Design of Observer Schemes for One-sided Lipschitz Noisy Nonlinear Systems

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***Abstract*— In this paper, the problem of designing observers for one-sided Lipschitz nonlinear systems considering noise and unknown inputs is addressed. In particular, the paper deals with two observer structures: Luenberger Observer (LO) and Unknown Input Observer (UIO). The observer synthesis procedures are formulated as convex optimization problems. Sufficient conditions for observer gain synthesis are shown to be equivalent to solve finite sets of Linear Matrix Inequalities (LMIs) and Linear Matrix Equalities (LMEs). An illustrative example is used to illustrate how the proposed observer design approaches are used to design the LO and UIO schemes. The obtained simulation results are presented to assess the proposed methods.**

I. Introduction

The supervision, control and predictive maintenance of a system generally requires the knowledge of its states. However, economic or technical constraints require the reduction of the number of physical sensors. Moreover, the evolution of the system can be influenced by unmeasured perturbations (usually modelled as unknown inputs). Thus, in the literature there are several studies focusing on reconstructing the state of a system from known input and output in the presence of unknown inputs. Such a task involves the synthesis of an Unknown Input Observers (UIO). On the other hand, when there are no unknown inputs, the Luenberger Observer (LO) scheme is often used. In the literature, many contributions are focused in the design of UIO and LO for linear systems (see [5], [6]). However, less works are devoted to nonlinear systems (NLS). Inside the family of NLS, there are nonlinear Lipschitz systems, for which there are several contributions related to the design of UIO, LO, see as e.g. in [8], [9], [7] and [3] The UIO and LO synthesis problems for nonlinear Lipschitz systems is formulated in the Linear Matrix Inequalitiy (LMI) framework, obtaining sufficient conditions to guarantee the convergence of the error dynamics. Another family of systems which is less restrictive than the Lipschitz one is family of One-sided Lipschitz Nonlinear Systems (OSL). This type of systems has attracted the attention of the scientific community, due to the fact that the bounding condition of the derivative must only be satisfied in the growth rate, but not in the decay rate (in the Lipschitz systems both must be satisfied) making it less restrictive. Examples of UIO design for OSL systems can be found in [2], [11] and [12] where they assume that the system under study is OSL and also quadratically inner bounded (QIB). UIO convergence according to the development of the authors previously cited is also reduced to a set of LMIs as the conventional Lipschitz case. The obtained results also include the cases with model uncertainties, robustness to noise and delays in the dynamics. With respect to LO design, authors of [14] assume OSL condition arriving at a Bilinear Matrix Inequality solution, while [10] design a LO assuming OSL and QIB condition under uncertainties in the nonlinearity and time delays.

Results regarding LO and UIO synthesis for OSL system (that are not QIB) are divided in two main categories. The first one considers some supplementary assumption on the Lyapunov used to establish the convergence of the observer. Typically the Lyapunov function is required to be the identity (or a multiple of the identity); The second category is to directly assume that the product of the Lyapunov function and the nonlinearity is OSL.

This paper belongs to the first category where the requirement of the Lyapunov function to be the identity is relaxed. Here, an UIO and a LO are designed using an LMI approach for one-sided Lipschitz nonlinear systems. From a methodological point of view, the main contribution of this work consists in the reformulation of the UIO and LO synthesis problem for one-sided Lipschitz non-linear system (without considering the QIB condition) as a convex optimization problem. This allows to solve the LO and UIO synthesis problem for non Lipsichtz system (since Lipschitz and QIB are the same family of fuctions [15]), as it formulate the observer synthesis problem as a convex optimization problem by combining matrix inequalities and equalities. Using this approach, it becomes clear that sufficient conditions for UIO synthesis are identical to the one considered for classical observer synthesis using LMIs (as considered for instance in [13]) combined with decoupling conditions between the observer error and the unknown input.

The structure of the paper is the following: In Section II, the problem statement is presented. Section III and IV introduces the LO design procedures without and with noise. Section III and IV introduces the UIO designs procedure without and with noise. Section VII show the obtained results applying the proposed approaches using an illustrative example. Finally, Section VIII draws the main conclusions and points several future research paths.

