VALUATIONS WITH AN INFINITE LIMIT-DEPTH

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Abstract. We construct a field K and a valuation-algebraic valuation on $K[x]$, whose underlying Maclane–Vaquié chain consists of an infinite (countable) number of limit augmentations.

Introduction

Let (K, v) be a valued field. In a pioneering work, Maclane studied the extensions of the valuation v to the polynomial ring $K[x]$ in one indeterminate, in the case v discrete of rank one [11]. He proved that all extensions of v to $K[x]$ can be obtained as a kind of limit of chains of augmented valuations:

$$(1) \qquad \mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \xrightarrow{\phi_2,\gamma_2} \cdots \longrightarrow \mu_{n-1} \xrightarrow{\phi_n,\gamma_n} \mu_n \longrightarrow \cdots \longrightarrow \mu$$

involving the choice of certain *key polynomials* $\phi_n \in K[x]$ and elements γ_n belonging to some extension of the value group of v.

These chains of valuations contain relevant information on μ and play a crucial role in the resolution of many arithmetic-geometric tasks in number fields and function fields of curves [3, 4].

For valuations of arbitrary rank, different approaches to this problem were developed by Alexandru-Popescu-Zaharescu [1], Kuhlmann [9], Herrera-Mahboub-Olalla-Spivakovsky [5, 6] and Vaquié [16, 18].

In this general context, *limit augmentations* and the corresponding *limit key polynomials* appear as a new feature. In the henselian case, limit augmentations are linked with the existence of *defect* in the extension μ/v [17]. Thus, they are an obstacle for local uniformization in positive characteristic.

A chain as in (1) is said to be a *MacLane–Vaquié chain* if it is constructed as a mixture of ordinary and limit augmentations, and satisfies certain additional technical condition (see Section 1.5). In this case, the intermediate valuations μ_n are essentially unique and contain intrinsic information about the valuation μ [13, Thm. 4.7].

In particular, the number of limit augmentations of any MacLane–Vaquié chain of μ is an intrinsic datum of μ, which is called the *limit-depth* of μ.

In this paper, we exhibit an example of a valuation with an infinite limit-depth, inspired in a construction by Kuhlmann [10] and Blaszczok [2] of infinite towers of Artin-Schreier extensions with defect.

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1. Maclane–Vaquié chains of valuations on $K[x]$

In this section we recall some well-known results on valuations on a polynomial ring, mainly extracted from the surveys [12] and [13].

Let (K, v) be a valued field, with valuation ring $\mathcal{O}_v$ and residue class field k. Let $\Gamma = v(K^*)$ be the value group and denote by $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ the divisible hull of Γ. In the sequel, we write $\Gamma_{\mathbb{Q}}\infty$ instead of $\Gamma_{\mathbb{Q}} \cup \{\infty\}$.

Consider the set $\mathcal{T}$ of all $\Gamma_{\mathbb{Q}}$-valued extensions of v to the field $K(x)$ of rational functions in one indeterminate. That is, an element $\mu \in \mathcal{T}$ is a valuation on $K[x]$,

$$\mu\colon K[x] \longrightarrow \Gamma_{\mathbb{Q}}\infty,$$

such that $\mu_{|K} = v$ and $\mu^{-1}(\infty) = \{0\}$. Let $\Gamma_\mu = \mu(K(x)^*)$ be the value group and k_μ the residue field.

This set $\mathcal{T}$ admits a partial ordering. For $\mu, \nu \in \mathcal{T}$ we say that $\mu \le \nu$ if

$$\mu(f) \le \nu(f), \quad \forall\, f \in K[x].$$

As usual, we write $\mu < \nu$ to indicate that $\mu \le \nu$ and $\mu \ne \nu$.

This poset $\mathcal{T}$ has the structure of a tree; that is, all intervals

$$(-\infty, \mu\,] := \{\rho \in \mathcal{T} \mid \rho \le \mu\}$$

are totally ordered [13, Thm. 2.4].

A node $\mu \in \mathcal{T}$ is a *leaf* if it is a maximal element with respect to the ordering $\le$. Otherwise, we say that μ is an *inner node*.

The leaves of $\mathcal{T}$ are the *valuation-algebraic* valuations in Kuhlmann's terminology [9]. The inner nodes are the *residually transcendental* valuations, characterized by the fact that the extension k_μ/k is transcendental. In this case, its transcendence degree is necessarily equal to one [9].

1.1. **Graded algebra and key polynomials.** Take any $\mu \in \mathcal{T}$. For all $\alpha \in \Gamma_\mu$, consider the $\mathcal{O}_v$-modules:

$$\mathcal{P}_\alpha = \{g \in K[x] \mid \mu(g) \ge \alpha\} \supset \mathcal{P}_\alpha^+ = \{g \in K[x] \mid \mu(g) > \alpha\}.$$

The *graded algebra of* μ is the integral domain:

$$\mathcal{G}_\mu = \bigoplus\nolimits_{\alpha \in \Gamma_\mu} \mathcal{P}_\alpha / \mathcal{P}_\alpha^+.$$

There is an *initial coefficient* mapping $\operatorname{in}_\mu\colon K[x] \to \mathcal{G}_\mu$, given by $\operatorname{in}_\mu 0 = 0$ and

$$\operatorname{in}_\mu g = g + \mathcal{P}_{\mu(g)}^+ \quad \text{for all nonzero } g \in K[x].$$

If $\mu < \nu$ in $\mathcal{T}$, there is an homomorphism of graded algebras $\mathcal{G}_\mu \to \mathcal{G}_\nu$, defined by

$$\operatorname{in}_\mu f \longmapsto \begin{cases} \operatorname{in}_\nu f, & \text{if } \mu(f) = \nu(f), \\ 0, & \text{if } \mu(f) < \nu(f). \end{cases}$$

The following definitions translate properties of the action of μ on $K[x]$ into algebraic relationships in the graded algebra $\mathcal{G}_\mu$.

