Automated Off-line Generation of Stable Variable Impedance Controllers According to Performance Specifications

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A. Extended Proof of Proposition 1

1) LMI conditions for Lyapunov stability for polytopic LPV models: According to Lyapunov theory, if there exists a discrete-time candidate function $V(\mathbf{x}_k)$ such that $\forall k \geq 0$:

1) $V(0) = 0$;
2) $V(\mathbf{x}_k) > 0,\ \forall \mathbf{x}_k \neq 0$
3) $V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0, \forall \mathbf{x}_k \neq 0$

the equilibrium point $\mathbf{x}_k = 0$ is stable in the sense of Lyapunov. As stated in [13], applying the LMI paradigm for a generic quadratic candidate function $V(\mathbf{x}_k) = \mathbf{x}_k^T \mathbf{P} \mathbf{x}_k$, above conditions are fulfilled iff there exist a solution matrix $\mathbf{P} = \mathbf{P}^T > 0$ that fulfils the following inequality:

$$\mathbf{A}_k \mathbf{P} \mathbf{A}_k{}^T - \mathbf{P} < 0 \tag{14}$$

For the polytopic LPV model of the VIC in Eq. (7), defined by the system evaluated at the convex hull Θ, stability implies that solution $\mathbf{P}$ is common to all the vertex state matrices $\mathbf{A}_i$, turning (14) into:

$$\mathbf{A}_i \mathbf{P} \mathbf{A}_i{}^T - \mathbf{P} < 0 \tag{15}$$

2) Equivalence between H_2 index and quadratic criterion: First, we have to introduce an auxiliary output $\mathbf{y}_k$ such that the complete system including the discrte-time form of system. (2) is:

$$\left\{ \begin{array}{r} \mathbf{x}_{k+1} = \mathbf{A}_0\, \mathbf{x}_k + \mathbf{B}_w\, \mathbf{u}_k + \mathbf{B}_F\, F_k \\ \mathbf{y}_k = \mathbf{C}\, \mathbf{x}_k + \mathbf{D}\, \mathbf{u}_k \end{array} \right. \tag{16}$$

Considering that a state-feedback control is being applied, i.e. $\mathbf{u}_k = \mathbf{W}_k\, \mathbf{x}_k$, system (16) turns into:

$$\left\{ \begin{array}{r} \mathbf{x}_{k+1} = \mathbf{A}_k\, \mathbf{x}_k + \mathbf{B}_F\, F_k \\ \mathbf{y}_k = (\mathbf{C} + \mathbf{D}\, \mathbf{W}_k) \mathbf{x}_k \end{array} \right.$$

where we can define $\mathbf{S} = \mathbf{C} + \mathbf{D}\mathbf{W}_k$. The effect of the exogenous input F_k in $\mathbf{y}_k$ (which is a modulation of $\mathbf{x}_k$) can be determined through the H_2 norm over the infinite horizon of the transfer function $G_{(y,F),k}$:

$$\begin{aligned} \|G_{(y,F)}\|_2 &= \left(\sum_{k=0}^{\infty} G_{(y,F)}{}^T\, G_{(y,F)} \right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} \mathbf{x}_k{}^T \mathbf{S}^T\, \mathbf{S} \mathbf{x}_k \right)^{1/2} \end{aligned}$$

Thus, choosing $\mathbf{C} = [\mathbf{Q}_\eta{}^{1/2}\ 0]^T$ and $\mathbf{D} = [0\ \mathbf{R}_\eta{}^{1/2}]^T$ we obtain

$$\begin{aligned} \|G_{(y,F),k}\|_2 &= \left(\sum_{k=0}^{\infty} \mathbf{x}_k{}^T \begin{bmatrix} \mathbf{Q}_\eta{}^{1/2} \\ \mathbf{R}_\eta{}^{1/2} \mathbf{W}_k \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_\eta{}^{1/2} \\ \mathbf{R}_\eta{}^{1/2} \mathbf{W}_k \end{bmatrix} \mathbf{x}_k \right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} \mathbf{x}_k{}^T (\mathbf{Q}_\eta + \mathbf{W}_k{}^T \mathbf{R}_\eta \mathbf{W}_k) \mathbf{x}_k \right)^{1/2} \end{aligned}$$

from which we can state the following equivalence with the quadratic criterion from Eq. (9).