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Notations and definitions

- v^T denote the transpose of v for either a matrix or a vector.
- $\mathcal{H}(A) = A + A'$
- For a symmetric matrix the symbol $*$ denotes the elements induced by symmetry: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ will be denoted $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$
- I_n is the identity of size n
- For a square matrix $P \succ 0$ (resp. $P \prec 0$) means that P is positive definite (resp. negative definite).
- For a square matrix $P \succeq 0$ (resp. $P \preceq 0$) means that P is positive semi-definite (resp. negative semi-definite).

Lemma 1 [16]: (The Schur complement)
Given matrices $Q = Q^T$, $R = R^T$ and S with appropriate dimensions, the following propositions are equivalent:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}, Q \prec 0, Q = Q^T, R = R^T \tag{1}$$

$$R \prec 0, Q - SR^{-1}S^T \prec 0 \tag{2}$$

Lemma 2 [16]: (The S-procedure)
For $P \succ 0$, $\forall \psi \neq 0$ and π satisfying $\pi^T\pi \leq \psi^T CC^T \psi$

$$\begin{bmatrix} \psi \\ \pi \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB \\ * & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \pi \end{bmatrix} \tag{3}$$

is equivalent to $\exists \tau \geq 0$ such that

$$\begin{bmatrix} A^TP + PA + \tau C^TC & PB \\ * & -\tau I_n \end{bmatrix} \tag{4}$$

II. System Definition and Problem Statement

A. System under consideration

Let us consider a noisy non-linear time-invariant system with unknown input

$$\begin{aligned} \dot{x} &= Ax + Bu + D_f f(Hx) + Dv \\ y &= Cx + Rh \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $v \in \mathbb{R}^p$ is an unknown input, $h \in \mathbb{R}^j$ is the noise in the measurement, while $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{n\times q}$, $H \in \mathbb{R}^{s\times n}$, $R \in \mathbb{R}^{p\times j}$ are the system matrices. The non-linearity $f(Hx)$ is assumed to be a one-sided Lipschitz function (OSL), fulfilling the following condition [2]:

$$< f(H\hat{x}) - f(Hx), H(\hat{x} - x) > \leq \rho \|H(\hat{x} - x)\|^2, \tag{6}$$

Which writing $\Delta f = f(H\hat{x}) - f(Hx)$ is equivalent to:

$$\Delta f^T He \leq \rho e^T H^T He. \tag{7}$$

where ρ is the OSL constant, and $< x, y >= x^Ty$ is the inner product between x and y.

B. Observer definition

1) Luenberger Observer (LO): For system (5) , the following Luenberger like observer is proposed

$$\dot{\hat{x}} = A\hat{x} + Bu + D_f f(H\hat{x}) + K_L C(\hat{x} - x) \tag{8}$$

where K_L is the observer gain.

2) Unknown Input Observer (UIO): For the system (5), the following UIO is proposed

$$\begin{aligned} \dot{z} &= Nz + TBu + Gy + TD_f f(H\hat{x}) \\ \hat{x} &= z - Ey \end{aligned} \tag{9}$$

where z is the classical auxiliary variable and $\hat{x}$ is the estimation of the system state. The matrices N, J, L, E and $T = I_n + EC$ are the observer gains.

C. Problem definition

Since both the system under consideration and the observation schemes have been defined, we are ready to state the observation problem to be solved.
Definition 1 (Exponential observer): An observer $\hat{x}$ is an exponential observer of x with convergence rate β and overshoot Ω if the following inequality holds for all $t \geq 0$:

$$\|x(t) - \hat{x}(t)\| \leq \Omega \|x(0) - \hat{x}(0)\| e^{-\beta t} \tag{10}$$

Definition 2 (Continuous time robust observer): A state observer $\hat{x}$ of x with $\mathcal{L}_2$ gain μ is said to be robust with respect to perturbation v, if :

$$\int_0^\infty e(s)^T e(s) - \mu^2 v(s)^T v(s) ds < \infty \tag{11}$$

where $e(s)$ corresponds to the error of estimation $(\hat{x}(s) - x(s))$

III. Luenberger Observer synthesis for One Sided Nonlinear Lipschitz Systems without noise

In this section, we consider the system (5) without measurement noise and unknown input i.e.

$$\begin{aligned} \dot{x} &= Ax + Bu + D_f f(Hx) \\ y &= Cx \end{aligned} \tag{12}$$

