Twenty Questions on Zonogons, Zonohedra and Zonoids

by Anton Hanegraaf, ELST (Netherlands)

Structural Topology #4, 1980

Abstract

We present a series of problems arising in the theoretical investigation of the use of zonoid surfaces as building elements. The questions put below concern mainly the basic morphological aspects of this investigation. Special attention is paid to the possibility of occurrence of not strictly convex and of nonconvex regions.

Topologie Structurale #4, 1980

Résumé

Nous présentons une série de problèmes survenant lors de la recherche théorique de l'utilisation de surfaces zonoïdales en tant qu'éléments de construction. Les questions énoncées plus bas traitent principalement des aspects morphologiques de base de cette recherche. Une attention toute particulière est portée à la possibilité de régions qui ne sont pas formellement connexes et d'autres qui sont non-convexes.

Un zonoïde est défini ici comme étant simplement une surface polyédrique (ou une partie de surface polyédrique) dont les faces peuvent être complétées par des parallélogrammes.

Introduction

This review will report about the investigations on zonoidal surfaces which were started in the autumn of 1973 by two simultaneously-operating research teams «Metaform» and «ABT-Zono», in which the author was involved. Up to the end of 1976 there was a period of intense activity and cooperative team effort. Most of the results obtained date back to that period.

Recently, after a forced three year standstill, the author has once more taken up this line of research. In order to give the an impression of some of the subjects which are waiting to be tackled, we have selected for summary in this review some 20 from an extensive list of problems.

The presentation is visual. The figures and captions are meant to illustrate some known results and to replace definitions. The statements in the text are kept brief. No proofs are given. Not all terms, notions and notations are explained. A single diagram may be used to illustrate several problems, so some captions may initially pose a problem to the reader. No attempt has been made to reproduce the extensive bibliography on the subject, nor to refer to the names of the numerous top geometers who should be given credit for their beautiful theorems. In fact, this review is necessarily incomplete and imperfect. The only aim of it is to show to a larger audience how someone was struck by an intriguing concept and now is trying, in an elementary and modest way, to understand more about it.

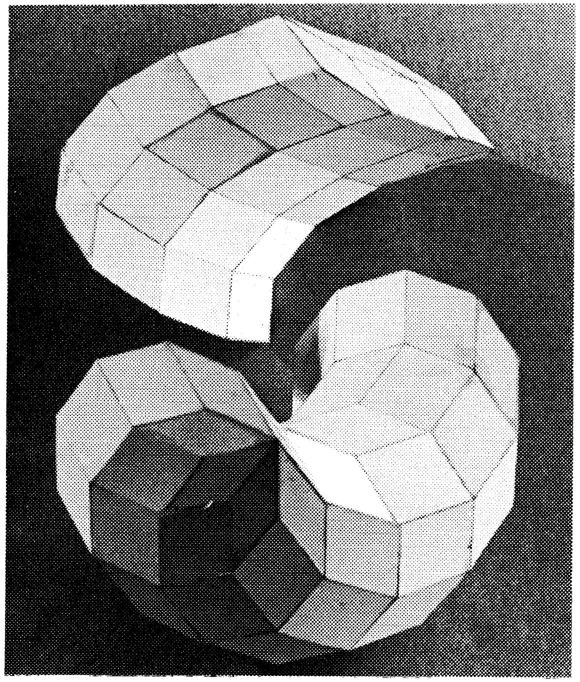


Figure 1. A zonohedron partitioned into two identical parts by the surface of a saddle-shaped zonoid. The surface of the saddle shape is obtained by taking a set of adjacent zones of the original zonohedron and by replacing these zones in exactly reversed order. This procedure has been called **reciprocation**. In fact, this figure was the start and the stimulant for our research on zonoids.

Preliminary Statements

A **zonoid** is here defined as a polyhedral surface (or a piece of one), the faces of which can be filled up with parallelograms. A strip of parallel arranged adjacent parallelograms is called a **zone**. Such a zone can be considered as a part of a prismatic cylinder, and a zonoid thus can be described as a system of intersecting cylinders. We call a zonoid **pure** or **mixed** if or if not all faces are already parallelograms. A plane zonoid, convex or nonconvex, is called a **zonogon**. For a closed strictly convex zonoid we maintain its familiar name **zonohedron**.

Relatives of a zonoid are obtained by means of changing the widths or the arrangements of the zones. A special type of relative is obtained by taking away one or more zones completely and uniting the remaining parts by translation; such a relative is called a **minor**.