$$J = \|G_{(y,F)}\|_2^2 < \gamma \tag{17}$$

3) Stability-H_2 LMI Conditions for VIC polytopic LPV model: First, we have to start from the formulation of the transfer function for each of the vertex systems $i = 1, \ldots, N$:

$$\begin{aligned} \|G_{i,(y,F)}\|_2^2 &= \sum_{k=0}^{\infty} G_{i,(y,F)}^T G_{i,(y,F)} \\ &= \sum_{k=0}^{\infty} \mathrm{trace}\{G_{i,(y,F)} G_{i,(y,F)}^T\} \\ &= \mathrm{trace}\left\{ \sum_{k=0}^{\infty} [\mathbf{S}_i (\mathbf{A}_i{}^{k-1} \mathbf{B}_{F,i})\, (\mathbf{A}_i{}^{k-1} \mathbf{B}_{F,i})^T \mathbf{S}_i{}^T \right\} \\ &= \mathrm{trace}\left\{ \mathbf{S}_i \left(\sum_{k=0}^{\infty} \mathbf{A}_i{}^{k-1} \mathbf{B}_{F,i} \mathbf{B}_{F,i}{}^T (\mathbf{A}_i{}^{k-1})^T \right) \mathbf{S}_i{}^T \right\} \end{aligned}$$

where $\mathbf{S}_i = \mathbf{C} + \mathbf{D}\mathbf{W}_i$. Considering that the controllability Gramian $\mathbf{X}_{C,i}$ is defined as

$$\mathbf{X}_{C,i} = \sum_{k=0}^{\infty} \mathbf{A}_i{}^{k-1} \mathbf{B}_{F,i} \mathbf{B}_{F,i}{}^T (\mathbf{A}_i{}^T)^{k-1} > 0 \tag{18}$$

we can obtain the following condition applying Eq. (17).

$$\|G_{i,(y,F)}\|_2^2 = \mathrm{trace}\{\mathbf{S_i} \mathbf{X}_{C,i} \mathbf{S_i}^T\} < \gamma \tag{19}$$

Additionally, $\mathbf{X}_{C,i}$ happens to be the solution of the following Lyapunov equality:

$$\mathbf{A}_i \mathbf{X}_{C,i} \mathbf{A}_i{}^T - \mathbf{X}_{C,i} + \mathbf{B}_{F,i} \mathbf{B}_{F,i}{}^T = 0 \tag{20}$$

To generalise this solution to the LMI framework we substitute $\mathbf{X}_{C,i}$ by a common $\mathbf{P} = \mathbf{P}^T > 0$ for all the vertex systems. Thus equation (20) turns into (10a) and condition (19) into (10b). Stability is also assessed in (10a) as it is equivalent to (15) considering that $\mathbf{B}_{F,i} \mathbf{B}_{F,i}{}^T > 0$.

B. Extended Proof of Proposition 2

Operational constraint (11a) can be defined considering the maximum squared norm:

$$\| \mathbf{u}_k \|_2^2 \leq \max_{k\geq 0} \| \mathbf{u}_k \|_2^2 \leq {u_{max}}^2 \tag{21}$$

Following [13], considering solution matrix $\mathbf{P}$ fulfilling conditions (10) and introducing intermediate variable $\mathbf{F} = \sum_{i=1}^{N} [\pi_i(\boldsymbol{\theta}(t))\mathbf{W}_i]\mathbf{P}$, the maximum squared norm of the control effort can be defined as follows:

$$\max_{k\geq 0} \| \mathbf{u}_k \|_2^2 = \max_{k\geq 0} \| \mathbf{F}\mathbf{P}^{-1}\mathbf{x}_k \|_2^2$$

which is upper bounded by the norm for the maximum $\mathbf{x}$

$$\max_{k\geq 0} \| \mathbf{u}_k \|_2^2 \leq \max_{\mathbf{x}} \| \mathbf{F}\mathbf{P}^{-1}\mathbf{x} \|_2^2$$

This equals to the maximum eigenvalue $\overline{\sigma}$ for all the $\mathbf{x}$ contained in ${\mathbf{x}_k}^T\mathbf{P}\mathbf{x}_k$, i.e. the ellipsoid defined by $\mathbf{P}$:

$$\max_{k\geq 0} \| \mathbf{u}_k \|_2^2 \leq \overline{\sigma}(\mathbf{P}^{-1/2}\mathbf{F}^T\mathbf{F}\mathbf{P}^{-1/2}) \tag{22}$$

Using definition in (21) and applying Schur lemma [14] for the polytopic formulation leads to (12a), where u_{max} can be substituted by $u_{max,i}$.

Similarly, for operational constraint (11b):

$$\| \mathbf{x}_k \|_2^2 \leq \max_{k\geq 0} \| \mathbf{x}_k \|_2^2 \leq {x_{max}}^2 \tag{23}$$

Following [13], with $\mathbf{P}$ fulfilling conditions (10), the maximum squared norm of the state leads to:

$$\max_{k\geq 0} \| \mathbf{x}_k \|_2^2 = \max_{k\geq 0} \| \mathbf{A}\mathbf{x}_{k-1} \|_2^2 \leq \max_{\mathbf{x}} \| \mathbf{A}\mathbf{x} \|_2^2$$

Again, last term equals to the maximum eigenvalue $\overline{\sigma}$ for all the $\mathbf{x}$ contained in the ellipsoid defined by $\mathbf{P}$:

$$\max_{k\geq 0} \| \mathbf{x}_k \|_2^2 \leq \overline{\sigma}(\mathbf{P}^{1/2}\mathbf{A}^T\mathbf{A}\mathbf{P}^{1/2}) \tag{24}$$

Using definition in (23) and applying Schur lemma [14] for the polytopic formulation leads to (12b).