Classification of Singularities in Kinematics of Mechanisms

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Abstract In this short article we will discuss methods of finding and classifying singularities of planar mechanisms. The key point is to observe that the configuration spaces of the mechanisms can be understood as *analytic and algebraic varieties*. The set of singular points of an algebraic variety is itself an algebraic variety and of lower dimension than the original one. The singular variety can be computed using the *Jacobian criterion*. Once the singular points are obtained their nature can be investigated by investigating the *localization* of the constraint ideal at *the local ring* at this point. This will tell us if the singularity is an intersection of several motion modes or a singularity of a particular motion mode. The nature of the singularity can be then analyzed further by computing *the tangent cone* at this point.

Key words: Kinematical singularities, Planar Mechanisms, Algebraic geometry, Local rings, Tangent cone.

1 Introduction

The equations of motion arising from Lagrangian mechanics for multibody systems are usually DAE equations where algebraic equations in our case determine the holonomic constraints. The constraint equations in Lagrangian mechanics in holonomic case are generally of form $g(u) = 0$ and $u : I \mapsto g^{-1}(0) \subset \mathbb{R}^k$ is the trajectory of the system which is the solution of the particular DAE. The set $g^{-1}(0)$ is the analytic or algebraic variety defin-

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ing the kinematical properties of the system. In this article we will introduce methods of algebraic geometry to study the configuration spaces as algebraic varieties and investigate the nature of the possible singularities. The concept of a tangent cone has previously been used for singularity analysis in [1]. More classical methods to study singularities can be found from [2, 3, 4] and [5] where also the concept of singularities is little bit wider. One of the earliest works to use algebraic geometry in kinematical analysis was [6] and we have used it previously for example in[7, 8, 9].

List of Abbreviations

$\mathbb{A} = \mathbb{K}[x_1, \ldots, x_k]$ denotes the *ring of polynomials* with coefficient field $\mathbb{K}$.

$\mathcal{I} = \langle g_1, \ldots, g_n \rangle \subset \mathbb{A}$ denotes the *ideal* generated by polynomials $g_i \in \mathbb{A}$.

$\sqrt{\mathcal{I}}$ denotes the *radical* of the ideal $\mathcal{I}$.

$\mathbb{K}(\mathrm{V}(\mathcal{I})) = \{[f] \mid f \in \mathbb{A}\}$ is the *coordinate ring* defined by $\mathcal{I}$.

$\mathrm{V}(\mathcal{I}) \subset \mathbb{K}^k$ is the *algebraic variety* defined by $\mathcal{I}$.

O_p is the *localization of* $\mathbb{A}$ at p. Also called a *local ring* at p.

$\mathcal{O}_{V,p} \subset O_p$ is the *localization of* $\mathrm{V}(\mathcal{I})$ at p.

$\mathsf{I}_l(dg)$ is the ℓ:th *Fitting ideal* of matrix dg generated by its $\ell \times \ell$ minors.

$\Sigma(\mathrm{V}(\mathcal{I}))$ is the *singular variety* of $\mathrm{V}(\mathcal{I})$.

$C_p(\mathrm{V}(\mathcal{I}))$ is the *tangent cone* of $\mathrm{V}(\mathcal{I})$ at p.

2 Preliminary definitions

In our terminology the dimension of a constraint variety is simply the mobility of the mechanism and the singular points are singular points of the corresponding variety. Let us present shortly the relevant definition and theorems in order to compute our examples. Remember that the *embedding dimension* of an algebraic variety is the minimal number of generators of M_p and $\mathrm{edim}(\mathcal{O}_{V,p}) = \dim_K(M_p/M_p^2)$. Particularly important is that the Krull dimension of an ideal can be easily computed if the elements of Gröbner basis of an ideal are known [10, 11].

Definition 1 (Singular and regular points of a variety). Let $\mathcal{I} \subset \mathbb{A}$ be a radical ideal. The local ring $\mathcal{O}_{V,p}$ is a *regular local ring* if

$$\dim_K(\mathcal{O}_{V,p}) = \mathrm{edim}(\mathcal{O}_{V,p}) = \dim(T_p\mathrm{V}(\mathcal{I})). \tag{1}$$

If the point p is not regular it is *singular*.

The last equation in definition 3 with *Krull's principal ideal theorem* [10] gives us actual means to compute the singular points [11, 10].