The synthesis of the Luenberger type observer will be done considering the Lyapunov stability theorem [1], where the convergence of the error will be imposed by solving a set of LMIs. The detailed steps of the above mentioned objective are condensed in Theorem 1.
Theorem 1: If there exists matrices $P \succ 0$ and Q and an scalar α such that the following relations holds

$$\begin{aligned} \Psi + 2\rho\alpha H^T H + \beta P \prec 0 \\ D_f^T P = \alpha H \end{aligned} \tag{13}$$

where

$$\Psi = (A^TP + Q^TC^T + PA + QC)$$

then observer (8) is an exponential observer for the system (12) with convergence rate $\frac{\beta}{2}$ and overshoot $\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$. The observer gain is given by $K_L = P^{-1}Q$.

Proof: The dynamics of the error can be obtained by making the difference between the states and their estimates

$$\dot{e} = \dot{\hat{x}} - \dot{x}$$

$$\dot{e} = Ae + D_f \Delta f + K_L Ce \tag{14}$$

with $\Delta f = f(H\hat{x}) - f(Hx)$ and the error dynamics are defined for the Luenberger like observer (8). Using the quadratic Lyapunov function (LF) $V = e^T Pe$ and taking its derivative, it is possible to find the following expression

$$\Psi = (A^T P + K_L^T C^T P + PA + PK_L C)$$

$$\dot{V} = e^T \Psi e + \Delta f^T D_f^T Pe + e^T PD_f \Delta f \tag{15}$$

Using the OSL condition (7) and the assumption $D_f^T P = \alpha H$, is possible to construct the following expression for the LF derivative:

$$\dot{V} \leq e^T \Psi e + 2\rho\alpha e^T H^T He \tag{16}$$

$$\dot{V} \leq e^T \Psi e + 2\rho\alpha e^T H^T He + \beta e^T Pe \tag{17}$$

From (13), we can obtain the exponential convergence rate definition

$$\dot{V} \leq -\beta V,$$

therefore

$$e(t)^T Pe(t) \leq e(0)^T Pe(0)e^{-\beta t}.$$

that leads to

$$||e(t)|| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} e^{-\frac{\beta}{2}} ||e(0)||.$$

This conclude the proof of Theorem 1. ■

Remark 1: Note that conditions (13) include both linear inequality and equality constraints that are the constraints of a convex optimization problem. In the proposed framework, one might also choose to minimize trace of P in order to reduce the overshoot $\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$.

IV. Luenberger Observer synthesis for One Sided Nonlinear Lipschitz Systems considering noise

In this section, we follow a similar methodology to the one of Theorem 1 with respect to the structure of the model. However, the error dynamics are modified by considering the measurement noise affecting the system. This noise effect is intended to be mitigated by minimizing the $\mathcal{L}_2$ gain generated by the noise.

Theorem 2: If there exists matrices $P \succ 0$, Q and two scalars $\alpha, \tau_1 > 0$ such that the following relations holds

$$\begin{bmatrix} \mathcal{H}(A^T P + C^T Q^T) + \tau_1 I_n + 2\rho\alpha H^T H & QR \\ * & -\tau_1 \mu^2 I_n \end{bmatrix} \prec 0$$
$$D_f^T P = \alpha H \tag{18}$$

then observer (8) is a robust observer of system and the gain is given by $K_L = P^{-1}Q$.

Proof:

Considering the noise h in the output is possible to obtain the following expression for the error dynamics:

$$\dot{e} = Ae + D_f \Delta f + K_L Ce - K_L Rh \tag{19}$$

Considering the quadratic Lyapunov function $V = e^T Pe$, and the OSL condition with the assumption $D_f^T P = \alpha H$, its time derivative is

$$\dot{V} \leq e^T \Psi e + 2\rho\alpha e^T H^T He - h^T R^T K_L^T Pe - e^T PK_L Rh \tag{20}$$

Using the definition (11) is possible to rewrite the expression (20) as follows

$$\dot{V} \leq e^T \Psi e + 2\rho\alpha e^T H^T He - h^T R^T K_L^T Pe - e^T PK_L Rh + e^T e - \mu^2 h^T h - (e^T e - \mu^2 h^T h) \tag{21}$$