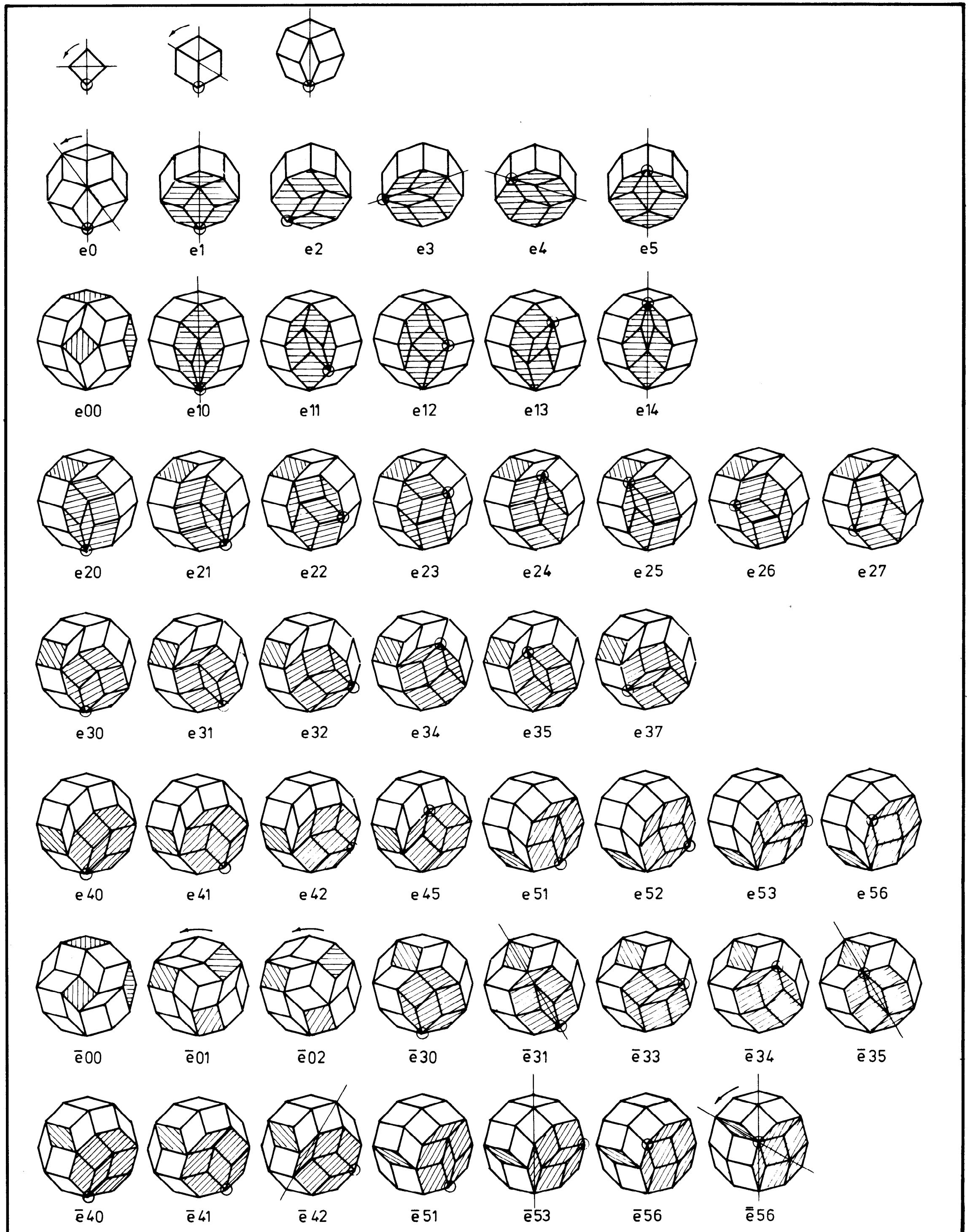


Figure 2. Modified representations for all types of strictly convex zonogon fillings up to order n = 6.

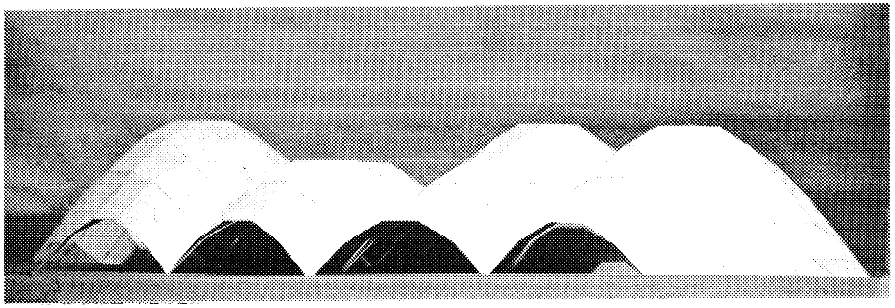


Figure 3. A zonoidal translation complex: On most of all zonohedra there occur translational caps. Such a cap and its minors may be considered to be parts of some translational zonoid. Major and minor caps can be fused along corresponding parts of the generators of the zonoids. The complex shown in this figure is built up from the translational caps of primary zonohedra.

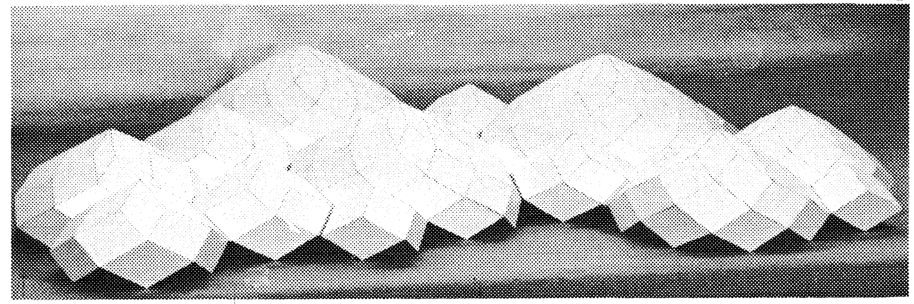


Figure 4. A zonoidal tesselation complex: Equatorial caps of major and minor zonohedra are placed in face to face contact in some arbitrary tesselation arrangement. The major polyhedron is a bipolar dodecazonohedron, the other polyhedra are copies of its minors. All edges are in the same inclination with respect to a vertical axis. The whole complex is built up with copies of just four distinct rhombic panel types.

Zonogons

A zonogon of order n is a polygon with 2n edges, in which each edge has an opposite parallel equal edge.

Question 1. *What is the number of topologically distinct types of zonogonfillings of order n?* This number is only known for low order zonogons: Strictly convex zonogons of order 2, 3, 4, 5 and 6 have 1, 1, 1, 6 and 43 types of filling respectively (**Figure 2**). For n=7 the number is 922, for n=8 it is about 40,000! It seems to make little sense to try to find all types for large n, but probably special types are found more easily:

Question 2. *Which types of strictly convex zonogonfillings contain no other convex zonogons, except for those of order n⩽3?* For instance n=5 has one such type, n=6 has four. The number does not seem to grow very fast (Observe the shadings in **Figure 2**).

Question 3. *Which types of (convex or nonconvex) zonogonfillings of order n have metric realizations with some kind of symmetry?* For strictly convex fillings of low order these types can be found in **Figure 2**.

Question 4. *Which types of nonconvex zonogonfillings can have a convex, but not necessarily strictly convex, representation?* These types occur in or may serve as simplified views of zonohedra. Several instances are shown in **Figures 5 through 8.**