Theorem 1 (Jacobian Criterion).
Let $\mathcal{I} = \langle g_1, \dots g_n\rangle \subset \mathbb{K}[x_1, \dots, x_k]$ be a radical ideal let $\overline{\mathbb{K}}$ be closed extension field of $\mathbb{K}$ and suppose that $\mathrm{V}(\mathcal{I}) \subset \overline{\mathbb{K}}^k$ is equidimensional and dim$(\mathrm{V}(\mathcal{I})) = k - \ell$. Then the singular variety of $\mathrm{V}(\mathcal{I})$ is

$$\Sigma(\mathrm{V}(\mathcal{I})) = \mathsf{V}\big(\mathcal{I} + \mathsf{I}_l(dg)\big) = \mathsf{V}\big(\mathcal{I}\big) \cap \mathsf{V}\big(\mathsf{I}_l(dg)\big) \subset \overline{\mathbb{K}}^k. \tag{2}$$

In other words if $p \in S(\mathrm{V}(\mathcal{I}))$ then $\mathcal{O}_{V,p}$ is not a regular local ring. Moreover if $1 \in \mathcal{I} + \mathsf{I}_l(dg)$ then the variety $\mathrm{V}(\mathcal{I})$ is naturally smooth since $\mathrm{V}(1) = \emptyset$.

Let us then introduce an other important object in our analysis the *tangent cone*[12]. With Taylor's formula we can expand any polynomial with respect to any point $p \in \mathbb{K}^k$ and present $f \in \mathbb{A}$ by total degree d as a linear combination

$$f = f_{p,0} + \ldots + f_{m,j}$$
$$f_{p,d} = \sum_{|s|=d} a_s(x-p)^s,$$

where $s = (s_1, \dots, s_k)$ and $s_1 + \ldots + s_k = d$. The polynomial $f_{p,min}$ is the smallest part for which $f_{p,j} \neq 0$ in previous expansion.

Definition 2 (Tangent cone). Suppose that $\mathrm{V}(\mathcal{I}) \subset \mathbb{K}^k$ is an affine variety and let $p \in \mathrm{V}(\mathcal{I})$. The *Tangent cone* of $\mathrm{V}(\mathcal{I})$ at p_i, denoted by $C_p(\mathrm{V}(\mathcal{I}))$, is the variety

$$C_p(\mathrm{V}(\mathcal{I})) = \mathrm{V}(f_{p,min} \mid f \in I(\mathrm{V}(\mathcal{I}))), \tag{3}$$

Note that if we make the coordinate transformation of p to origin $C_0(\mathrm{V}(\mathcal{I}))$ is the best approximation of $\mathrm{V}(\mathcal{I})$ at 0 with variety of an homogeneous ideal of same dimension as $\mathrm{V}(\mathcal{I})$. The following theorem allows us also to distinguish between singular and regular points[12].

Theorem 2. *Assume that $\mathbb{K}$ is closed and $p \in \mathrm{V}$. Then the following conditions are equivalent*

$$p \in \mathrm{V} \textit{ is regular point of } \mathrm{V} \;\Leftrightarrow\; \dim(C_p(\mathrm{V})) = \dim(T_p\mathrm{V}) \;\Leftrightarrow\; C_p(\mathrm{V}) = T_p\mathrm{V}. \tag{4}$$

The next theorem allows us to recognize certain types of singularities[13].

Theorem 3. *Suppose that* $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_k]$ *is an ideal where* $\mathbb{K}$ *is algebraically closed. Let* $p \in \mathrm{V}(\mathcal{I})$ *be a singular point of* $\mathrm{V}(\mathcal{I})$ *and* $\mathcal{O}_p$ *be the local ring at* p. *If the prime decomposition of the radical of* $\mathcal{O}_{V,p}$ *in the local ring is*

$$\sqrt{\mathcal{O}_{V,p}} = \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_r \subset \mathcal{O}_p \tag{5}$$

then the corresponding irreducible varieties $\mathrm{V}(\mathcal{I}_i)$ *of prime ideals* $\mathcal{I}_i$ *represent varieties passing through the singular point* p *and they intersect at this point. However if the prime decomposition is* $\sqrt{\mathcal{O}_{V,p}} = \mathcal{O}_{V,p}$, *then* $\mathcal{O}_{V,p}$ *is an integral domain and the point* p *is a singularity of an irreducible variety* $\mathrm{V}(\mathcal{I})$.

3 Examples

In this section we will apply previous theorems to two relatively easy examples. Let us look at planar N-bar slider crank mechanism and planar closed one loop N-bar mechanism.