By means of (21), we can write our LMI condition

$$\Lambda = \mathcal{H}(A^T P + C^T Q^T) + \tau_1 I_n + 2\rho\alpha H^T H$$

$$\begin{bmatrix} e^T \\ h^T \end{bmatrix} \begin{bmatrix} \Lambda & QR \\ * & -\tau_1 \mu^2 I_n \end{bmatrix} \begin{bmatrix} e \\ h \end{bmatrix} \prec 0 \tag{22}$$

Looking at the equation (22), together with the definition (11) it could be inferred that the variable μ^2 is an $\mathcal{L}_2$ gain for the observer. With (22) the proof of Theorem 2 is finalized ■

Remark 2: According to (22), it is possible to minimize the impact of the noise on the system by replacing $\tau_1 \mu$ by a new LMI variable ϵ such that $\frac{\epsilon}{\tau_1}$ is minimized. This minimizes the $\mathcal{L}_2$ gain μ). While $\frac{\epsilon}{\tau_1}$ is not a convex function of the optimization variable, an approximate solution can be found by replacing the cost function by $\epsilon - \tau_1$ such that the resulting optimization problem is convex.

V. Unknown Input Observer synthesis for One-sided Nonlinear Lipschitz Systems without Noise

In this section, analogous to the Luenberger observer, we will synthesize convergence conditions for an observer with unknown inputs. Initially, the noise-free case (12) will be considered. Defining the error of estimation for the UIO, equation (23) is deduced.

$$e = \hat{x} - x = z - ECx - x = z - Tx \tag{23}$$
$$T = (I_n + EC)$$

Taking time derivative of (23), the error dynamics are found:

$$\dot{e} = \dot{z} - T\dot{x} \tag{24}$$
$$\dot{e} = Ne + TD_f\Delta f + (NT + GC - TA)x - TDv$$

Once the dynamics of the error are known, we proceed to do the synthesis of the UIO based on what was developed in [3]. In the following the gains of the observer are stated as expression of other matrix (that weill be used a matrix variable in LMEs and LMIs expressions):

$$\begin{aligned} E &= P^{-1}S \\ K &= P^{-1}Q \\ N &= TA - KC \\ G &= K + KCE - TAE \end{aligned} \tag{25}$$

Theorem 3: If there exists matrices $P \succ 0$, S , Q and scalars $\alpha \in \mathbb{R}, \beta > 0$ such that the following relations holds:

$$\mathcal{H}(A^TP + A^TC^TS^T + C^TQ^T) + 2\alpha\rho H^TH + \beta P \prec 0 \tag{26}$$

$$D_f^TP + D_f^TC^TS^T - \alpha H = 0 \tag{27}$$

$$PD + SCD = 0 \tag{28}$$

then observer (9) is an exponential observer of system and its gains are given by (25).

Proof: By assumption $PD + SCD = 0$, so since P is invertible $D + P^{-1}SCD = 0$, and $P^{-1}S = E$ therefore $(I + EC)D = 0$, this leads to $TD = 0$. Furthermore

$$\begin{aligned} NT + GC - TA &= N + NEC + GC - TA \\ &= -KC + NEC + GC \\ &= NEC + (KCE - TAE)C \\ &= (N + KC - TA)EC = 0 \end{aligned}$$

With the above considerations, we have a simplified error dynamics

$$\dot{e} = Ne + TD_f\Delta f \tag{29}$$

Considering the error dynamics (29), the quadratic Lyapunov function (LF) $V = e^TPe$ is proposed. Taking LF derivative, the following expression is found:

$$\dot{V} = e^T[N^TP + PN]e + \Delta f^TD_f^TT^TPe + e^TPTD_f\Delta f \tag{30}$$

The OSL condition (6) implies the following expression for the non-linearity Δf [2]:

$$\rho e^TH^THe \geq \Delta f^THe \tag{31}$$

With the assumption $D_f^TP + D_f^TC^TS^T = D_f^TT^TP = \alpha H$, it is possible to conclude that the equation (32) corresponds to an upper bound of the Lyapunov function

$$\dot{V} \leq e^T[N^TP + PN]e + 2\alpha\rho e^TH^THe + e^T\beta Pe \tag{32}$$

The additional expression βP was considered with the aim of improving the observer speed of convergence, since if $\dot{V} \leq -\beta V$ then $V \leq e^{-\beta t}V(0)$.
Therefore

$$\dot{V} \leq -\beta V \leftrightarrow N^TP + PN + 2\alpha\rho H^TH + \beta P \preceq 0.$$

Using the equality stated in (25),

$$\mathcal{H}(A^TP + A^TC^TS^T) + 2\alpha\rho H^TH + \beta P \prec 0 \tag{33}$$

With the exponential convergence rate definition (such as Theorem 1), the proof of Theorem 3 is finished.