Definition. Let g, $h \in K[x]$.

We say that g is *μ-divisible* by h, and we write $h \mid_\mu g$, if $\operatorname{in}_\mu h \mid \operatorname{in}_\mu g$ in $\mathcal{G}_\mu$.

We say that g is *μ-irreducible* if $\operatorname{in}_\mu g$ is a prime element; that is, the homogeneous principal ideal of $\mathcal{G}_\mu$ generated by $\operatorname{in}_\mu g$ is a prime ideal.

We say that g is *μ-minimal* if $g \nmid_\mu f$ for all nonzero $f \in K[x]$ with $\deg(f) < \deg(g)$.

Let us recall a well-known characterization of μ-minimality [12, Prop. 2.3].

Lemma 1.1. *A polynomial $g \in K[x] \setminus K$ is μ-minimal if and only if μ acts as follows on g-expansions:*

$$f = \sum\nolimits_{0 \le n} a_n g^n \qquad \Longrightarrow \qquad \mu(f) = \min\left\{\mu(a_n g^n) \mid 0 \le n\right\},$$

where $a_n \in K[x]$ are polynomials of degree $\deg(a_n) < \deg(g)$, for all $n \ge 0$.

Definition. A *(Maclane-Vaquié) key polynomial* for μ is a monic polynomial in $K[x]$ which is simultaneously μ-minimal and μ-irreducible. The set of key polynomials for μ is denoted $\mathrm{KP}(\mu)$.

All $\phi \in \mathrm{KP}(\mu)$ are irreducible in $K[x]$. For all $\phi \in \mathrm{KP}(\mu)$ let $[\,\phi\,]_\mu \subset \mathrm{KP}(\mu)$ be the subset of all key polynomials $\varphi \in \mathrm{KP}(\mu)$ such that $\mathrm{in}_\mu \varphi = \mathrm{in}_\mu \phi$.

Lemma 1.2. [16, Thm. 1.15] *Let $\mu < \nu$ be two nodes in $\mathcal{T}$. Let $\mathbf{t}(\mu,\nu)$ be the set of monic polynomials $\phi \in K[x]$ of minimal degree satisfying $\mu(\phi) < \nu(\phi)$. Then, $\mathbf{t}(\mu,\nu) \subset \mathrm{KP}(\mu)$ and $\mathbf{t}(\mu,\nu) = [\,\phi\,]_\mu$ for all $\phi \in \mathbf{t}(\mu,\nu)$.*

Moreover, for all $f \in K[x]$, the equality $\mu(f) = \nu(f)$ holds if and only if $\phi \nmid_\mu f$.

The existence of key polynomials characterizes the inner nodes of $\mathcal{T}$.

Theorem 1.3. *A node $\mu \in \mathcal{T}$ is a leaf if and only if $\mathrm{KP}(\mu) = \emptyset$.*

Definition. The *degree* $\deg(\mu)$ of an inner node $\mu \in \mathcal{T}$ is defined as the minimal degree of a key polynomial for μ.

1.2. **Depth zero valuations.** For all $a \in K$, $\gamma \in \Gamma_\mathbb{Q}$, consider the *depth-zero* valuation

$$\mu = \omega_{a,\delta} = [v;\, x-a, \gamma] \in \mathcal{T},$$

defined in terms of $(x-a)$-expansions as

$$f = \sum\nolimits_{0 \le n} a_n (x-a)^n \implies \mu(f) = \min\{v(a_n) + n\gamma \mid 0 \le n\}.$$

Note that $\mu(x-a) = \gamma$. Clearly, $x-a$ is a key polynomial for μ of minimal degree and $\Gamma_\mu = \langle\, \Gamma, \gamma \,\rangle$. In particular, μ is an inner node of $\mathcal{T}$. One checks easily that

$$\omega_{a,\delta} \le \omega_{b,\epsilon} \iff v(a-b) \ge \delta \ \text{ and } \ \epsilon \ge \delta. \tag{2}$$

1.3. **Ordinary augmentation of valuations.** Let $\mu \in \mathcal{T}$ be an inner node. For all $\phi \in \mathrm{KP}(\mu)$ and all $\gamma \in \Gamma_\mathbb{Q}$ such that $\mu(\phi) < \gamma$, we may construct the *ordinary* augmented valuation

$$\mu' = [\mu;\, \phi, \gamma] \in \mathcal{T},$$

defined in terms of ϕ-expansions as

$$f = \sum\nolimits_{0 \le n} a_n \phi^n \implies \mu'(f) = \min\{\mu(a_n) + n\gamma \mid 0 \le n\}.$$

Note that $\mu'(\phi) = \gamma$, $\mu < \mu'$ and $\mathbf{t}(\mu,\mu') = [\phi]_\mu$.

By [12, Cor. 7.3], ϕ is a key polynomial for μ' of minimal degree. In particular, μ' is an inner node of $\mathcal{T}$ too.