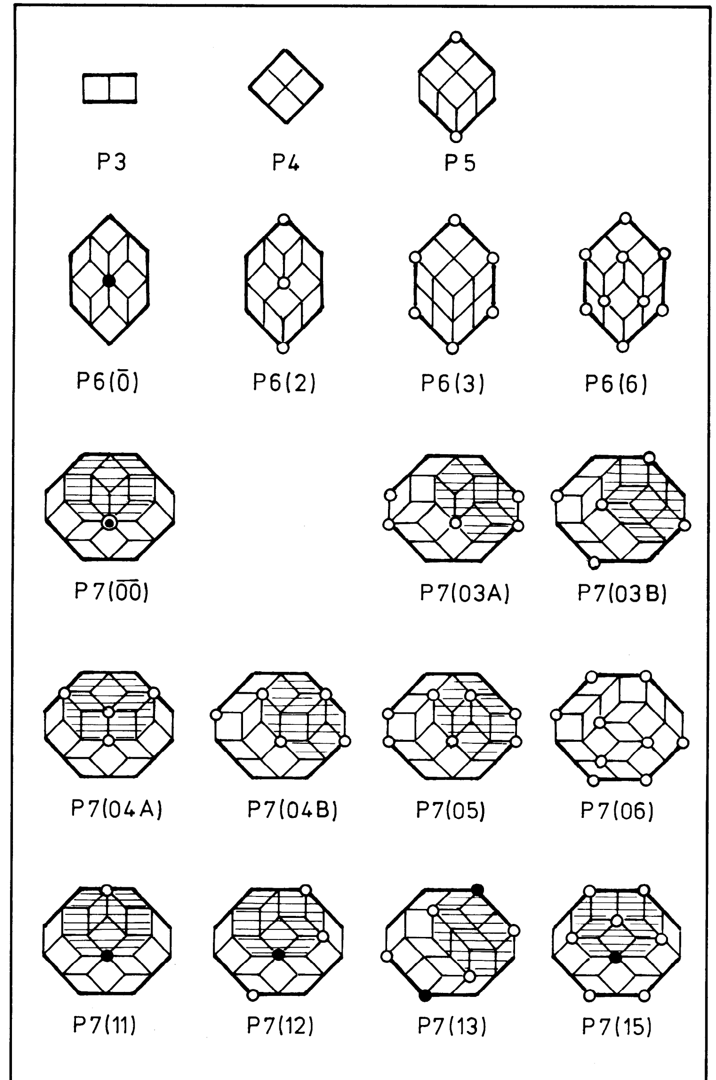


Figure 5. Simplified representations for all types of pure zonohedra up to order n=7. Pentavalent and hexavalent vertices are indicated by white and black circles respectively.

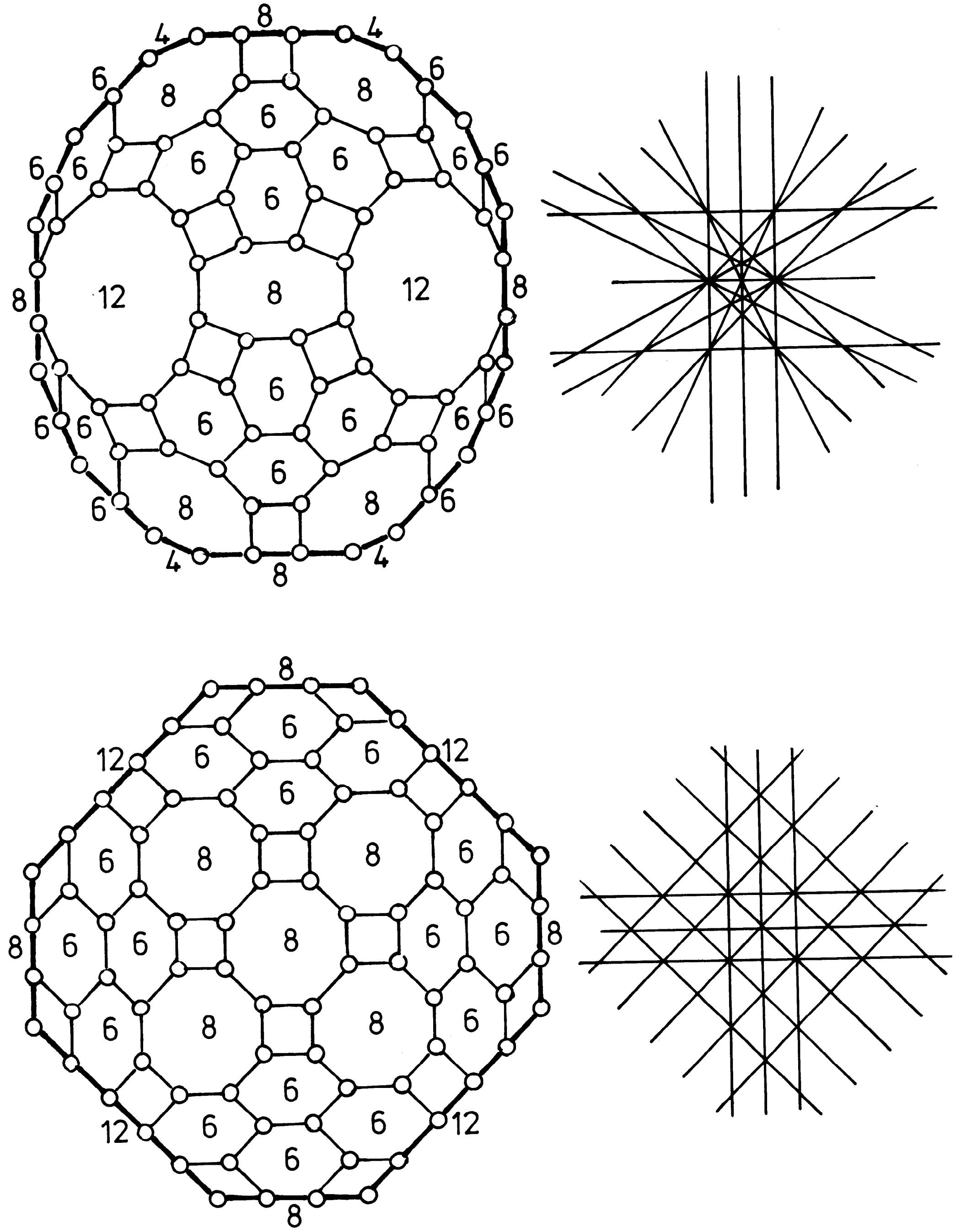


Figure 6. The effect of simultaneously **rolling over** a zonohedron and its corresponding system of great circles on a concentric sphere, represented by projective diagrams. The zonohedral views are given in modified representations. In both views one zone is supposed to be perpendicular to the plane of the paper. The corresponding great circles then lie in the plane of the paper and their projective representations are the (invisible) lines at infinity.

Zonohedra

The **order** of a zonohedron is its number of zones. A zonohedron, like a convex zonogon, has **central symmetry**. Each part of a zonohedron has an opposite identical component. Each cylinder which carries a zone has a centre of symmetry incident with the centre of the zonohedron. All planes through the centre perpendicular to the axes of the cylinders mark on a concentric sphere a **system of great circles,** which can be considered as a dual representation of the zonohedron. A view of this system of circles may conveniently be represented by a projective diagram.

Question 5. *How can we decide the equivalence of two views of a zonohedron or of their corresponding projective diagrams?* This is not always easily done. A simple, but probably too cumbersome, manner is the **rolling over** procedure of **Figure 6.**

It is not advisable to make an attempt to enumerate all types of zonohedra of order ten, or higher. It is more significant to look for special types:

Question 6. *Which types of pure zonohedra of even order do not contain an equatorial chain?* By which is meant a closed chain of faces which are connected only by opposite vertices. Zonohedra on which occur one or more such chains may be represented by more simplified views. Their corresponding projective diagrams simplify simultaneously, for all lines occur in parallel pairs (see **Figure 7**). The nonequatorial cases seem to be exceptional.