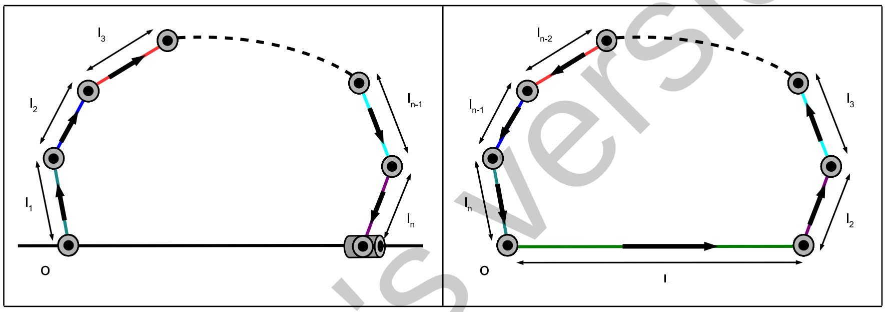

Fig. 1 On the left: Planar N-bar slider-crank mechanism. On the right: Planar closed one-loop N-bar mechanism.

In general case it is perhaps shorter to treat the configuration spaces first as analytic varieties when proving necessary conditions for singularities[1].

Theorem 4. *Suppose that we have either Planar* N*-bar slider-crank mechanism or planar closed one loop* N*-bar mechanism so that the lengths of the bars are* $l_1, \ldots, l_n$ *as in Fig.*1. *Then the necessary condition for existence of kinematical singularities is*

$$f(l_1, \ldots, l_n) = l_1 \pm l_2 \pm \ldots \pm l_n = 0, \quad l_i > 0 \;\forall\; 1 \leq i \leq n. \tag{6}$$

[1] This could have been done by transforming the analytic variety to algebraic variety but let us do that later.

Proof. In the case of the N-bar slider-crank mechanism the constraint map is $g : \mathbb{R}^n \mapsto \mathbb{R}$

$$g(\theta_1, \ldots, \theta_n) = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) + \ldots + l_n \sin(\theta_n).$$

The Jacobian of g is then the gradient $dg = \nabla g = (l_1 \cos(\theta_1), \ldots, l_n \cos(\theta_n))$. Now if $\theta^* = (\theta_1, \ldots, \theta_n)$ is singular point $\Leftrightarrow$ rank$(dg(\theta^*)) = 0$ and $\cos(\theta_i) = 0 \ \forall \ 1 \leq i \leq n$ which implies $\theta_i = \pi(1/2 + n), \ n \in \mathbb{Z}$. Substituting this to constraint equation $g(\theta^*) = 0$ implies directly

$$g(\theta^*) = l_1 \pm l_2 \pm \ldots \pm l_n = 0.$$

In the case of N-bar planar single closed loop mechanism the constraint map is $\hat{g} := (g_1, g_2) : \mathbb{R}^n \mapsto \mathbb{R}^2$

$$\begin{cases} g_1(\theta_1, \ldots, \theta_n) = l_1 \cos(\theta_1) + \ldots + l_n \cos(\theta_n) \\ g_2(\theta_1, \ldots, \theta_n) = l_1 \sin(\theta_1) + \ldots + l_n \sin(\theta_n). \end{cases}$$

Now the Jacobian of $\hat{g}$ is the $2 \times n$ matrix

$$d\hat{g} = \begin{pmatrix} -l_1 \sin(\theta_1) & \ldots & -l_n \sin(\theta_n) \\ l_1 \cos(\theta_1) & \ldots & l_n \cos(\theta_n) \end{pmatrix}.$$

If $\theta^* = (\theta_1, \ldots, \theta_n)$ is singular point $\Leftrightarrow$ rank$(d\hat{g}(\theta^*)) < 2$ which is equivalent to the fact that all the 2×2 minors of $d\hat{g}$ have to vanish

$$\begin{vmatrix} -l_j \sin(\theta_j) & -l_i \sin(\theta_i) \\ l_j \cos(\theta_j) & l_i \cos(\theta_i) \end{vmatrix} = l_i l_j \sin(\theta_j - \theta_i) = 0 \ \ \forall \ \ 1 \leq i, j \leq n, \ i \neq j.$$

This is equivalent to $\theta_j = \theta_i + n\pi, \ n \in \mathbb{Z}$ and like in Fig.1 without loss of generality in kinematical analysis we can choose $\theta_1 = 0$ so that $\theta_j = n\pi, \ n \in \mathbb{Z}, \ \forall \ 1 \leq j \leq n$ and substituting this we get automatically $g_2(\theta^*) = 0$ and the first equation reveals the condition

$$g_1(\theta^*) = l_1 \pm l_2 \pm \ldots \pm l_n = 0.$$

4 Local analysis of examples

Let us then investigate the singularities locally first in slider-crank mechanism when $n = 2$ and $n = 3$. The configuration space was the analytic variety $g^{-1}(0)$. With substitutions $c_i = \cos(\theta_i), \ s_i = \sin(\theta_i)$ the general constraint equations take form

$$p_1 = l_1 s_1 + \ldots + l_n s_n = 0, \ p_{i+1} = c_i^2 + s_i^2 - 1 = 0, \ 1 \leq i \leq n. \tag{7}$$