1.4. **Limit augmentation of valuations.** Consider a totally ordered family of residually transcendental valuations,

$$\mathcal{C} = (\rho_i)_{i\in A} \subset \mathcal{T}.$$

not containing a maximal element. We assume that $\mathcal{C}$ is parametrized by a totally ordered set A of indices such that the mapping $A \to \mathcal{C}$ determined by $i \mapsto \rho_i$ is an isomorphism of totally ordered sets.

If $\deg(\rho_i)$ is stable for all sufficiently large $i \in A$, we say that $\mathcal{C}$ has *stable degree*, and we denote this stable degree by $\deg(\mathcal{C})$.

We say that $f \in K[x]$ is $\mathcal{C}$*-stable* if for some index $i \in A$, it satisfies

$$\rho_i(f) = \rho_j(f), \qquad \text{for all } \ j > i.$$

Lemma 1.4. *A nonzero $f \in K[x]$ is $\mathcal{C}$-stable if and only if* $\operatorname{in}_{\rho_i} f$ *is a unit in* $\mathcal{G}_{\rho_i}$ *for some* $i \in A$.

Proof. Suppose that $\operatorname{in}_{\rho_i} f$ is a unit in $\mathcal{G}_{\rho_i}$ for some $i \in A$. Take any $j > i$ in A, and let $\mathbf{t}(\rho_i, \rho_j) = [\varphi]_{\rho_i}$. By Lemma 1.2, $\varphi \in \operatorname{KP}(\rho_i)$, so that $\operatorname{in}_{\rho_i} \varphi$ is a prime element and $\varphi \nmid_{\rho_i} f$. This implies $\rho_i(f) = \rho_j(f)$, again by Lemma 1.2.

Conversely, if f is $\mathcal{C}$-stable, there exists $i_0 \in A$ such that $\rho_{i_0}(f) = \rho_i(f)$ for all $i > i_0$. Hence, $\operatorname{in}_{\rho_i} f$ is the image of $\operatorname{in}_{\rho_{i_0}} f$ under the canonical homomorphism $\mathcal{G}_{\rho_{i_0}} \to \mathcal{G}_{\rho_i}$. By [13, Cor. 2.6], $\operatorname{in}_{\rho_i} f$ is a unit in $\mathcal{G}_{\rho_i}$. □

We obtain a *stability function* $\rho_{\mathcal{C}}$, defined on the set of all $\mathcal{C}$-stable polynomials by

$$\rho_{\mathcal{C}}(f) = \max\{\rho_i(f) \mid i \in A\}.$$

If all polynomials in $K[x]$ are $\mathcal{C}$-stable, then $\rho_{\mathcal{C}}$ is a valuation. Indeed, for any given $f, g \in K[x]$ there exists $i \in A$ large enough so that $\rho_{\mathcal{C}}$ coincides with the valuation ρ_i on f, g, $f+g$ and fg. Thus, $\rho_{\mathcal{C}}(f+g) \ge \min\{\rho_{\mathcal{C}}(f), \rho_{\mathcal{C}}(g)\}$ and $\rho_{\mathcal{C}}(fg) = \rho_{\mathcal{C}}(f)+\rho_{\mathcal{C}}(g)$.

Definition. If all polynomials in $K[x]$ are $\mathcal{C}$-stable, we say that the valuation $\rho_{\mathcal{C}}$ is the *stable limit* of $\mathcal{C}$ and we write

$$\rho_{\mathcal{C}} = \lim(\mathcal{C}) = \lim_{i\in A} \rho_i.$$

If $\mathcal{C}$ has no stable limit, let $\operatorname{KP}_\infty(\mathcal{C})$ be the set of all monic $\mathcal{C}$-unstable polynomials of minimal degree. The elements in $\operatorname{KP}_\infty(\mathcal{C})$ are said to be *limit key polynomials* for $\mathcal{C}$. Since the product of stable polynomials is stable, all limit key polynomials are irreducible in $K[x]$.

Definition. We say that $\mathcal{C}$ is an *essential continuous family* of valuations if it has stable degree and it admits limit key polynomials whose degree is greater than $\deg(\mathcal{C})$.

For all limit key polynomials $\phi \in \operatorname{KP}_\infty(\mathcal{C})$, and all $\gamma \in \Gamma_{\mathbb{Q}}$ such that $\rho_i(\phi) < \gamma$ for all $i \in A$, we may construct the *limit augmented* valuation

$$\mu = [\mathcal{C};\, \phi, \gamma] \in \mathcal{T}$$

defined in terms of ϕ-expansions as:

$$f = \sum\nolimits_{0\le n} a_n\phi^n \implies \mu(f) = \min\{\rho_{\mathcal{C}}(a_n) + n\gamma \mid 0 \le n\}.$$

Since $\deg(a_n) < \deg(\phi)$, all coefficients a_n are $\mathcal{C}$-stable. Note that $\mu(\phi) = \gamma$ and $\rho_i < \mu$ for all $i \in A$. By [12, Cor. 7.13], ϕ is a key polynomial for μ of minimal degree, so that μ is an inner node of $\mathcal{T}$.