Question 7. *Which types of pure zonohedra do not contain convex octagon circuits?* We can call these exceptional types **nontranslational**, for the absence of translational caps (see **Figure 8**).

Question 8. *Which combinations of numbers of k-valent vertices are incompatible?* For instance, pure zonohedra of order $n = 6$ do not exist with 1, 4 or 5 pairs of pentavalent vertices.

Question 9. *Can we find a suitable set of frames to define all types of pure zonohedra of order n?* By a **frame** is meant a mixed zonohedron with low order faces ($n \leqslant 5$). By some filling of the faces of such a frame we obtain a convex, but not strictly convex, pure zonohedron which may be topologically equivalent with a strictly convex pure zonohedron. It may be

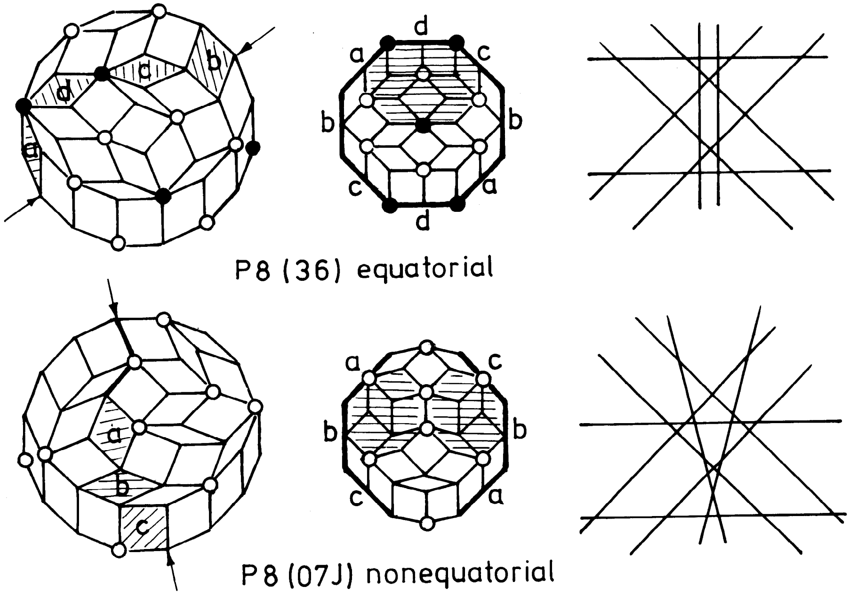


Figure 7. Two types of pure octazonohedra. For each type is shown one of its general views, one simplified view and the projective diagram corresponding to the latter. On all but one of the 135 types of pure octazonohedra there occur equatorial chains.

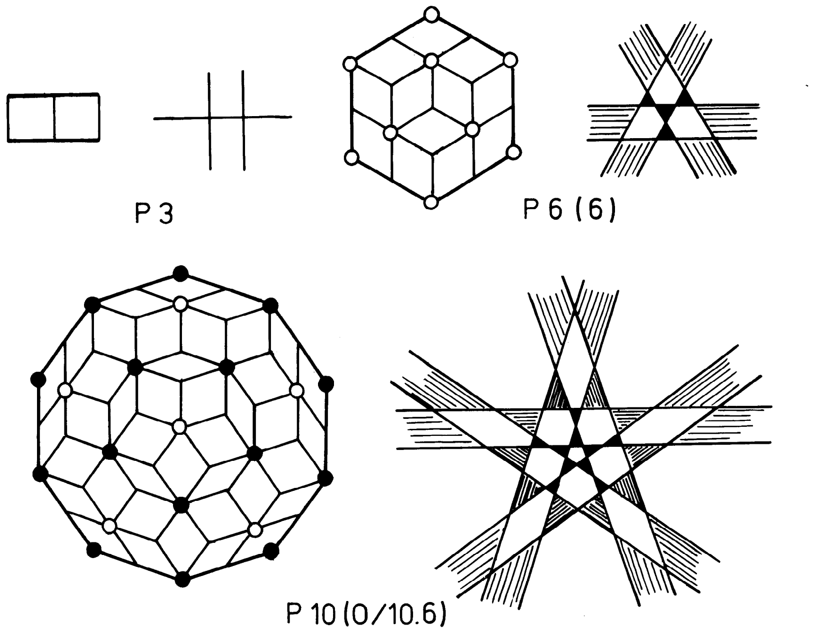


Figure 8. Nontranslational pure zonohedra, satisfying the condition $2v_5 + 3v_6 + \ldots + (k\text{-}3)v_k = n(n\text{-}1) - 6$, in which v_j means the number of j-valent vertices.

