c, R\rangle\|_{F,W} = \|R\|_{F,W}$.

Definition 2: (Zonotopic quadratic performance [17]): Given an SPD matrix $S \in \mathbb{R}^{n\times n}$ and the unknown but zonotopically bounded vector $x \in \langle c, R\rangle \subset \mathbb{R}^n$, the performance of x is evaluated according to $Q\left[x^T S x\right] = c^T S c + \|R\|^2_{F,S} = c^T S c + \mathrm{Tr}[SP]$, with $P = RR^T$.

III. SWITCHED ZONOTOPIC KALMAN FILTER

In order to obtain the unmeasured states of the uncertain switched system (1), we would like to estimate and bound the uncertain system states $x(k), \forall k \geq 0$ in a zonotopic set using a SZKF. An optimal switched filter gain $\lambda_{\sigma(k)}$ is used to reduce the state estimation error with a measured output $y(k)$.

Theorem 1: (Estimation-type Switched Zonotopic Kalman Filter) Consider the discrete-time switched system (1), let $\hat{\mathcal{X}}(k) = \langle c(k), R(k)\rangle \in \mathbb{R}^{n_x}$ be the zonotopic estimated state, where $c(k) \in \mathbb{R}^{n_x}$ and $R(k) \in \mathbb{R}^{n_x\times n_r}$ represent the center and shape matrix, respectively. $\downarrow_{q,W} R(k)$ is the reduced generator matrix that is computed according to [4]. Assume that the initial state $\hat{x}(k)$ belongs to the set $\hat{\mathcal{X}}(0) = \langle c(0), R(0)\rangle$, the estimated state can be propagated as follows:

$$c(k+1) = \bar{c}(k+1) + \lambda_{\sigma(k)}\left(y(k+1) - C_{\sigma(k)}\bar{c}(k)\right) \tag{2a}$$

$$R(k+1) = \begin{bmatrix}\Lambda_{\sigma(k)}\bar{R}(k+1) & \lambda_{\sigma(k)}F_{\sigma(k)}\end{bmatrix} \tag{2b}$$

$$\bar{c}(k+1) = A_{\sigma(k)}c(k) + B_{\sigma(k)}u(k) \tag{3a}$$

$$\bar{R}(k+1) = \begin{bmatrix}A_{\sigma(k)} \downarrow_{q,W} R(k) & E_{\sigma(k)}\end{bmatrix} \tag{3b}$$

where $\Lambda_{\sigma(k)} \triangleq I_{n_x} - \lambda_{\sigma(k)}C_{\sigma(k)}$, $\left\langle \bar{c}(k), \bar{R}(k)\right\rangle \in \mathbb{R}^{n_x}$ is the zonotopic predicted state, $\bar{c}(k) \in \mathbb{R}^{n_x}$ and $\bar{R}(k) \in \mathbb{R}^{n_x\times n_{\bar{r}}}$ represent the predicted center and shape matrix, respectively.

Proof: Considering the inclusion $x(k) \in \langle c(k), R(k)\rangle$ and assuming that the $\downarrow_{q,W}$ preserves inclusion $x(k) \in \langle c(k), \downarrow_{q,W} R(k)\rangle$, the zonotopic predicted set can be computed as $\langle\bar{c}(k+1), \bar{R}(k+1)\rangle = A_{\sigma(k)}\left\langle c(k), \downarrow_{q,W} R(k)\right\rangle \oplus \langle B_{\sigma(k)}u(k), 0\rangle \oplus \langle 0, E_{\sigma(k)}\rangle$. Thus, the predicted state set (3a, 3b) is derived. Then, the estimated state is updated as $\langle c(k+1), R(k+1)\rangle = \langle\bar{c}(k+1), \bar{R}(k+1)\rangle \oplus \lambda_{\sigma(k)}\langle y(k+1), F_{\sigma(k)}\rangle \oplus \lambda_{\sigma(k)}C_{\sigma(k)}\langle -\bar{c}(k+1), -\bar{R}(k+1)\rangle$. Therefore, the estimated state set (2a, 2b) is derived. ∎

Theorem 2: Given the switched system (1), $\forall i \in \mathcal{I}$, the optimal switched filter gain λ_i can be obtained if there exists a matrix $W_i \in \mathbb{R}^{n_x\times n_y}$, a positive definite matrix $\Gamma \in \mathbb{R}^{n_x\times n_x}$

