Single Exponential Motion and Its Kinematic Generators

Guanfeng Liu, Yuanqin Wu, and Xin Chen

Abstract Both constant velocity (CV) joints and zero-torsion parallel kinematic machines (PKMs) possess special geometries in their subchains. They are studied as two different subjects in the past literature. In this paper we provide an alternative analysis method based on the symmetric product on $SE(3)$ (the Special Euclidean group). Under this theoretical framework CV joints and zero-torsion mechanisms are unified into *single exponential motion generators* (SEMG). The properties of single exponential motion are studied and sufficient conditions are derived for the arrangement of joint screws of a serial chain so that the motion pattern of the resulting mechanism is indeed a single exponential motion generator.

Key words: Constant velocity transmission, zero torsion,symmetric product, single exponential motion generator.

1 Introduction

Constant velocity (CV) joints have found applications in a variety of domains, ranging from car drive chains to rotation transmissions in DELTA parallel robot. They have received great research interests from the robotics and mechanism community since 1970s. Hunt [7] developed a general theory for analysis and synthesis of CV joints using screw theory . He found that CV couplings could be realized by

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Guanfeng Liu
Guangdong University of Technology e-mail: guanfeng.liu@innovationteam.cn
Yuanqin Wu
Guangdong University of Technology and Hong Kong University of Science and Technology e-mail: troy.wu@gmail.com
Xin Chen
Guangdong University of Technology e-mail: chenx@gdut.edu.cn
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kinematic chains with special geometry that their joint axes form a symmetric arrangement with respect to a plane. Carricato [4] examined the computational detail of three different types of CV couplings:(i) $\mathscr{U}-\mathscr{U}$; (ii) $\mathscr{R}-\mathscr{S}-\mathscr{R}$ where $\mathscr{R}$ stands for revolute joint, and $\mathscr{S}$ for spherical or ball joint; (iii) $\mathscr{R}-\mathscr{P}L-\mathscr{R}$ where $\mathscr{P}L$ denotes planar gliding joint. He also showed the important roles of CV couplings in the construction of close-chain orientational manipulators with simple and diagonal velocity Jacobian, for which he coined the term *decoupled and homokinemtic transmission*.

Recently, not only the spatial structure of the CV joints but also their motion pattern[1], usually described by the set of motions of the output shaft with respect to the input shaft, received the attention of robotics researchers. Bonev [3] proposed a modified Euler angle parametrization, the tilt and torsion angle, for studying a class of CV joints with PKM structures. He noticed that the torsion angles for these mechanisms are always zero (hence the name zero-torsion mechanisms), and showed that their forward kinematics map as well as their singularity loci have closed form [3, 1]. Zero-torsion property seems a more general concept than CV coupling although the latter necessarily implies the former. Besides the CV joints, there are different examples exhibiting zero-torsionness. The first example comes from the study of humuan eye movement. Donders (1848) first noticed that human eyes only have 2 DoFs because its orientation is uniquely determined by the line of sight [6]. This 2-DoF motion is zero-torsion because its instantaneous velocity satisfies the Listing's law [2][6]. Another example is human shoulder, whose motion pattern is not simply a ball-in-socket joint. Rosheim [9] noticed that human should should be modeled, instead of a ball-in-socket joint, as an omni-wrist, which employs a parallel kinematic structure with 4 identical $\mathscr{U}-\mathscr{U}$ subchains, where $\mathscr{U}$ stands for universal joints. This omni-wrist uses CV couplings, and are therefore zero-torsion.

The goal of this paper is to extend the theory about CV joints and zero-torsion PKMs with the purpose to put them in a unified theoretical framework, and develop tools for analyzing high-dimensional counterparts. First we found that the symmetric arrangement of joint screws of a serial chain implies a symmetric product of screw motions in its forward kinematics. Then we show that except at singularities symmetric products of screw motions for twists in some special classes of subspaces of the Lie algebra $se(3)$ could be turned into a single exponential on $SE(3)$. Finally we show the sufficient conditions for a serial chain being a single exponential motion generator.

2 Exponential Map, POE, and Zero-Torsion Mechanisms

It is well known that the Special Euclidean group $SE(3)$ is a 6-dimensional Lie group. It could be used to describe the relative position and orientation of the end-

[1] Sometimes motion pattern is also called motion type.