■

VI. Unknown Input Observer Synthesis for One-sided Nonlinear Noisy Lipschitz Systems

Considering the same structure as (5) but adding the noise in the output $y = Cx + Rh$, the new error dynamics are found:

$$\dot{e} = Ne + (GR - NER)h + TD_f\Delta f - ER\dot{h} \tag{34}$$

From this equation, one can already observe that the dynamics of the observation error is influenced by both the noise h and its derivative $\dot{h}$. In order to reduce noise, the constraints $SR = QR = 0$ are added to Theorem 3 giving rise to Theorem 4.
Theorem 4: If there exists matrices $P \succ 0$, S and Q such that the following relations holds:

$$\mathcal{H}(A^TP + A^TC^TS^T + C^TQ^T) + 2\alpha\rho H^TH \prec 0 \tag{35}$$

$$D_f^TP + D_f^TC^TS^T - \alpha H = 0 \tag{36}$$

$$PD + SCD = 0 \tag{37}$$

$$SR = 0 \tag{38}$$

$$QR = 0 \tag{39}$$

Then observer (9) is an exponential observer of system and its gains are given by (25) .
Proof: Since $GR - NER$ must be 0, it implies that $SR = QR = 0$. For the LMI (35) and the matrix equality's (36)-(37) the methodology is the same as the case without noise (Theorem 3).

■

While the previous theorem give sufficient conditions to be insensitive to noise, the proposed conditions might be too demanding in practice. However, it is possible to mitigate the noise effect in the derivative while minimizing the impact of the noise (as measured by minimizing an $\mathcal{L}_2$ gain). Such method will be stated in the next theorem.
Theorem 5: If there exists matrices $P \succ 0$, S, Q and an scalar τ_1, such that the following relations hold:

$$\begin{bmatrix} \mathcal{H}(a_1) + 2\rho H^TH + \tau_1 I_n & QR \\ * & -\mu^2\tau_1 I_n \end{bmatrix} \prec 0 \tag{40}$$

$$D_f^TP + D_f^TC^TS^T - \alpha H = 0 \tag{41}$$

$$PD + SCD = 0 \tag{42}$$

$$SR = 0 \tag{43}$$

$$a_1 = A^T P + A^T C^T S^T + C^T Q^T \quad (44)$$

Then, observer (9) is a robust observer with $\mathcal{L}_2$ gain μ and its gains are given by (25) .

Proof: By means of the stability analysis based on the error dynamics which include noise (34), and considering the quadratic Lyapunov function $V = e^T Pe$ is possible to derive the expression (45) for $\dot{V}$, assuming that $\eta^T = [e^T h^T]$ and using the definition of $\mathcal{L}_2$ gain, the LF derivative is:

$$\eta^T \Lambda \eta = e^T [N^T P + PN]e + 2\rho e^T H^T He + b_1 \quad (45)$$

$$b_1 = h^T R^T G^T Pe + e^T PGRh + \tau_1(e^T e - \mu^2 hh) \quad (46)$$

$$\dot{V} \le \eta^T \Lambda \eta - \tau_1(e^T e - \mu^2 hh) \quad (47)$$

$$\begin{bmatrix} e^T \\ h^T \end{bmatrix} \begin{bmatrix} \mathcal{H}(a_1) + 2\rho H^T H + \tau_1 I_n & QR \\ * & -\mu^2 \tau_1 I_n \end{bmatrix} \begin{bmatrix} e \\ h \end{bmatrix} \prec 0$$

$$a_1 = A^T P + A^T C^T S^T + C^T Q^T \quad (48)$$

Looking at the equation (47), together with the definition could be inferred that the variable μ^2 is an $\mathcal{L}_2$ gain for the observer. Thus, we can proceed in the same way as Theorem 2 and the proof is finalized. ■