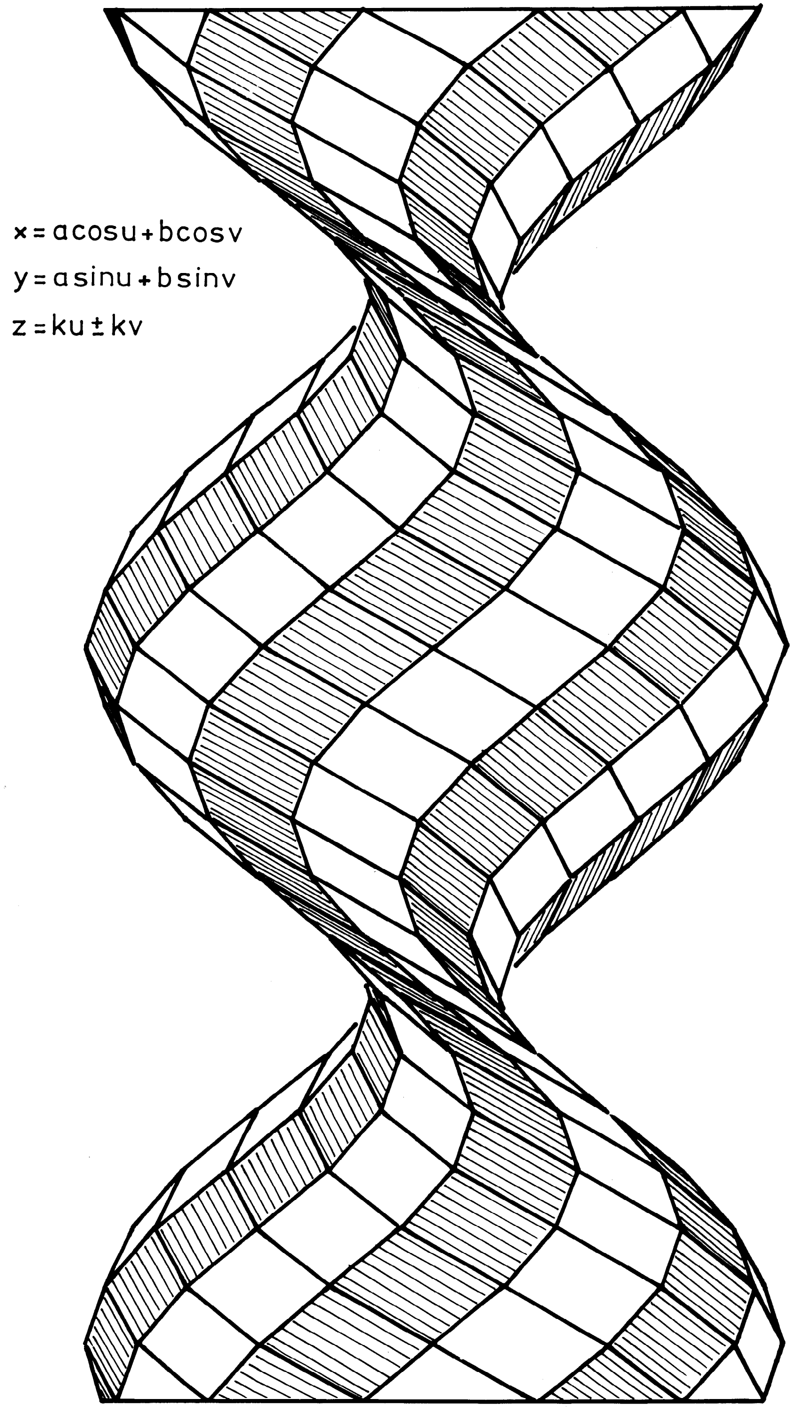


Figure 9. This single illustration serves as the front view of two related but very differently shaped periodical zonoids: Either a helicoidal zonoid or a rotational zonoid. Both are related by the procedure of reciprocation. The formulas which represent their carrying surfaces differ in only one sign!

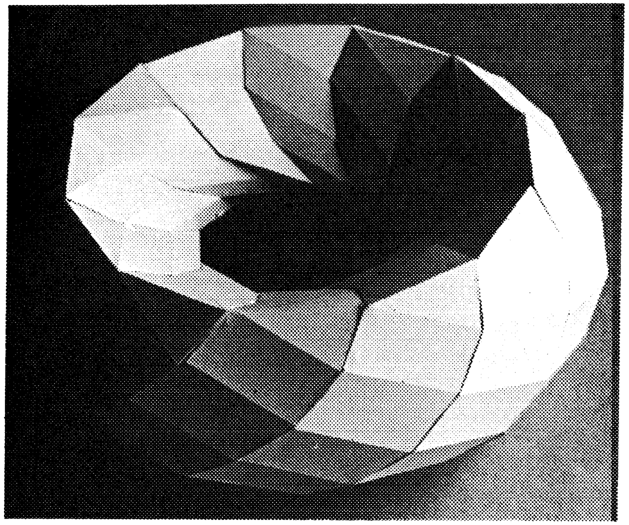
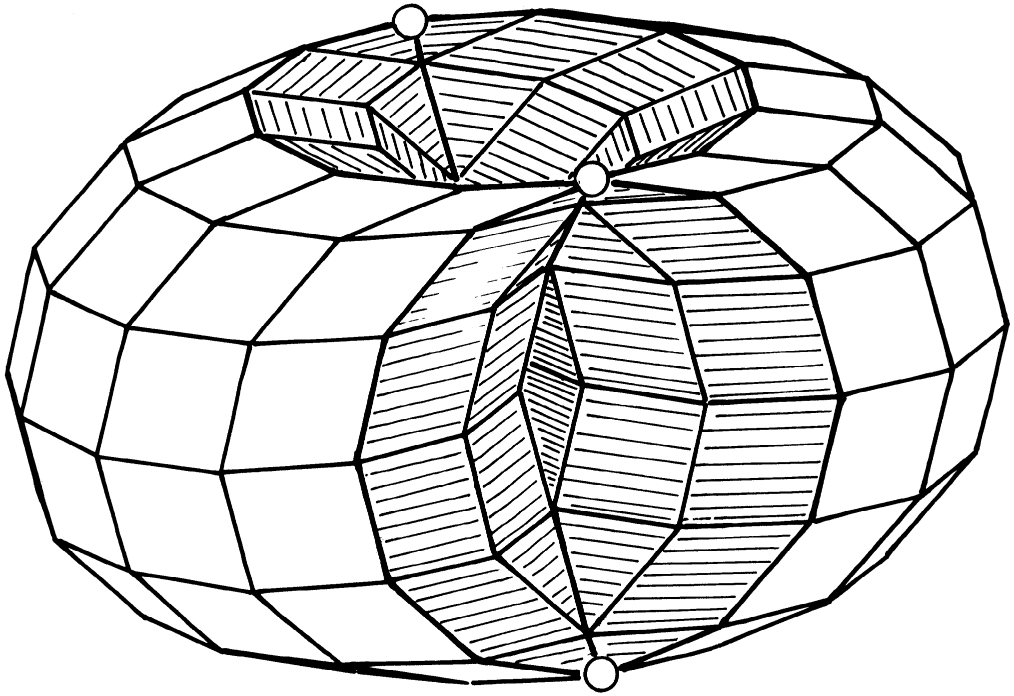
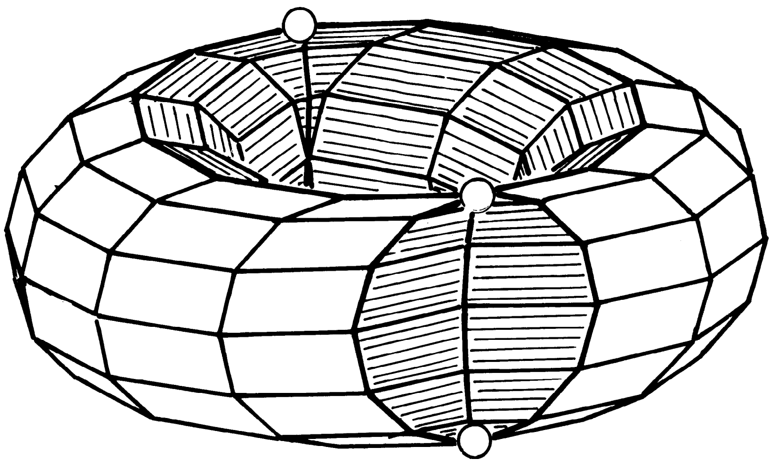


Figure 10. A toroidal «zonoid», obtained as the intersection of two congruent rotational zonoids.