1.5. **Maclane–Vaquié chains.** Consider a countable chain of valuations in $\mathcal{T}$:

$$v \xrightarrow{\phi_0,\gamma_0} \mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \xrightarrow{\phi_2,\gamma_2} \cdots \xrightarrow{\phi_n,\gamma_n} \mu_n \longrightarrow \cdots \tag{3}$$

in which $\phi_0 \in K[x]$ is a monic polynomial of degree one, $\mu_0 = [v;\, \phi_0, \gamma_0]$ is a depth zero valuation, and each other node is an augmentation of the previous node, of one of the two types:

Ordinary augmentation: $\mu_{n+1} = [\mu_n;\, \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \mathrm{KP}(\mu_n)$.

Limit augmentation: $\mu_{n+1} = [\mathcal{C}_n;\, \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \mathrm{KP}_\infty(\mathcal{C}_n)$, where $\mathcal{C}_n$ is an essential continuous family whose first valuation is μ_n.

Therefore, for all n such that $\gamma_n < \infty$, the polynomial ϕ_n is a key polynomial for μ_n of minimal degree.

Definition. A chain of mixed augmentations as in (3) is said to be a *MacLane–Vaquié (MLV) chain* if every augmentation step satisfies:

- If $\mu_n \to \mu_{n+1}$ is ordinary, then $\deg(\mu_n) < \deg(\mu_{n+1})$.
- If $\mu_n \to \mu_{n+1}$ is limit, then $\deg(\mu_n) = \deg(\mathcal{C}_n)$ and $\phi_n \notin \mathbf{t}(\mu_n, \mu_{n+1})$.

In this case, we have $\phi_n \notin \mathbf{t}(\mu_n, \mu_{n+1})$ for all n. As shown in [13, Sec. 4.1], this implies that $\mu(\phi_n) = \gamma_n$ and $\Gamma_{\mu_n} = \left\langle \Gamma_{\mu_{n-1}}, \gamma_n \right\rangle$ for all n.

The following theorem is due to Maclane, for the discrete rank-one case [11], and Vaquié for the general case [16]. Another proof may be found in [13, Thm. 4.3].

Theorem 1.5. *Every node $\mu \in \mathcal{T}$ falls in one, and only one, of the following cases.*

(a) It is the last valuation of a finite MLV chain.

$$\mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \xrightarrow{\phi_2,\gamma_2} \cdots \longrightarrow \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r = \mu.$$

(b) It is the stable limit of a continuous family $\mathcal{C} = (\rho_i)_{i\in A}$ of augmentations whose first valuation μ_r falls in case (a):

$$\mu_0 \xrightarrow{\phi_1,\gamma_1} \mu_1 \xrightarrow{\phi_2,\gamma_2} \cdots \longrightarrow \mu_{r-1} \xrightarrow{\phi_r,\gamma_r} \mu_r \xrightarrow{(\rho_i)_{i\in A}} \rho_{\mathcal{A}} = \mu.$$

Moreover, we may assume that $\deg(\mu_r) = \deg\, \mathbf{t}(\mu_r, \nu)$ *and* $\phi_r \notin \mathbf{t}(\mu_r, \nu)$.

(c) It is the stable limit, $\mu = \lim_{n\to\infty} \mu_n$, of an infinite MLV chain.

The main advantage of MLV chains is that their nodes are essentially unique, so that we may read in them several data intrinsically associated to the valuation μ.

For instance, the sequence $(\deg(\mu_n))_{n\ge 0}$ and the character "ordinary" or "limit" of each augmentation step $\mu_n \to \mu_{n+1}$, are intrinsic features of μ [13, Sec. 4.3].

Thus, we may define order preserving functions

$$\mathrm{depth}, \text{lim-depth}\colon \mathcal{T} \longrightarrow \mathbb{N}\infty,$$

where $\mathrm{depth}(\mu)$ is the length of the MLV chain underlying μ, and $\text{lim-depth}(\mu)$ counts the number of limit augmentations in this MLV chain.

It is easy to construct examples of valuations on $K[x]$ of infinite depth. In the next section, we show the existence of valuations with an infinite limit-depth too. Their construction is much more involved.

2. A valuation with an infinite limit-depth

In this section, we exhibit an example of a valuation with an infinite limit-depth, based on explicit constructions by Kuhlmann, of infinite towers of field extensions with defect [10].

For a prime number p, let $\mathbb{F}$ be an algebraic closure of the prime field $\mathbb{F}_p$. For an indeterminate t, consider the fields of Laurent series, Newton-Puiseux series and Hahn series in t, respectively:

$$\mathbb{F}((t)) \subset K = \bigcup_{N\in\mathbb{N}} \mathbb{F}((t^{1/N})) \subset H = \mathbb{F}((t^{\mathbb{Q}})).$$

For a generalized power series $s = \sum_{q\in\mathbb{Q}} a_q t^q$, its support is a subset of $\mathbb{Q}$:

$$\operatorname{supp}(s) = \{q \in \mathbb{Q} \mid a_q \neq 0\}.$$

The Hahn field H consists of all generalized power series with well-ordered support. The Newton-Puiseux field K contains all series whose support is included in $\frac{1}{N}\mathbb{Z}_{\geq m}$ for some $N \in \mathbb{N}$, $m \in \mathbb{Z}$.

From now on, we denote by $\operatorname{Irr}_K(b)$ the minimal polynomial over K of any $b \in \overline{K}$.