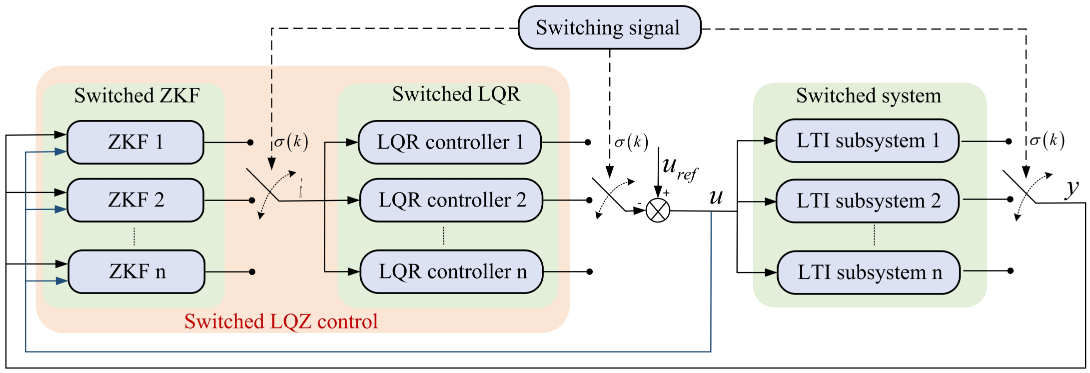


Fig. 1: Diagram of Switched Linear Quadratic Zonotopic control.

and a scalar γ that are obtained by solving the following LMI optimization problem

$$\min \gamma, \tag{4a}$$

$$s.t., \begin{bmatrix} \gamma I_{n_x} & I_{n_x} \\ I_{n_x} & \Gamma \end{bmatrix} > 0 \tag{4b}$$

$$\begin{bmatrix} -\Gamma & * & * & * \\ A_i^T(\Gamma - W_iC)^T & -\Gamma & * & * \\ E_i^T(\Gamma - W_iC)^T & 0 & -I_{n_w} & * \\ W_i^T & 0 & 0 & -M_i^{-1} \end{bmatrix} \leq 0 \tag{4c}$$

where $\lambda_i = \Gamma^{-1}W_i$, $M_i = F_iF_i^T$.

Proof: As mentioned before, the optimal filter gain aims to minimize the F_W-radius of the state bounding zonotope, i.e., $\min\ J_e = \|R(k)\|_{F,W}^2$. According to [4], the optimal observer gain is independent of the weighting matrix W. The optimal observer gain is equivalently obtained through minimising the following cost function

$$\min\ J_e = \|R(k)\|_F^2 = \mathrm{Tr}[P(k)] = \sum_{i=1}^{n} \gamma_i \tag{5}$$

where $P(k) = R(k)R^T(k)$, $\gamma_1, \gamma_2, \ldots, \gamma_n$ represent the eigenvalues of P. Therefore, a possible approach to minimize the cost function J_e is to minimize a given scalar γ, s.t., $\gamma_i < \gamma$. It follows that

$$P(k) - \gamma I_{n_x} < 0 \tag{6}$$

Recalling that

$$R(k+1) = [\Lambda_{\sigma(k)}A_{\sigma(k)}R(k)\ \Lambda_{\sigma(k)}E_{\sigma(k)}\ \lambda_{\sigma(k)}F_{\sigma(k)}],$$

it follows that

$$\begin{aligned} P(k+1) = &\Lambda_{\sigma(k)}E_{\sigma(k)}E_{\sigma(k)}^T\Lambda_{\sigma(k)}^T + \lambda_{\sigma(k)}F_{\sigma(k)}F_{\sigma(k)}^T\lambda_{\sigma(k)}^T \\ &+ \Lambda_{\sigma(k)}A_{\sigma(k)}P(k)A_{\sigma(k)}^T\Lambda_{\sigma(k)}^T \end{aligned} \tag{7}$$