$x = \cos u \quad y = \sin u + \cos v \quad z = \sin v$

$x = \cos u \quad y = \sin u + k\cos v \quad z = k\sin v$



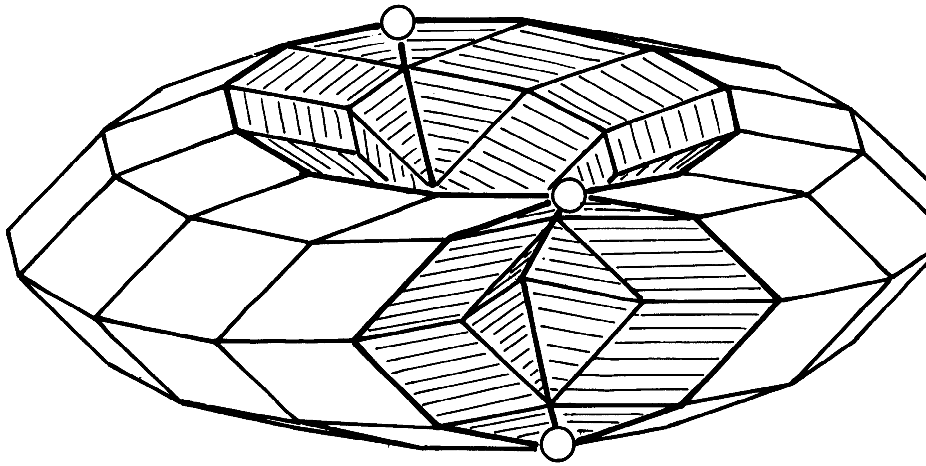


Figure 11. Bohemian zonoids. These generalizations of bohemian vaults are self-reciprocal and self-intersecting zonoids.

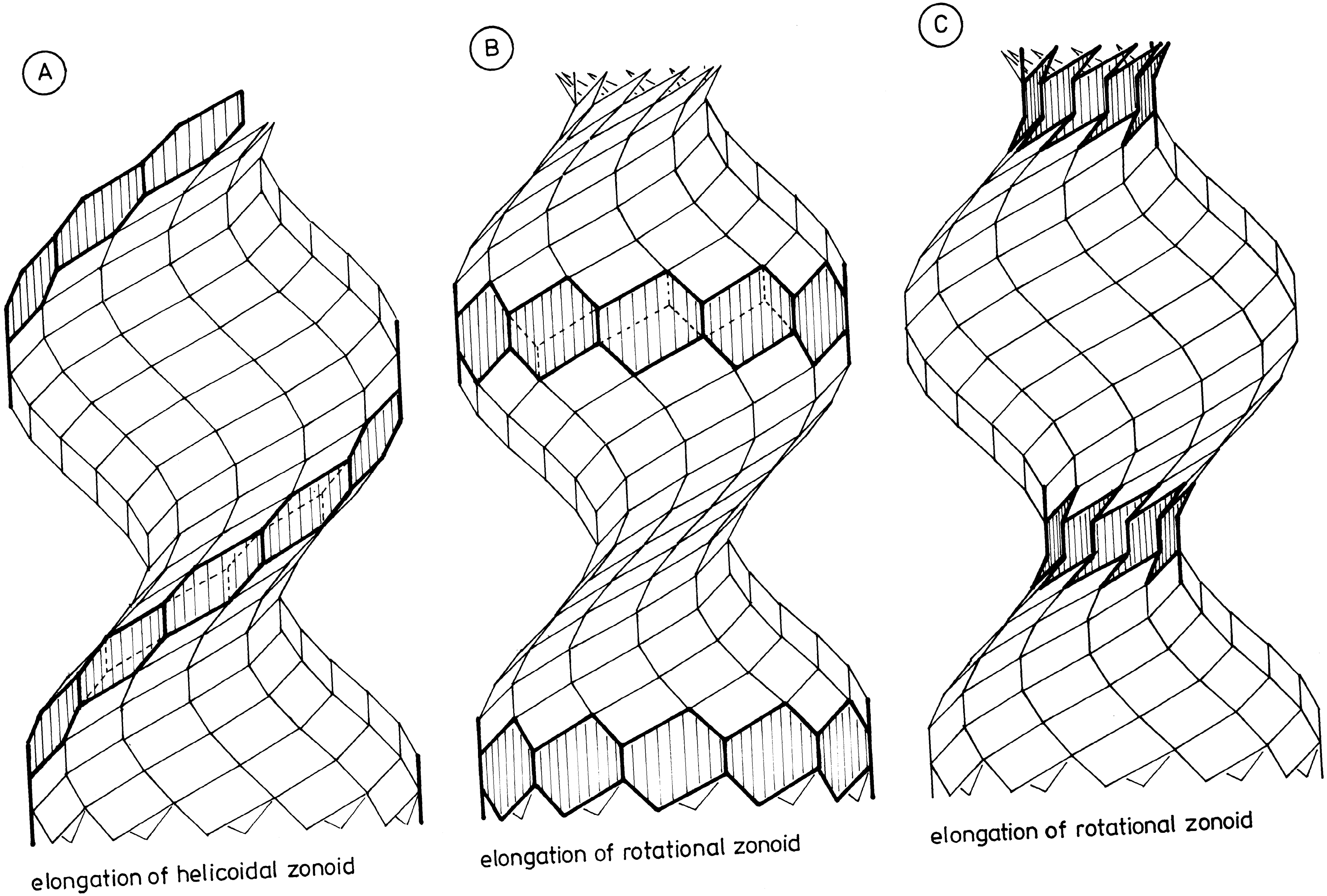


Figure 12. The process of elongation applied to a zonoid does not always result in a zonoid: The elongations A and B, in contrast with C, still remain zonoids (no longer pure and proper). Each of the hexagonal faces can be filled, in two ways, with parallelograms (some are indicated). These surfaces can be covered, in many distinct ways, by systems of zones. The elongation C, however, is not a zonoid. The concave hexagons cannot be filled with parallelograms. There are no continued zones. Surfaces like this one could probably be called «quasi-zonoids».