On these three fields we may consider the valuation v defined as

$$v(s) = \min(\operatorname{supp}(s)),$$

which clearly has,

$$v\left(\mathbb{F}((t))^*\right) = \mathbb{Z}, \qquad v(K^*) = v(H^*) = \mathbb{Q}.$$

The valued field $(\mathbb{F}((t)), v)$ is henselian, because it is the completion of the discrete rank-one valued field $(\mathbb{F}(t), v)$. Since the extension $\mathbb{F}((t)) \subset K$ is algebraic, the valued field (K, v) is henselian too.

The Hahn field H is algebraically closed. Thus, it contains an algebraic closure $\overline{K}$ of K. The algebraic generalized power series have been described by Kedlaya [7, 8]. Let us recall [7, Lem. 3], which is essential for our purposes.

Lemma 2.1. *If $s \in H$ is algebraic over K, then it is contained in a tower of Artin-Schreier extensions of K. In particular, s is separable over K and $\deg_K s$ is a power of p.*

Any $s \in H$ determines a valuation on $H[x]$ extending v:

$$v_s \colon H[x] \longrightarrow \mathbb{Q}\infty, \qquad g \longmapsto v_s(g) = v(g(s)).$$

We are interested in the valuation on $K[x]$ obtained by restriction of v_s, which we still denote by the same symbol v_s. If s is algebraic over K and $f = \operatorname{Irr}_K(s) \in K[x]$, we have $v_s(f) = \infty$. Hence, v_s cannot be extended to a valuation on $K(x)$.

On the other hand, suppose that $s = \sum_{q\in\mathbb{Q}} a_q t^q \in H$ is transcendental over K and all its truncations

$$s_r = \sum_{q<r} a_q t^q, \qquad r \in \mathbb{R},$$

are algebraic over K and have a bounded degree over K. Then, it is an easy exercise to check that v_s falls in case (b) of Theorem 1.5.

Therefore, our example of a valuation with infinite limit-depth must be given by a transcendental $s \in H$, all whose truncations are algebraic over K and have unbounded degree over K. In this case, v_s will necessarily fall in case (c) of Theorem 1.5. We want to find an example such that, moreover, all steps in the MLV chain of v_s are limit augmentations.

By Lemma 2.1, the truncations of s must belong to some tower of Artin-Schreier extensions of K. Let us use a concrete tower constructed by Kuhlmann [10, Ex. 3.14].

2.1. **A tower of Artin-Schreier extensions of K.** Let $\mathrm{AS}(g) = g^p - g$ be the Artin-Screier operator on $K[x]$. It is $\mathbb{F}_p$-linear and has kernel $\mathbb{F}_p$.

Let us start with the classical Abhyankar's example

$$s_0 = \sum_{i\geq 1} t^{-1/p^i} \in H,$$

which is a root of the polynomial $\varphi_0 = \mathrm{AS}(x) - t^{-1} \in K[x]$. Since the denominators of the support of s_0 are unbounded, we have $s_0 \notin K$. Since the roots of φ_0 are $s_0 + \ell$, for ℓ running on $\mathbb{F}_p$, the polynomial φ_0 has no roots in K. Hence, φ_0 is irreducible in $K[x]$, because all irreducible polynomials in $K[x]$ have degree a power of p.

Now, we iterate this construction to obtain a tower of Artin-Schreier extensions

$$K \subsetneq K(s_0) \subsetneq K(s_1) \subsetneq \cdots \subsetneq K(s_n) \subsetneq \cdots$$

where $s_n \in H$ is taken to be a root of $\varphi_n = \mathrm{AS}(x) - s_{n-1}$. The above argument shows that φ_n is irreducible in $K(s_{n-1})$ as long as $s_n \notin K(s_{n-1})$, which is easy to check.

More generally, we could have taken s_0 to be the root of $\mathrm{AS}(x)-a$, for any $a \in \mathbb{F}((t))$ such that $v(a) < 0$ and $p \nmid v(a)$. However, for Abhyankar's example we can give more explicit formulas for the s_n. Indeed, from the algebraic relationship $\mathrm{AS}(s_n) = s_{n-1}$ we may deduce a concrete choice for all s_n:

$$s_n = \sum_{j\geq n} \binom{j}{n} t^{-1/p^{j+1}}, \qquad \text{for all } n \geq 0,$$

which follows from the well-known identity

$$\binom{j+1}{n+1} = \binom{j}{n+1} + \binom{j}{n}, \qquad \text{for all } j \geq n.$$

In particular,

$$\deg_K s_n = p^{n+1}, \qquad v(s_n) = -1/p^{n+1}, \qquad \text{for all } n \geq 0.$$

For all $n \geq 0$, we have $\mathrm{Irr}_K(s_n) = \mathrm{AS}^{n+1}(x) - t^{-1}$, and the set of roots of this polynomial is

$$Z\left(\mathrm{Irr}_K(s_n)\right) = s_n + \mathrm{Ker}(\mathrm{AS}^{n+1}) \subset s_n + \mathbb{F}. \tag{4}$$

In particular, the support of all these conjugates of s_n is contained in $(-1, 0]$, and Krasner's constant of s_n is zero:

$$\Delta(s_n) = \max\left\{v(s_n - \sigma(s_n)) \mid \sigma \in \mathrm{Gal}(\overline{K}/K),\ \sigma(s_n) \neq s_n\right\} = 0. \tag{5}$$

We are ready to define our transcendental $s \in H$ as:

$$s = \sum_{n \geq 0} t^n s_n.$$

Let us introduce some useful notation to deal with the support of s and its truncations. Consider the well-ordered set

$$S = \left\{ (n,i) \in \mathbb{Z}^2_{\mathrm{lex}} \mid \ 0 \leq n \leq i, \quad p \nmid \binom{i}{n} \right\}.$$

The support of s is the image of the following order-preserving embedding

$$\delta \colon S \hookrightarrow \mathbb{Q}, \qquad (n,i) \longmapsto \delta(n,i) = n - \frac{1}{p^{i+1}}.$$

The limit elements in S are (n,n) for $n \geq 0$. These elements have no immediate predecessor in S. On the other hand, all elements in S have an immediate successor:

$$(n,i) \ \rightsquigarrow \ (n, i+m),$$

where m is the least natural number such that $p \nmid \binom{i+m}{n}$.