As LMI provides the solution for steady state, then, $P(k+1) = P(k) = P_{ARE}$ satisfies the following Algebraic Riccati Equation (ARE)

$$\begin{aligned} &\Lambda_{\sigma(k)}A_{\sigma(k)}P_{ARE}(\Lambda_{\sigma(k)}A_{\sigma(k)})^T + \Lambda_{\sigma(k)}E_{\sigma(k)}E_{\sigma(k)}^T\Lambda_{\sigma(k)}^T \\ &- P_{ARE} + \lambda_{\sigma(k)}F_{\sigma(k)}F_{\sigma(k)}^T\lambda_{\sigma(k)}^T = 0 \end{aligned} \tag{8}$$

According to [19], $\forall\sigma(k) = i \in \mathcal{I}$ the optimal value of the following inequality equals the solution to the above ARE,

$$\begin{aligned} &(I - \lambda_iC_i)A_iP((I - \lambda_iC_i)A_i)^T - P + \\ &(I - \lambda_iC_i)E_iE_i^T(I - \lambda_iC_i)^T + \lambda_iF_iF_i^T\lambda_i^T \leq 0 \end{aligned} \tag{9}$$

Let $\Gamma = P^{-1}$, $W_i = \Gamma\lambda_i$, $M_i = F_iF_i^T$, and multiplying by Γ on the left and right of equation (9) yields to

$$\begin{aligned} &(\Gamma - W_iC_i)A_i\Gamma^{-1}A_i^T(\Gamma - W_iC_i)^T - \Gamma + \\ &(\Gamma - W_iC_i)E_iE_i^T(\Gamma - W_iC_i)^T + W_iM_iW_i^T \leq 0 \end{aligned} \tag{10}$$

which is equivalent to

$$\begin{aligned} &-\Gamma - [(\Gamma - W_iC_i)A_i \quad (\Gamma - W_iC_i)E_i \quad W_i] \\ &\begin{bmatrix} -\Gamma^{-1} & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -R_i \end{bmatrix} \begin{bmatrix} A_i^T(\Gamma - W_iC_i)^T \\ E_i^T(\Gamma - W_iC_i)^T \\ W_i^T \end{bmatrix} \leq 0 \end{aligned} \tag{11}$$

Now, applying Schur complement, it follows (4c). Considering $\Gamma = P^{-1}$, the previous condition (6) becomes $\gamma I_{n_x} - \Gamma^{-1} > 0$, i.e., $\begin{bmatrix} \gamma I_{n_x} & I_{n_x} \\ I_{n_x} & \Gamma \end{bmatrix} > 0$. Then, the optimization problem (4) should be solved by minimizing the value of γ. Hence, we complete the proof. ■

IV. Switched Linear Quadratic Zonotopic Control

The SLQZ control scheme is based on the optimal feedback of the estimated bounding state $\hat{x}(k) \in \langle c(k), R(k)\rangle$ to the system input via the optimal controller gain $K_{\sigma(k)}$, as depicted by Fig. 1. Now, a zonotopic quadratic cost function is given for the proposed control scheme

$$J = Q\left[\sum_{k=0}^{\infty}\left(x^T(k)W_{\sigma(k)}^x x(k) + u^T(k)W_{\sigma(k)}^u u(k)\right)\right]. \tag{12}$$

Usually, the switched weighting matrices $W_{\sigma(k)}^x$ and $W_{\sigma(k)}^u$ are implemented as diagonal matrices.

A. Separation Principle

In the following, it is proved that the proposed switched filter gain $\lambda_{\sigma(k)}$ and controller gain $K_{\sigma(k)}$ can be designed separately for the following closed-loop system:

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + w_{\sigma(k)}, \\ y(k) = C_{\sigma(k)}x(k) + v_{\sigma(k)}, \\ c(k+1) = \Lambda_{\sigma(k)}(A_{\sigma(k)}c(k) + B_{\sigma(k)}u(k)) + \lambda_{\sigma(k)}y(k+1), \\ R(k+1) = [\Lambda_{\sigma(k)}A_{\sigma(k)}R(k)\ \Lambda_{\sigma(k)}E_{\sigma(k)}\ \lambda_{\sigma(k)}F_{\sigma(k)}], \\ u(k) = -K_{\sigma(k)}c(k) \end{cases}$$

In this regard, the SLQZ design is decomposed into two independent problems: 1) the optimization of the switched LQR control of the nominal plant; 2) the optimization of the estimated state $\hat{x}$ from the measurements y.