VII. Illustrative Example

In this section, an example will be proposed to test the performance of the observers against each of the design criteria presented in the methodological part of the paper. The technique selected to measure performance is the absolute integral of the error IAE.

$$IAE = \int_0^t \|e\| dt \quad (49)$$

where e is the error of estimation. Considering the example given in [17]

$$\begin{aligned} \dot{x} &= Ax + Bu + D_f f(Hx) + Dv \\ y &= Cx \end{aligned} \quad (50)$$

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad (51)$$

$$D = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T H = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} 0 & 0 & -x_3^{1/3} \end{bmatrix} \quad (52)$$

$$D_f = H^T \quad (53)$$

the methodology proposed in [15] allow us to analyze the characteristics of dynamical system (51)-(53). Initially, the first step is to obtain the Jacobian of then nonlinear part for system (51)-(53)

$$J_f(f(Hx)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{3x_3^{2/3}} \end{bmatrix} \quad (54)$$

Then, according to [15], the following optimization problem must be solved, in order to obtain the upper γ_u and lower γ_l Lipschitz constants for (51)-(53).

$$\gamma_u = \max_{(x,u)} \quad \lambda_{max}(\tfrac{1}{2}(J_f + J_f^T)) \quad (55)$$

$$\gamma_l = \min_{(x,u)} \quad \lambda_{min}(\tfrac{1}{2}(J_f + J_f^T)) \quad (56)$$

computing the optimization problems (55)-(56) one can see that $\gamma_u = 0$ and $\gamma_l = -\infty$. Since the lower Lipschitz constant is unbounded one can conclude that the system is OSL due the fact that it decreases faster than a linear system. Then, calculating the OSL constant according to [15] its value is equal to $\gamma_{OSL} = \rho = 0$.

By implementing the observer (8) in the system (51)-(53), the result presented in Figure 1 is obtained.

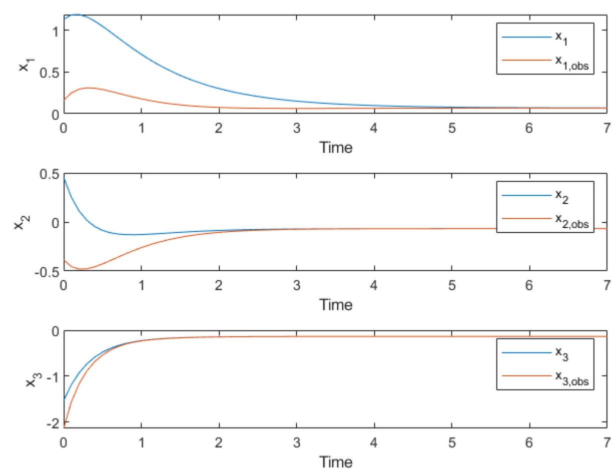

Fig. 1. Implementation of Luenberger observer, $IAE = 3.11$

Looking at this figure, it can be inferred that the estimation error is asymptotically stable, and has convergence in finite time. On the other hand, results in Figure 2 suggests that by minimizing the gain $\mathcal{L}_2$, the observer gains is able to filter the noise allowing the estimation to reduce the uncertainty related to the sensor.

To evaluate the UIO formulations, the considered unknown input is $v = 0.5sin(5t)$. The first simulation case shown in Fig. 3 does not consider the noise in the output, i.e. $R = 0$.

For the previous example, the improved speed of convergence constant is $\beta = 10$. Analyzing the results obtained in Figure 3, it can be inferred that the observer performance is not affected by the unknown input, so the estimator design specifications are satisfactorily achieved. Additionally, a fast convergence speed is presented since the signal of the estimated variable manages to efficiently reconstruct the output $y = x_3$ in finite time.

Analogously, the performance of the observer is tested with an output contaminated with Gaussian noise as shown in Fig 4.