For all $(n,i) \in S$ we consider the truncations of s determined by the rational numbers $\delta(n,i)$:

$$s_{n,i} := s_{\delta(n,i)} = \sum_{m=0}^{n-1} t^m s_m + t^n \sum_{j=n}^{i-1} \binom{j}{n} t^{-1/p^{j+1}}.$$

For the limit indices $(n,n) \in S$ the truncations are:

$$s_{n,n} = \sum_{m=0}^{n-1} t^m s_m.$$

Since $(0,0) = \min(S)$, the truncation $s_{0,0} = 0$ is defined by an empty sum.

All truncations of s are algebraic over K. Their degree is

$$\deg_K s_{n,i} = p^n, \qquad \text{for all } (n,i) \in S,$$

because s_{n-1} has degree p^n, and all other summands have strictly smaller degree. For instance, the "tail" $t^n \sum_{j=n}^{i-1} \binom{j}{n} t^{-1/p^{j+1}}$ belongs to K.

The unboundedness of the degrees of the truncations of s is not sufficient to guarantee that s is transcendental over K. To this end, we must analyze some more properties of these truncations.

For any pair $(a,\delta) \in \overline{K} \times \mathbb{Q}$, consider the ultrametric ball

$$B = B_\delta(a) = \{ b \in \overline{K} \mid v(b-a) \geq \delta \}.$$

We define the *degree* of such a ball over K as

$$\deg_K B = \min\{ \deg_K b \mid b \in B \}.$$

Lemma 2.2. *For all $n \geq 1$, we have* $\deg_K B_{n-1}(s_{n,n}) = p^n$.

Proof. Denote $B = B_{n-1}(s_{n,n})$. From the computation in (5), we deduce that Krasner's constant of $s_{n,n}$ is $\Delta(s_{n,n}) = n-1$. Any $u \in B$ may be written as

$$u = s_{n,n} + \ell\, t^{n-1} + b, \qquad \ell \in \mathbb{F}, \quad b \in \overline{K}, \quad v(b) > n-1.$$

Let $z = s_{n,n} + \ell\, t^{n-1}$. Since $\ell\, t^{n-1}$ belongs to K, we have

$$\deg_K z = p^n, \qquad \Delta(z) = n-1.$$

Since $v(u-z) > \Delta(z)$, we have $K(z) \subset K(u)$ by Krasner's lemma. Hence, $\deg_K u \geq p^n$. Since B contains elements of degree p^n, we conclude that $\deg_K B = p^n$. □

Corollary 2.3. *The element $s \in H$ is transcendental over K.*

Proof. If s were algebraic over K, it would belong to $B_{n-1}(s_{n,n})$ for all n. This is impossible, because $\deg_K s$ would be unbounded, by Lemma 2.2. □

2.2. **A MLV chain of v_s as a valuation on $\overline{K}[x]$.** For all $(s,i) \in S$, we have

$$v_s(x - s_{n,i}) = v(s - s_{n,i}) = n - \frac{1}{p^{i+1}} = \delta(n,i).$$

Let $v_{n,i} = \omega_{s_{n,i},\delta(n,i)}$ be the depth zero valuation on $\overline{K}[x]$ associated to the pair $(s_{n,i}, \delta(n,i)) \in \overline{K} \times \mathbb{Q}$; that is,

$$v_{n,i}\left(\sum_{0\le\ell} a_\ell\,(x - s_{n,i})^\ell\right) = \min_{0\le\ell}\left\{v_s\left(a_\ell\,(x-s_{n,i})^\ell\right)\right\} = \min_{0\le\ell}\left\{v\left(a_\ell\right) + \ell\,\delta(n,i)\right\}.$$

Lemma 2.4. *For all $(n,i), (m,j) \in S$ we have $v_{n,i}(x - s_{m,j}) = \min\{\delta(n,i), \delta(m,j)\}$. In particular, $v_{n,i} < v_s$ for all $(n,i) \in S$.*

Proof. The computation of $v_{n,i}(x - s_{m,j})$ follows immediately form the definition of $v_{n,i}$. The inequality $v_{n,i} \le v_s$ follows from the comparison of the action of both valuation on $(x - s_{n,i})$-expansions. Finally, if we take $\delta(n,i) < \delta(m,j)$, we get

$$v_{n,i}(x - s_{m,j}) = \delta(n,i) < \delta(m,j) = v_s(x - s_{m,j}).$$

This shows that $v_{n,i} < v_s$. □

Lemma 2.5. *The family $\mathcal{C} = (v_{n,i})_{(n,i)\in S}$ is a totally ordered family of valuations on $\overline{K}[x]$ of stable degree one, admitting v_s as its stable limit.*

Proof. Let us see that $\mathcal{C}$ is a totally ordered family of valuations. More precisely,

$$(n,i) < (m,j) \implies \delta(n,i) < \delta(m,j) \implies v_{n,i} < v_{m,j} < v_s.$$

Indeed, this follows from (2) because $v(s_{n,i} - s_{m,j}) = v(s_{n,i} - s) = \delta(n,i)$.