Theorem 3: The optimal control of the system (1), with the cost functional (12), is obtained by taking the optimal control law $u(k) = -K_{\sigma(k)}x_c(k)$, calculated for the corresponding nominal system,

$$x_c(k+1) = A_{\sigma(k)}x_c(k) + B_{\sigma(k)}u(k), \tag{13}$$

with the cost function

$$J_c = \sum_{k=0}^{\infty}\left(x_c^T(k)W^x_{\sigma(k)}x_c(k) + u^T(k)W^u_{\sigma(k)}u(k)\right). \tag{14}$$

The nominal plant state $x_c(k)$ is then replaced by its optimal estimate $c(k)$, obtained by the SZKF (3-2) with the optimal filter gain $\lambda_{\sigma(k)}$ calculated using the minimization of the cost function

$$J_e = \|R(k)\|^2_{F,W} \tag{15}$$

Proof: Denote the estimation error as $e(k+1) = x(k+1) - c(k+1)$, considering the estimated center

$$c(k+1) = \Lambda_{\sigma(k)}(A_{\sigma(k)}c(k) + B_{\sigma(k)}u(k)) + \lambda_{\sigma(k)}y(k+1)$$

the error dynamics is given by

$$e(k+1) = \Lambda_{\sigma(k)}A_{\sigma(k)}e(k) + \Lambda_{\sigma(k)}w_{\sigma(k)} - \lambda_{\sigma(k)}v_{\sigma(k)}.$$

Replacing the input by $u(k) = -K_{\sigma(k)}c(k)$, the system state equation becomes $x(k+1) = A_{\sigma(k)}x(k) - B_{\sigma(k)}K_{\sigma(k)}c(k) + w_{\sigma(k)} = (A_{\sigma(k)} - B_{\sigma(k)}K_{\sigma(k)})x(k) + B_{\sigma(k)}K_{\sigma(k)}e(k) + w_{\sigma(k)}$. Therefore, the composite system (controller+filter) is described as follows.

$$\begin{aligned}\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} &= \begin{bmatrix} A_{\sigma(k)} - B_{\sigma(k)}K_{\sigma(k)} & B_{\sigma(k)}K_{\sigma(k)} \\ 0 & \Lambda_{\sigma(k)}A_{\sigma(k)} \end{bmatrix}\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ \Lambda_{\sigma(k)} & -\lambda_{\sigma(k)} \end{bmatrix}\begin{bmatrix} w_{\sigma(k)} \\ v_{\sigma(k)} \end{bmatrix}\end{aligned} \tag{16}$$

It reveals that the dynamics of the estimation error is completely decoupled from that of the state. It is notable that $x(k) = c(k) + e(k) \in \langle c(k), R(k)\rangle$. According to Definition 2, the cost function (12) can be rewritten as

$$\begin{aligned} J &= Q\left[\sum_{k=0}^{\infty}\left(x^T(k)W^x_{\sigma(k)}x(k) + u^T(k)W^u_{\sigma(k)}u(k)\right)\right] \\ &= \sum_{k=0}^{\infty}\left(c^T(k)W^x_{\sigma(k)}c(k) + u^T(k)W^u_{\sigma(k)}u(k)\right) \\ &+ \sum_{k=0}^{\infty}\|R(k)\|^2_{F,W^x_{\sigma(k)}} \end{aligned} \tag{17}$$

The first part in expression (17) yields the part which will be minimized by a suitable choice of the sequence of control signals $u(k)$. The second part will be minimized as well by selecting a suitable filter gain $\lambda_{\sigma(k)}$ to get the optimal estimate of $x(k)$. ■

B. Optimal Switched Controller Gain Design

In order to stabilize the switched system (1) for any admissible switching signal $\sigma(k)$, the following Theorem 4 is proposed for computing the optimal robust controller $u(k) = -K_{\sigma(k)}x_c(k)$ through minimizing the cost function (14).

Theorem 4: Given the nominal switched system (13), $\forall i \in \mathcal{I}$, the optimal controller gain K_i can be obtained if there exists a matrix $Z_i \in \mathbb{R}^{n_x \times n_u}$, a positive definite matrix $\Upsilon \in \mathbb{R}^{n_x \times n_x}$ and a scalar γ that are obtained by solving the following optimization problem

$$\min \gamma \tag{18a}$$

$$s.t., \begin{bmatrix} \gamma I_{n_x} & I_{n_x} \\ I_{n_x} & \Upsilon \end{bmatrix} > 0 \tag{18b}$$

$$\begin{bmatrix} -\Upsilon & * & * & * \\ A_i\Upsilon - B_iZ_i & -\Upsilon & * & * \\ H_i\Upsilon & 0 & -I_{n_x} & * \\ Z_i & 0 & 0 & -W_i^{u-1} \end{bmatrix} < 0 \tag{18c}$$

with $K_i = Z_i\Upsilon^{-1}$, $H_i^TH_i = W_i^x$.