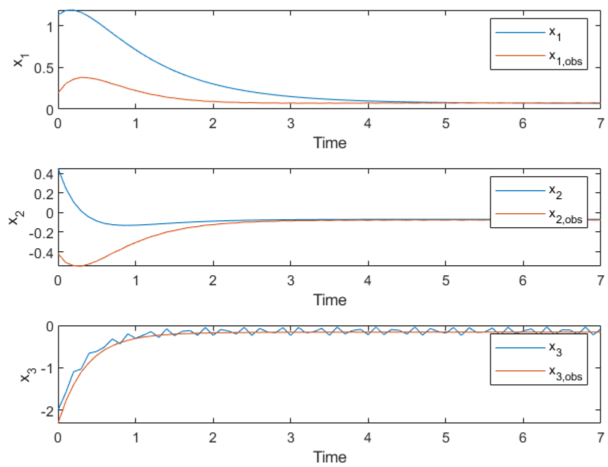

Fig. 2. Luenberger observer performance considering noise, $IAE = 1.8$

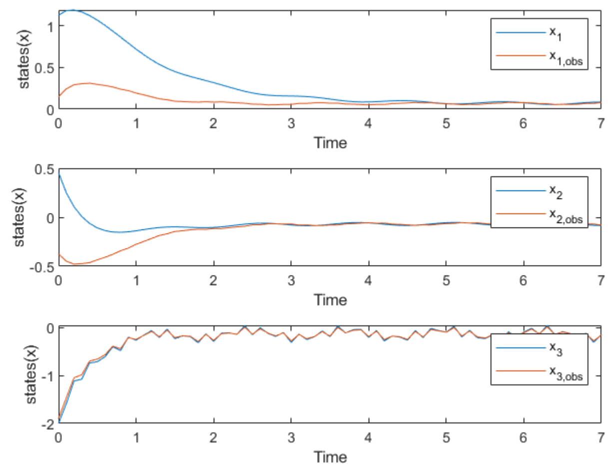

Fig. 4. Unknown Input Observer considering the measurement contaminated with noise, $R = 1$, $IAE = 5.93$

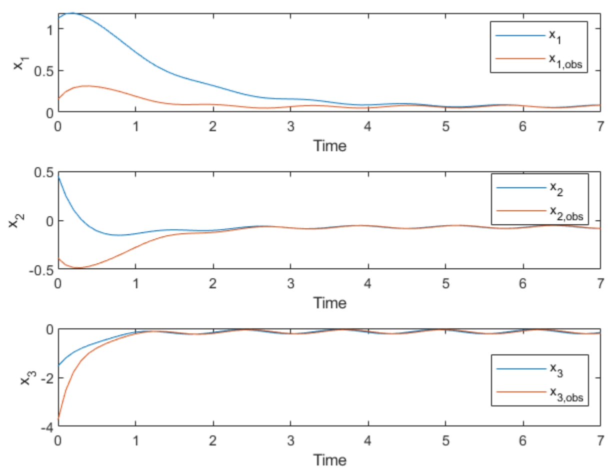

Fig. 3. Unknown Input Observer considering the measurement free of noise , $R = 0$. $IAE = 7.02$

Fig 4 shows that the observer manages to effectively filter out the noisy measurement, as the noise does not propagate into the other state variables. Another characteristic that can be seen is that a fast convergence is achieved when following the measured variable, and the stability of the observer is not affected by the noise. Observing the performance of each of the observers with respect to the established design conditions, it could be concluded that minimizing the $\mathcal{L}_2$ gain improves the performance of the estimator, since higher convergence speeds are presented, noise is filtered out and the IAE is lower.

A. Discussion

From the results previously obtained, it can be concluded that the observers presents a satisfactory performance with respect to the design conditions initially established.
The result shown in Fig 3,1, suggests that the LMI conditions are satisfactorily fulfilled, achieving the convergence of the observer. From Figure 4,2, it can be observed that the $\mathcal{L}_2$ gain generated by the noise is minimized, leading to the observer error dynamics converge efficiently to the origin.

VIII. Conclusions

In this paper, the problem of designing observers for one-sided Lipschitz nonlinear systems considering noise and unknown inputs have been addressed considering two observer structures: Luenberger Observer (LO) and Unknown Input Observer (UIO). The observer synthesis procedures has been formulated as convex optimization problems in the LMI framework. An illustrative example has been used to illustrate how the proposed observer design approaches are used to design the LO and UIO schemes. The obtained simulation results are presented to assess the proposed methods. The results shown in the present paper could be considered novel, because the observer design only includes the OSL condition, while other works with the same objective, make the assumption that the nonlinearity is QIB or Lipschitz.

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