Clearly, $\mathcal{C}$ contains no maximal element, and all valuations in $\mathcal{C}$ have degree one. Let us show that all polynomials $x - a \in \overline{K}[x]$ are $\mathcal{C}$-stable, and the stable value coincides with $v_s(x-a) = v(s-a)$.

Since s is transcendental over K, we have $s \neq a$ and $q = v(s-a)$ belongs to $\mathbb{Q}$. For all $(n,i) \in S$ such that $\delta(n,i) > q$ we have

$$v_{n,i}(x-a) = \min\{v(a - s_{n,i}), \delta(n,i)\} = \min\{q, \delta(n,i)\} = q = v_s(x-a).$$

This ends the proof of the lemma. □

Therefore, v_s falls in case (b) of Theorem 1.5, as a valuation on $\overline{K}[x]$. A MLV chain of v_s is, for instance,

$$v_{0,0} \xrightarrow{\mathcal{C}} v_s = \lim(\mathcal{C}).$$

In order to obtain a MLV chain of v_s as a valuation of $K[x]$, we need to "descend" this result to $K[x]$. In this regard, we borrow some ideas of [18].

2.3. A MLV chain of v_s as a valuation on $K[x]$.

We say that $(a, \delta) \in \overline{K} \times \mathbb{Q}$ is a *minimal pair* if $\deg_K B_\delta(a) = \deg_K a$. This concept was introduced in [1]. By equation (2), for all $b \in \overline{K}$ we have

$$\omega_{a,\delta} = \omega_{b,\delta} \iff b \in B_\delta(a).$$

However, only the minimal pairs (a, δ) of this ball contain all essential information about the valuation on $K[x]$ that we obtain by restriction of $\omega_{a,\delta}$.

Lemma 2.6. [18, Prop. 3.3] *For $(a, \delta) \in \overline{K} \times \mathbb{Q}$, let μ be the valuation on $K[x]$ obtained by restriction of the valuation $\omega = \omega_{a,\delta}$ on $\overline{K}[x]$. Then, for all $g \in K[x]$, $\operatorname{in}_\mu g$ is a unit in $\mathcal{G}_\mu$ if and only if $\operatorname{in}_\omega g$ is a unit in $\mathcal{G}_\omega$.*

The following result was originally proved in [15]; another proof can be found in [14, Thm. 1.1].

Lemma 2.7. *For a minimal pair $(a, \delta) \in \overline{K} \times \mathbb{Q}$, let μ be the valuation on $K[x]$ obtained by restriction of the valuation $\omega_{a,\delta}$ on $\overline{K}[x]$. Then, $\operatorname{Irr}_K(a)$ is a key polynomial for μ, of minimal degree.*

We need a last auxiliary result.

Lemma 2.8. *For all $(n, i) \in S$ the pair $(s_{n,i}, \delta(n, i))$ is minimal.*

Proof. All $(s_{0,i}, \delta(0, i))$ are minimal pairs, because $\deg_K s_{0,i} = 1$. For $n > 0$, denote $B_{n,i} = B_{\delta(n,i)}(s_{n,i})$. Since $B_{n,i} \subset B_{n-1}(s_{n,n})$, Lemma 2.2 shows that

$$\deg_K B_{n,i} \geq \deg_K B_{n-1}(s_{n,n}) = p^n.$$

Since the center $s_{n,i}$ of the ball $B_{n,i}$ has $\deg_K s_{n,i} = p^n$, we deduce $\deg_K B_{n,i} = p^n$. Thus, $(s_{n,i}, \delta(n, i))$ is a minimal pair. □

Notation. Let us denote the restriction of $v_{n,i}$ to $K[x]$ by

$$\rho_{n,i} = (v_{n,i})_{|K[x]}\,.$$

Moreover, for the limit indices (n, n), $n \geq 0$, we denote:

$$\mu_n = \rho_{n,n}, \qquad \phi_n = \operatorname{Irr}_K(s_{n,n}), \qquad \gamma_n = v_s(\phi_n).$$

By Lemmas 2.4 and 2.5, the set of all valuations $(\rho_{n,i})_{(n,i)\in S}$ is totally ordered, and $\rho_{n,i} < v_s$ for all (n, i).

Proposition 2.9. *For all $n \geq 0$, the set $\mathcal{C}_n = (\rho_{n,i})_{(n,i)\in S}$ is an essential continuous family of stable degree p^n. Moreover, the polynomial ϕ_{n+1} belongs to $\mathrm{KP}_\infty(\mathcal{C}_n)$ and $\mu_{n+1} = [\mathcal{C}_n; \phi_{n+1}, \gamma_{n+1}]$.*

Proof. Let us fix some $n \ge 0$. By Lemmas 2.7 and 2.8, all valuations in $\mathcal{C}_n$ have degree p^n. Hence, $\mathcal{C}_n$ is a totally ordered family of stable degree p^n.

Let us show that all monic $g \in K[x]$ with $\deg(g) \le p^n$ are $\mathcal{C}_n$-stable. Let $u \in \overline{K}$ be a root of g. By Lemma 2.2, $u \notin B_n(s_{n+1,n+1})$, so that $v(s_{n+1,n+1} - u) < n$. Since $v(s - s_{n+1,n+1}) = \delta(n+1, n+1) > n$, we deduce that $v(s-u) < n$.