Proof: For all $\sigma(k) = i \in \mathcal{I}$, let us choose a common Lyapunov function: $V(x_c(k)) = x_c^T(k)Px_c(k))$, $P = P^T > 0$, satisfying the following conditions:

$$V(x_c(0)) < \gamma \tag{19}$$

$$V(x_c(k+1)) - V(x_c(k)) + x_c^T(k)W_i^xx_c(k) + u^T(k)W_i^uu(k) < 0 \tag{20}$$

Therefore,

$$\begin{aligned} &V(x_c(\infty)) - V(x_c(0)) + \\ &\sum_{k=0}^{\infty}\left(x_c^T(k)W_i^xx_c(k) + u^T(k)W_i^uu(k)\right) < 0, \end{aligned} \tag{21}$$

which we can rewrite as $J_c < V(x_c(0)) < \gamma$. In this context, the cost function J_c is minimized along with the objective scalar γ.

Recalling that

$$\begin{aligned} V(x_c(k+1)) &= x_c^T(k+1)Px_c(k+1) \\ &= ((A_i - B_iK_i)x_c(k))^TP((A_i - B_iK_i)x_c(k)), \end{aligned}$$

Inequality (20) is equivalent to

$$(A_i - B_iK_i)^TP(A_i - B_iK_i) - P + W_i^x + K_i^TW_i^uK_i < 0. \tag{22}$$

Applying Schur complementation, (22) can be rewritten as the following inequality:

$$\begin{bmatrix} -\Upsilon & * & * & * \\ A_i\Upsilon - B_iZ_i & -\Upsilon & * & * \\ H_i\Upsilon & 0 & -I & * \\ Z_i & 0 & 0 & -Q_i^{u-1} \end{bmatrix} < 0 \tag{23}$$

where $\Upsilon = P^{-1}$, $Z_i = K_i\Upsilon$. Furthermore, the condition (19) can be rewritten as $\gamma I_{n_x} - \Upsilon^{-1} > 0$, i.e., $\begin{bmatrix} \gamma I_{n_x} & I_{n_x} \\ I_{n_x} & \Upsilon \end{bmatrix} > 0$. Then, the switched state feedback matrix K_i can be obtained by solving the LMIs (18), by letting $K_i = Z_i\Upsilon^{-1}$. ■

Remark 1: It is worth noting that the optimal filter gain of the SLQZ control scheme is calculated directly using Theorem 2, as the weighting matrix $W^x_{\sigma(k)}$ is not related to the calculation of the optimal filter gain [4].

V. CASE STUDY

In this section, a vehicle lateral dynamics nonlinear model is given as the following state equation:

$$\begin{bmatrix} \dot{\beta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} -\frac{c_f+c_r}{mv_x} & \frac{c_r l_r - c_f l_f}{mv_x^2} - 1 \\ \frac{c_r l_r - c_f l_f}{I_z} & -\frac{c_r l_r^2 + c_f l_f^2}{I_z v_x} \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \frac{c_f}{mv_x} \\ \frac{c_f l_f}{Iz} \end{bmatrix} \delta_f \tag{24}$$

where β and $\dot{\psi}$ are vehicle sideslip angle and yaw rate, δ_f is the steering angle, m and I_z are the mass and the yaw moment, v_x is the longitudinal velocity, l_f and l_r are the distances from front and rear axle to the center of gravity, c_f, c_r are the cornering stiffness of front and rear tires. Note that the yaw rate $\dot{\psi}$ and longitudinal speed v_x can be measured online.