The configuration space is transformed to algebraic variety $\mathrm{V}(\langle p_1, \ldots, p_{n+1}\rangle)$ and the constraint mapping to $p : S^1 \times \ldots \times S^1 \subset \mathbb{R}^{2n} \mapsto \mathbb{R}^{n+1}$. In the case $n = 2$ we set $l_1 = l_2 = 1$ and compute singular variety using theorem (1)

$$\Sigma(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\mathcal{I} + \mathsf{I}_3(dp)) = \{(0, 1, 0, -1), (0, -1, 0, 1)\} = q_1 \cup q_2 \subset (S^1)^2.$$

Let us investigate $q_2 = (0, -1, 0, 1)$. After transformation to origin the constraint ideal defined by takes form

$$\hat{\mathcal{I}} = \langle q_1, q_2, q_3\rangle = \langle b_1 + b_2, a_1^2 + (b_1 - 1)^2 - 1, a_2^2 + (b_2 + 1)^2 - 1\rangle.$$

Now we can compute the tangent cone and get

$$C_0(\mathrm{V}(\hat{\mathcal{I}})) = \mathrm{V}(\langle b_1, b_2, a_1^2 - a_2^2\rangle).$$

Near origin the variety $\mathrm{V}(\hat{\mathcal{I}})$ looks like two lines $s_1 = t(1, 0, 1, 0)$, $s_2 = t(1, 0, -1, 0)$, $t \in \mathbb{R}$ intersecting in the plane $b_1 = b_2 = 0$. Let us then compute the prime decomposition of local ring $\mathcal{O}_{V,0}$. As expected we have $\mathcal{O}_{V,0} = H_1 \cap H_2$. By theorem (3) two irreducible varieties/motion modes pass through q_2. In fact it is easy to find out that the configuration space breaks to irreducible components/motion modes $\mathrm{V}(\mathcal{I}) = \mathrm{V}(\mathcal{I}_1) \cup \mathrm{V}(\mathcal{I}_2)$ and $S(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\mathcal{I}_1) \cap \mathrm{V}(\mathcal{I}_2)$.

Let us then do similar analysis for 3-bar slider crank mechanism. The constraint variety defined by (9) is now $\mathrm{V}(\mathcal{I}) = \mathrm{V}(\langle p_1, \ldots, p_4\rangle)$. First we set $l_1 = 2,\ l_2 = l_3 = 1$ and compute the singular variety again by theorem (1)

$$\Sigma(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\mathcal{I} + \mathsf{I}_4(dp)) = \{(0, 1, 0, -1, 0, -1), (0, -1, 0, 1, 0, 1)\} = q_1 \cup q_2.$$

Next we investigate q_2 locally and after transformation to origin we have

$$\hat{\mathcal{I}} = \langle b_1 + b_2 + b_3, a_1^2 + (b_1 - 1)^2 - 1, a_2^2 + (b_2 + 1)^2 - 1, a_3^2 + (b_3 + 1)^2 - 1\rangle.$$

When we compute the tangent cone at origin we find

$$C_0(\mathrm{V}(\hat{\mathcal{I}})) = \mathrm{V}(\langle b_1, b_2, b_3, 2a_1^2 - a_2^2 - a_3^2\rangle).$$

so near the origin the variety appears to have a cone type singularity $2a_1^2 - a_2^2 - a_3^2 = 0$ at the hyper plane $b_1 = b_2 = b_3 = 0$. Next we compute the prime decomposition of localization $\mathcal{O}_{V,0}$ of $\mathrm{V}(\mathcal{I})$ at $\mathcal{O}_0$ and get $\sqrt{\mathcal{O}_{V,0}} = \mathcal{O}_{V,0}$ The theorem (3) tells us that the singularity is not an intersection of different motion modes/irreducible varieties which in this case agrees with the nature of the nature of the tangent cone. When $n = 2$ the configuration space breaks to two parts and when $n = 3$ such separation does not exist. The phenomena is clearly visible from the plots of configuration spaces in Fig.2.