Therefore, we may find $j \ge n$ such that

$$v(s-u) < n - \frac{1}{p^{j+1}}$$

for all roots u of g. As we showed along the proof of Lemma 2.5, this implies

$$v_{n,i}(x-u) = v_s(x-u) \quad \text{for all } (n,i) \ge (n,j)$$

simultaneously for all roots u of g. Therefore, $\rho_{n,i}(g) = v_s(g)$ for all $(n,i) \ge (n,j)$ and g is $\mathcal{C}_n$-stable.

Now, let us show that ϕ_{n+1} is $\mathcal{C}_n$-unstable. For all $i \ge n$, we have

$$v(s_{n+1,n+1} - s_{n,i}) = \delta(n,i) = v_{n,i}(x - s_{n,i}).$$

By [12, Prop. 6.3], $x - s_{n+1,n+1}$ is a key polynomial for $v_{n,i}$; thus, $\operatorname{in}_{v_{n,i}}(x - s_{n+1,n+1})$ is not a unit in the graded algebra $\mathcal{G}_{v_{n,i}}$. Hence, $\operatorname{in}_{v_{n,i}} \phi_{n+1}$ is not a unit in $\mathcal{G}_{v_{n,i}}$ and Lemma 2.6 shows that $\operatorname{in}_{\rho_{n,i}} \phi_{n+1}$ is not a unit in $\mathcal{G}_{\rho_{n,i}}$. Since this holds for all i, Lemma 1.4 shows that ϕ_{n+1} is $\mathcal{C}_n$-unstable.

Since the irreducible polynomials in $K[x]$ have degree a power of p (Lemma 2.1), ϕ_{n+1} is an $\mathcal{C}_n$-unstable polynomial of minimal degree. Therefore, $\mathcal{C}_n$ is an essential continuous family and $\phi_{n+1} \in \mathrm{KP}_\infty(\mathcal{C}_n)$.

Since ϕ_{n+1} is $\mathcal{C}_n$-unstable, $\rho_{n,i}(\phi_{n+1}) < v_s(\phi_{n+1}) = \gamma_{n+1}$ for all i. Thus, it makes sense to consider the limit augmentation $\mu = [\mathcal{C}_n; \phi_{n+1}, \gamma_{n+1}]$. Let us show that $\mu = \mu_{n+1}$ by comparing their action on ϕ_{n+1}-expansions. For all $g = \sum_{0 \le \ell} a_\ell \phi_{n+1}^\ell$,

$$\mu_{n+1}(g) = \min_{0 \le \ell} \left\{ \mu_{n+1}\left(a_\ell \phi_{n+1}^\ell\right) \right\}, \qquad \mu(g) = \min_{0 \le \ell} \left\{ \mu\left(a_\ell \phi_{n+1}^\ell\right) \right\}. \tag{6}$$

Since $\deg(a_\ell) < p^{n+1} = \deg(\phi_{n+1})$, all these coefficients a_ℓ are $\mathcal{C}_n$-stable. Hence, $\rho_{n,i}(a_\ell) = v_s(a_\ell)$ for all (n,i) sufficiently large. Since $\rho_{n,i} < \mu_{n+1} < v_s$, we deduce

$$\mu(a_\ell) = \rho_{\mathcal{C}_n}(a_\ell) = \rho_{n,i}(a_\ell) = \mu_{n+1}(a_\ell) = v_s(a_\ell).$$

Finally, for all $i \ge n+1$, we have $v(s_{n+1,i} - s_{n+1,n+1}) = \delta(n+1, n+1)$, so that

$$v_{n+1,n+1}(x - s_{n+1,n+1}) = \delta(n+1,n+1) = v_{n+1,i}(x - s_{n+1,n+1}).$$

By (4), for all the other roots u of ϕ_{n+1}, the support of u is contained in $(-1, n]$. Thus, for all $i \ge n+1$ we get

$$v_{n+1,n+1}(x-u) = v(s_{n+1,n+1} - u) = v(s_{n+1,i} - u) = v_{n+1,i}(x-u).$$

Since $\mu_{n+1} = \rho_{n+1,n+1} < \rho_{n+1,i} < v_s$, [13, Cor. 2.5] implies

$$\mu_{n+1}(\phi_{n+1}) = \rho_{n+1,i}(\phi_{n+1}) = v_s(\phi_{n+1}) = \gamma_{n+1} = \mu(\phi_{n+1}).$$

By (6), we deduce that $\mu = \mu_{n+1}$. □

Therefore, we get a countable chain of limit augmentations

$$\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \longrightarrow \mu_{n-1} \xrightarrow{\phi_n, \gamma_n} \mu_n \longrightarrow \cdots$$

whose stable limit is v_s.

In order to see that this chain satisfies the MLV condition, we need to show that

$$\phi_n \notin \mathbf{t}(\mu_n, \mu_{n+1}) \quad \text{for all } n \geq 0.$$

This means $\mu_n(\phi_n) = \mu_{n+1}(\phi_n)$ for all n. Since $\mu_n < \mu_{n+1} < v_s$, the desired equality follows from $\mu_n(\phi_n) = \gamma_n = v_s(\phi_n)$.

As a consequence, v_s has an infinite limit-depth.

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