The bicycle model (24) is first discretized using the Euler's discretization method, with the sampling time T= 0.01s. Then, a switched representation of the model (24) is generated where each subsystem operates around a given constant longitudinal velocity value (for example, three subsystems defined for low, average and high longitudinal speed). Then, a switching signal depending on the measured longitudinal velocity v_x is considered as follows:

$$\sigma(k) = \begin{cases} 1 & if \quad 10m.s^{-1} < v_x \leq 13m.s^{-1} \\ 2 & if \quad 13m.s^{-1} < v_x \leq 16m.s^{-1} \\ 3 & if \quad 16m.s^{-1} < v_x \leq 20m.s^{-1} \end{cases} \tag{25}$$

and three local models are obtained with the considered switching law shown in Fig.2.

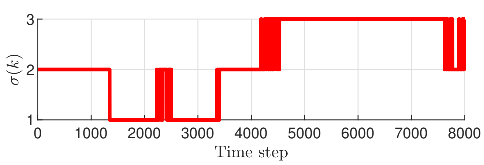


Fig. 2: Switching signal $\sigma(k)$

Note that environmental disturbances, as well as nonmodelled effects are added to the vehicle model through additive state disturbance and measurement noise vectors $w_{\sigma(k)}$ and $v_{\sigma(k)}$, satisfying $|w_{\sigma(k)}| \leq [0.002 \quad 0.01]^T$, $|v_{\sigma(k)}| \leq 0.03$.

As the objective is to design a reference tracking control of the vehicle lateral dynamics using the proposed SLQZ control. The goal of this controller is to ensure that the vehicle tracks a desired reference trajectory by regulating the steering angle δ_f. The desired sideslip angle and yaw rate are given as follows

$$\beta_{\text{ref}} = \frac{l_r - \frac{l_f m v_x^2}{C_r(l_f+l_r)}}{\ell_f + \ell_r + \frac{m v_x^2(\ell_r C_r - \ell_f C_f)}{C_f C_r(\ell_f+\ell_r)}} \delta_{ref} \tag{26}$$

$$\dot{\psi}_{\text{ref}} = \frac{v_x}{\ell_f + \ell_r + \frac{m v_x^2(\ell_r C_r - \ell_f C_f)}{C_f C_r(\ell_f+\ell_r)}} \delta_{ref} \tag{27}$$

As the trajectory planner has full-state information, a full-state feedback controller can be designed. By denoting the difference between the desired sideslip angle, yaw rate and steering angle $(\beta_{ref}, \dot{\psi}_{ref}, \delta_{ref})$ and the real vehicle sideslip angle, yaw rate and steering angle $(\beta, \dot{\psi}, \delta_f)$ as $x_e = \begin{bmatrix} \beta_{ref} \\ \dot{\psi}_{ref} \end{bmatrix} - \begin{bmatrix} \beta \\ \dot{\psi} \end{bmatrix}$, $u_e = \delta_{ref} - \delta_f$, the nominal reference tracking model can be obtained:

$$x_e(k+1) = A_{\sigma(k)} x_e(k) + B_{\sigma(k)} u_e(k) \tag{28}$$

Then, the real input of the vehicle is represented as $u(k) = K_{\sigma(k)} x_e(k) + u_{ref}(k)$, where $K_{\sigma(k)}$ is the switched controller gain to be designed. As the system states are not fully available, a SZKF is employed to obtain the optimal estimated states bounded by a zonotope set $\hat{x}(k) \in \langle c(k), R(k) \rangle$, as well as to deal with the uncertainty problem.

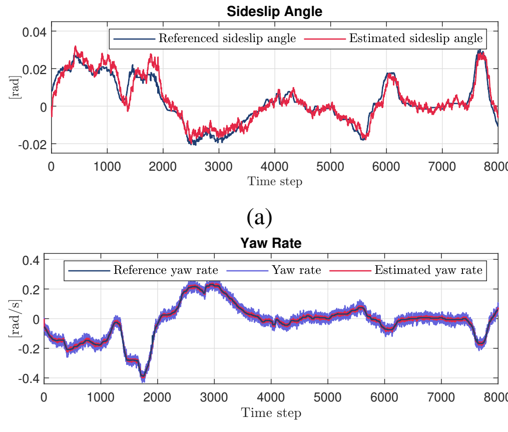
By assigning $W^x_{\sigma(k)} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $W^u_{\sigma(k)} = 0.01$, and solving the LMI optimization problem (18) and (4) by means of YALMIP toolbox, we have

$$K_1 = [0.6276 \; 3.2898], K_2 = [0.5988 \; 3.3257],$$
$$K_3 = [0.5793 \; 3.3472], \lambda_1 = [0.0076 \; 0.2603]^T,$$
$$\lambda_2 = [0.0071 \; 0.2661]^T, \lambda_2 = [0.0068 \; 0.2724]^T.$$