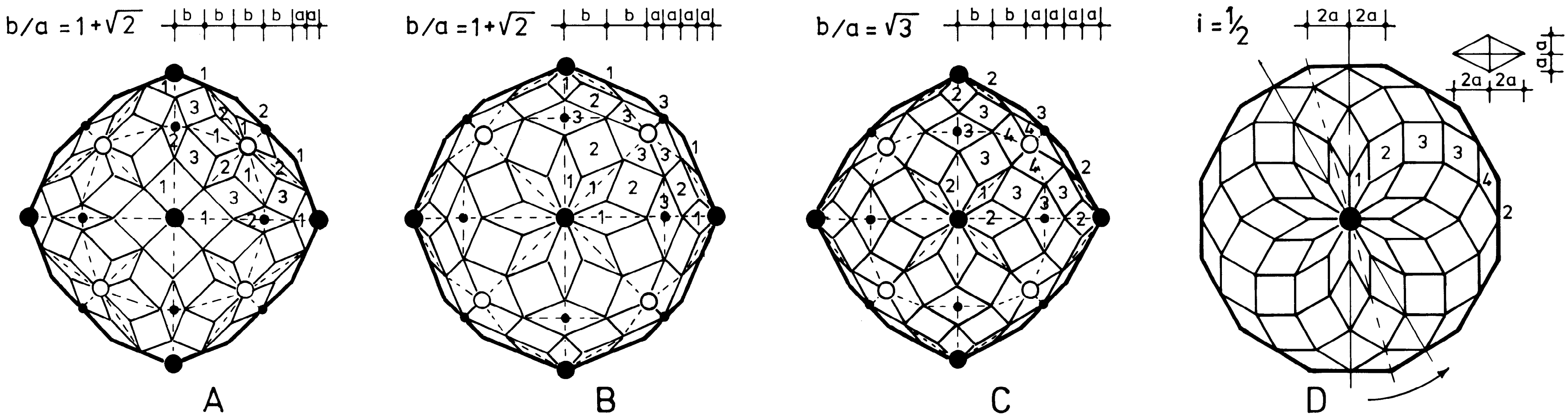


Figure 13. True equatorial views of four topologically distinct equilateral dodecazonohedra (12 zones, 132 faces). The zonohedra A, B, C have octahedral symmetry; the zonohedron D has an axis of twelvefold rotational symmetry. All four are metrically determined by the choice of the edge length and the choice of one particular ratio. The minimum numbers $m_A=3$, $m_B=3$, $m_C=4$, $m_D=4$, are obtained for the ratios indicated above. In each figure equally numberes faces are congruent. Moreover, the equally numbered faces of the zonohedra C and D are mutually congruent.

expected that classifying pure zonohedra in this way can be done with, proportionally, a few types of such frames. For instance, all but one pure heptazonohedron have an identical barrel-shaped frame (**Figure 5**). All pure octazonohedra have at least a pair of decagon-circuits or four nonintersecting octagon-circuits, by which eight frames are sufficient to describe all 135 types.

The following questions concern metric realizations of zonohedra:

Question 10. *Which types of zonohedra can have distinct metric realizations, each with another kind of symmetry?* A trivial example is the hexazonohedron p6(6), which has a realization with icosahedral symmetry: axes of twofold, threefold and fivefold rotational symmetry. Then there are also realizations with only one such axis.

Question 11. *Given some realization of a major zonohedron, how can we enumerate and describe all its metrically distinct minors?* This is easily done for some very special types, like the primary zonohedra (Hanegraaf 1975). But in most cases it seems to be less simple.

Question 12. *How can we control rapidly the convex-realizability of a metrically determined projected zonohedron?* For instance, the two types of multipolar zonohedra B and C of **Figure 13** differ only in the hexagonal regions with white circles. (In this case these hexagons become planar for the ratio $b/a=2$). Recall that, from order $n=9$ upwards, types of **pseudo-zonohedra** are known which cannot have convex realizations at all. The projective diagrams of these types cannot have all lines stretched.

Zonoids

Here we will pay attention to the aspect of non-convexity, in which the general concept of **zonoid** differs from the already treated cases. A zonoid is called **proper** or **compound** if or if not all vertices are tetravalent. A proper zonoid may be considered to be a discrete model of a **translational surface**, which is represented by equations like $x = u_1 + v_1$, $y = u_2 + v_2$, $z = u_3 + v_3$, in which u_i is any function of u alone and v_i is any function of v alone (see for instance the formulas in **Figures 9 and 11**).

Question 13. *Which types of analytical surfaces may be approximated by a zonoid?* First, we may look for special types of analytical surfaces, which are translational and at the same time either surfaces of revolution, helicoidal surfaces, ruled surfaces, or minimal surfaces, etc. Most of these are known from differential geometry and several of them are treated as zonoids in an earlier paper (Hanegraaf 1976a). However, the concept of zonoid surfaces is not restricted to proper analytical surfaces only and several interesting compounds can be found. For instance, we are still looking for zonoid approximations of a torus surface, better than those depicted in **Figures 10 and 11.**

Question 14. *Which types of zonoids are closed, at least in one direction?* Except for the obvious cases of zonohedra, which are so by definition, several other classes are known. We first can mention the pseudo-zonohedra, which are locally concave zonohedra. An interesting class consists of the helicoidal zonoids, which occur in reciprocal pairs. (A particular view of one of such a pair is shown in **Figure 9**.) There is also the class in which the bohemian zonoid (**Figure 11**) belongs. So far, we did not find a suitable classification.

Question 15. *On which conditions can convex zonoid surfaces contain convex zonogons of higher order?*

Question 16. *Which restrictions occur in trying to obtain minors of nonconvex zonoids?* Questions 15 and 16 are related. Only very poor results are known so far. (See also **Figure 12**.)