Let us then investigate planar closed one loop 4-bar mechanism. Let us fix again first bar to x-axis and without loss of generality choose $l_1 = 1$. With substitutions $c_i = \cos(\theta_i),\ s_i = \sin(\theta_i)$ the constraint equations take form

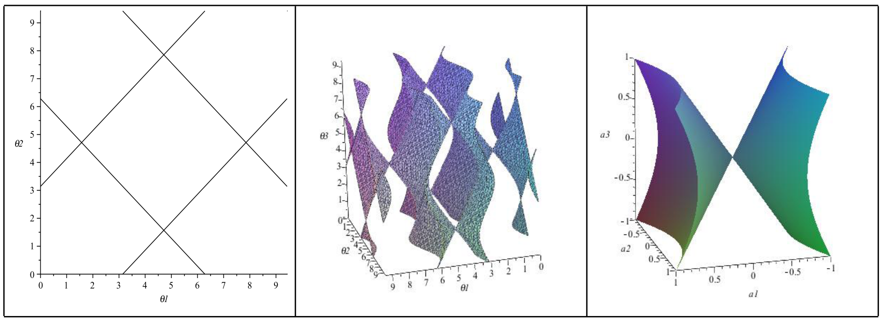

Fig. 2 On the left: Configuration space of 2-bar slider crank on (θ_1, θ_2)-space. In the middle: Configuration space of 3-bar slider crank on $(\theta_1, \theta_2, \theta_3)$-space. On the right: The nonlinear part $2a_1^2 - a_2^2 - a_3^2 = 0$ of the tangent cone $C_0(\mathrm{V}(\mathcal{I}))$ in (a_1, a_2, a_3)-space.

$$
\begin{aligned}
&p_1 = l_2 c_2 \ldots + l_n s_n + 1 = 0 \\
&p_2 = l_2 c_2 + \ldots + l_n c_n = 0,\ p_{i+1} = c_{i+1}^2 + s_{i+1}^2 - 1 = 0,\ 1 \leq i \leq n.
\end{aligned} \tag{8}
$$

The configuration space is transformed to algebraic variety $\mathrm{V}(\langle p_1, \ldots, p_{n+1} \rangle)$ and the constraint mapping to $p : S^1 \times \ldots \times S^1 \subset \mathbb{R}^{2n-2} \mapsto \mathbb{R}^{n+1}$. In the case $n = 4$ we set $l_2 = l_3 = l_4 = 1$ and compute again the singular variety

$$
\Sigma(\mathrm{V}(\mathcal{I})) = \mathrm{V}\big(\mathcal{I} + \mathsf{I}_5(dp)\big) = \{q_1, q_2, q_3\} \subset (S^1)^3,
$$

where $\{q_1, q_2, q_3\} = \{(1, 0, -1, 0, -1, 0), (-1, 0, 1, 0, -1, 0), (-1, 0, -1, 0, 1, 0)\}$. Now it is also straightforward to check that $\mathrm{V}(\mathcal{I})$ is union of three irreducible varieties/motion modes $\mathrm{V}(\mathcal{I}) = \mathrm{V}(\mathcal{I}_1) \cup \mathrm{V}(\mathcal{I}_2) \cup \mathrm{V}(\mathcal{I}_3)$. Let us still do the local analysis for $\mathrm{V}(\mathcal{I})$ for example at q_1. Let us then move q_1 to origin and denote the transformed constraint ideal as $\hat{\mathcal{I}}$. The tangent cone $C_0(\mathrm{V}(\hat{\mathcal{I}}))$ is

$$
C_0(\mathrm{V}(\hat{\mathcal{I}})) = \mathrm{V}(\langle a_2, a_3, a_4, b_2 + b_3 + b_4, b_3 b_4 \rangle)
$$

The singularity looks now again as two lines $s_1 = b_2(1, 0, -1)$, $b_2 \in \mathbb{R}$ and $s_2 = b_2(1, -1, 0)$, $b_2 \in \mathbb{R}$ intersecting at origin in the hyperplane $a_2 = a_3 = a_4 = 0$. The computation of prime decomposition of $\mathcal{O}_{V,0}$ confirms our previous computation $\mathcal{O}_{V,0} = H_1 \cap H_2$. Two irreducible varieties $\mathrm{V}(\mathcal{I}_1)$ and $\mathrm{V}(\mathcal{I}_3)$ intersect at point q_1 as the theorem (3) of suggests.

5 Conclusion

We applied computational algebraic geometry to three simple mechanism examples to find out the possible kinematical singularities using Jacobian criterion and further investigate their nature using concept of localization, local ring and tangent cone. Indeed we can conclude that the singularities in 2-bar slider crank and 4-bar mechanism are *removable singularities* in the sense that they are intersections of smooth assembly modes. However with 3-bar slider crank this is not the case and we can call these *essential singularities*. Although we investigated relatively simple examples the methods generalize to more complicated mechanisms as we will show in the future. For actual computations we have used a well established program Singular[14].

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