The derived controller and observer are applied to the switched linear system (24). The tracking control and state estimation performance are presented in Fig. 3-4. Fig. 3 depicts a comparison between the reference states and real/estimated states (center value of the zonotopic estimated states). Fig. 4 shows that the proposed SZKF, providing tight bounds for the estimation, is a successful extension of [4] to switched systems. To validate the robustness of the proposed architecture with respect to unmodeled vehicle dynamics, the designed switched controller and observer are tested using a high fidelity nonlinear vehicle model employing the nonlinear Magic Formula for lateral force modeling [20]. The results are depicted in Fig. 5. It reveals a comparison between the reference and the vehicle trajectories in the world coordinates, and the lateral and heading offsets. From this figure, it is evident that the proposed algorithm can achieve good performance in terms of trajectory tracking for both simplified and highly nonlinear vehicle models. It is worth noting that the lateral and heading offsets could be further minimized by employing a multiple Lyapunov function in the design procedure, thus mitigating the conservatism resulting from the use of a common function for all modes.

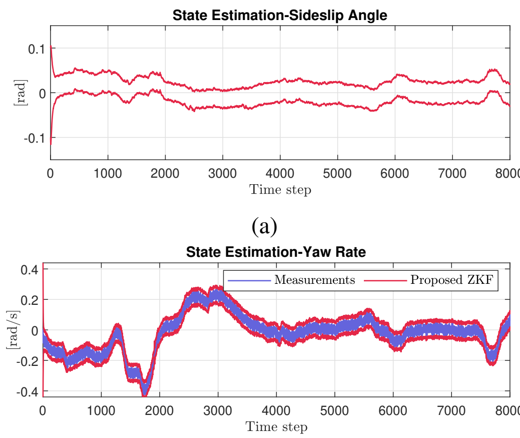
VI. CONCLUSIONS AND FUTURE WORKS

This letter proposed a zonotope-based feedback control scheme for uncertain switched systems. In order to obtain the unmeasurable states and achieve robustness, a ZKF has been employed and extended to switched systems. Moreover, the optimal filter gain is calculated offline by solving an LMI optimization problem. Then, an SLQZ control scheme has been proposed, in which the optimal switched controller and filter have been designed separately according to the separation principle. Finally, the proposed method has been validated in simulation via a reference tracking control of vehicle lateral dynamics. The simulation results revealed the effectiveness of the proposed SLQZ control scheme. Future work will explore a switched Lyapunov function for designing the controller/filter gain to reduce the conservativeness.



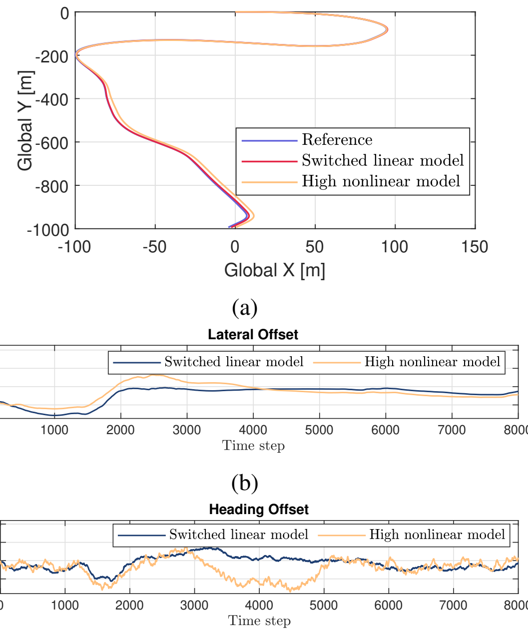
(a) (b)

Fig. 3: Tracking performance (a) Sideslip angle, (b) Yaw rate.



(a) (b)

Fig. 4: State estimation performance with the proposed ZKF. (a) Estimated sideslip angle, (b) Estimated yaw rate.



(a) (b) (c)

Fig. 5: Comparison of tracking control performance using the proposed approach. (a) Vehicle trajectory, (d) Lateral offset, (e) Heading offset.

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