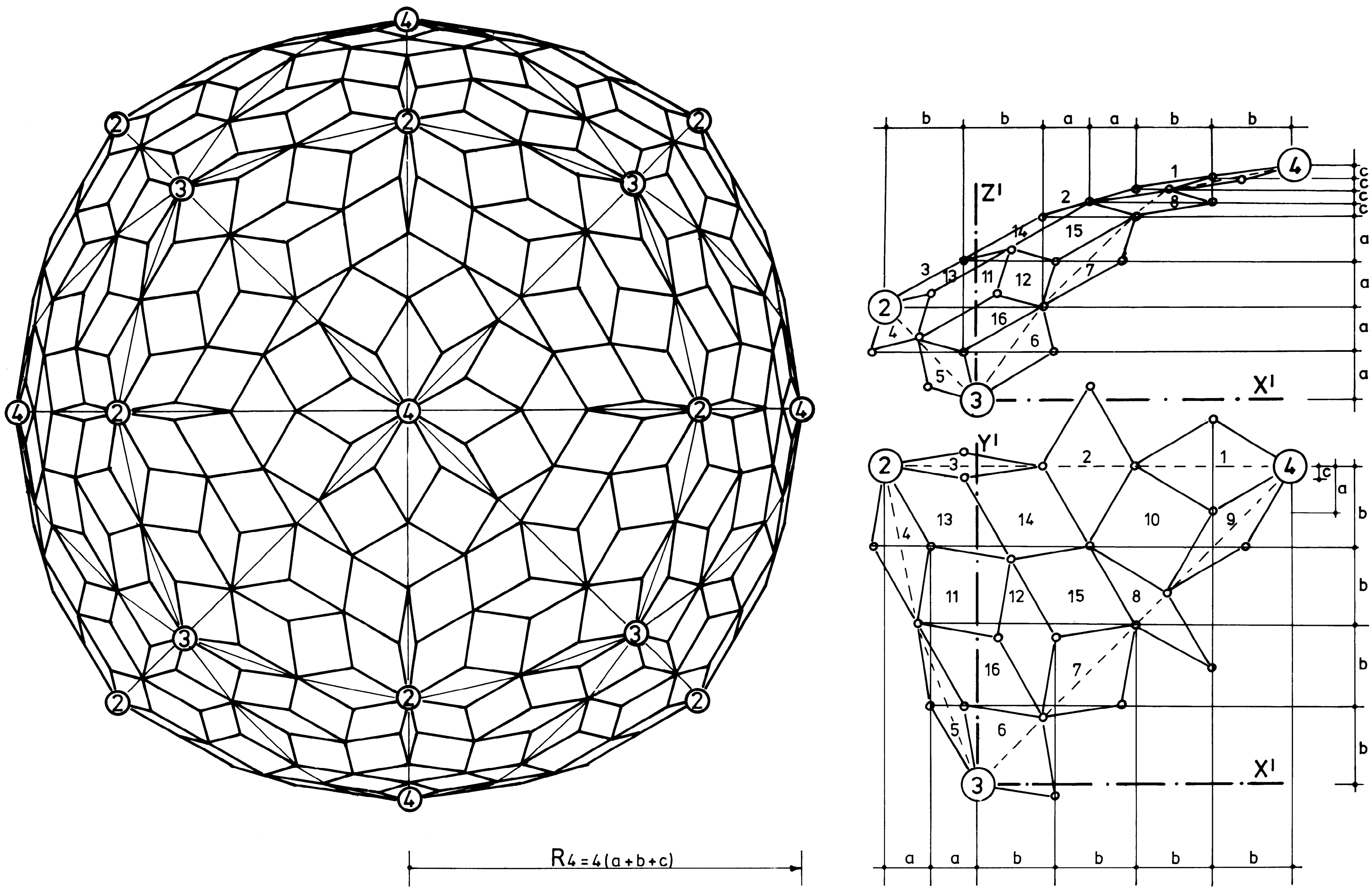


Figure 14. True equatorial view of an icositetrazonohedron (24 zones, 552 faces) with octahedral symmetry. The two views of the characteristic triangle show that with given edge length only two ratios b/a and c/a determine the surface metrically. If these ratios are arbitrarily chosen, the number of distinct faces is sixteen. However, for the special ratios $b/a = \sqrt{3}$, $c/a = 2-\sqrt{3}$, ten distinct types suffice: $R_2 = R_7 = R_{16}$, $R_4 = R_{13}$, $R_5 = R_9$, $R_6 = R_{10}$, $R_{11} = R_{15}$.

Maxima and Minima

By introducing certain regularities and symmetries the number of different types of components for a zonoid can be reduced.

Question 17. *How is the number of metrically distinct face types of a zonoid effected by the choice of an equilateral realization?* For zonohedra we can conclude that this choice is mostly favourable. This does not always apply to nonconvex zonoids. For instance, the helicoids can be built most economically from nonequilateral faces.

Question 18. *What is the maximum number of metrically distinct types of zonohedra, which can be built with copies of a given set of face types?*

Question 19. *What is the minimum number m of metrically distinct faces, such that with copies of this set we can build a zonohedron of prescribed order and type?* Questions 18 and 19 are related. **Figures 13 and 14** illustrate some possible starting points.

Question 20. *What is the absolute minimum M_n for zonohedra of order n?* M_n is the lowest number m (as defined in Question 19) possible for all zonohedra of order n. The hexazonohedron depicted in **Figure 8** can have a realization with all faces equal, so $M_3 = M_4 = M_5 = M_6 = 1$. The decazonohedron depicted in **Figure 8** can have a realization with $m=2$, so we expect that $M_7 = M_8 = M_9 = M_{10} = 2$. The dodecazonohedra depicted in **Figure 13a and 13b** have a realization with $m=3$, so we expect that $M_{11} = M_{12} = 3$.

Acknowledgement

We wish to thank most cordially Professor Henry Crapo for the patience and tolerance which he has shown by reading the first version of this paper. He has suggested many valuable corrections by which the final version of this paper is dramatically improved.

Bibliography

The code in the first block of each bibliographic item consists of three parts, separated by dashes. The first letter indicates whether the item is a

B ook
A rticle
P reprint, or
C ourse notes.

The middle letter(s) indicates whether the piece was intended primarily for an audience of

M athematicians,
A rchitects, or
E ngineers.

The final letter(s) indicates if the piece touches upon one or more of the principal themes of structural topology:

G eometry (in general),
P olyhedra,
J uxtaposition, or
R igidity.

The key words or other annotations in the third column are intended to show the relevance of the work to research in structural topology, and do not necessarily reflect its overall contents, or the intent of the author.

Hanegraaf 1975 Anton Hanegraaf, Onno van Dokkum P—AE—P	**Primary Zonohedra** Proceedings of the Second International Conference on Space Structures (W. J. Supple, editor) University of Surrey, Guildford, 1975. (Separate paper, not included in the preprinted proceedings volume) 15 pages.	Basic properties of zonohedra; special properties of primary zonohedra; tesselation patterns and translation patterns for fusing of major and minor bipolar zonohedra; translational zonoids.
Hanegraaf 1976a Anton Hanegraaf A—AE—P	**Zonoids and Zonoidal Complexes** Proceedings IASS World Conference on Space Enclosures (P. Fazio, G. Haider, A. Biron, editors) Concordia University, Montreal 1976, pages 73-82.	Principle of reciprocation; special types of zonohedra; parallelohedra, isozonohedra, bipolar and multipolar zonohedra; special types of zonoids (Unduloidal, tubular, etc.); periodic zonoid surfaces.
Hanegraaf 1976b Anton Hanegraaf A—AE—P	**Zonoid Structures** ABT, Adviesbureau voor Bouwtechniek BV, Arnheim, 1976. 5 pages.	A photographic review, compilation of both above-mentioned articles.