[2] The Listing's law about human eye movement, also called *the half-angle law*, states that the instantaneous velocity plane tilts exactly one half of that of the line of sight.

effector of a robot with respective to a fixed world frame. The tangent space $T_eSE(3)$ of $SE(3)$ at the identity element e consists of the set of feasible twists of the end-effector. $T_eSE(3)$ satisfies the conditions of being a Lie algebra, and is often denoted as $se(3)$. The exponential map:

$$\exp : se(3) \mapsto SE(3) : \hat{\xi} \mapsto e^{\hat{\xi}} \tag{1}$$

is a surjective map and gives a screw motion on $SE(3)$ [8]. The forward kinematics map of a serial articulated chain of lower pairs is given by the product of exponentials (POE) formula [8]

$$g_{wt} = e^{\hat{\xi}_1\theta_1} \cdot e^{\hat{\xi}_2\theta_2} \dots e^{\hat{\xi}_k\theta_k} \tag{2}$$

where $\hat{\xi}_i \in se(3)$ is the twist for joint i, and θ_i the corresponding joint angle. In other words the kinematic map (2) is a cascaded composition of screw motions. The definition for a zero-torsion mechanism is originally based on the formulation of the tilt-and-torsion parametrization of robot orientation R [2],

$$R = e^{\hat{\omega}\theta} \cdot e^{\hat{n}\alpha}. \tag{3}$$

where $e^{\hat{\omega}\theta}$ denotes a rotation of θ about the axis $\omega \in \mathbb{R}^3$, again in terms of the exponential on the rotation group $SO(3)$. ω lies in a plane with the normal vector n. θ and α are referred to as the tilt and torsion angles respectively. The set of rotations is zero-torsion if $\alpha \equiv 0$, i.e., the orientation set is described by a single exponential. For a 2-DoF orientational serial manipulator with two perpendicular joint axes, its torsion angle is obviously not always zero, but θ_2 as seen from its forward kinematic map

$$R_{wt} = e^{\hat{\omega}_1\theta_1} \cdot e^{\hat{\omega}_2\theta_2}. \tag{4}$$

In fact we could make the same conclusion as long as the two axes ω_1 and ω_2 are not parallel[3]. Although zero-torsionness is a concept originally defined for 2-DoF orientational mechanisms, the idea of using single exponential formulation (3) could be generalized and applied to manipulators with combined translational and rotational motion.

3 Single Exponential Motion and Its Kinematic Generators

Consider the set of motions defined by

[3] According to the Baker-Cambell-Hausdoff formula, we have

$$e^{\hat{\omega}_1\theta_1} e^{\hat{\omega}_2\theta_2} = e^{\hat{\omega}_1\theta_1 + \hat{\omega}_2\theta_2 + \frac{1}{2}[\hat{\omega}_1, \hat{\omega}_2]\theta_1\theta_2 + O(\theta_1^2, \theta_2^2)},$$

which is not a single exponential of a twist in the plane $\{\hat{\omega}_1, \hat{\omega}_2\}$, but a twist in the three-dimensional Lie algebra $\{\hat{\omega}_1, \hat{\omega}_2, [\hat{\omega}_1, \hat{\omega}_2]\}$, which is the Lie algebra $so(3)$ of the rotation group $SO(3)$.

$$e^{\Omega} \subset SE(3) \tag{5}$$

where Ω is a linear subspace of $se(3)$. It is trivial when Ω is a Lie subalgebra of $se(3)$ as e^{Ω} will be simply a Lie subgroup in this case. In other cases e^{Ω} may or may not be a submanifold depending on whether Ω satisfies some conditions.

3.1 Lie Triple System

Definition 1. A linear vector subspace Ω of $se(3)$ (not necessarily a Lie subalgebra) is said to be a Lie triple system (LTS) if it is closed under the double Lie bracket $[,[,]]$: [4]

$$\forall \hat{u}, \hat{v}, \hat{w} \in \Omega, [\hat{u}, [\hat{v}, \hat{w}]] \in \Omega \tag{6}$$

The Lie subalgebras of $se(3)$ are trivial examples of LTS. But there are nontrivial examples. Let $\{\hat{e}_1, \cdots, \hat{e}_6\}$ be the canonical basis of $se(3)$.

Example 1 (The Instantaneous 2R Motion Type). The subspace $\Omega_{2R} := \{\hat{e}_4, \hat{e}_5\}$ is a linear combination of the instantaneous rotations about the x and y axis. This subspace is a LTS, but not a Lie subalgebra of $se(3)$.

Example 2 (The Instantaneous 1T2R Motion Type). Consider adding an instantaneous translational DoF $\hat{e}_3$ into Ω_{2R} so a new subspace Ω_{1T2R} forms

$$\Omega_{1T2R} := \{\hat{e}_3, \hat{e}_4, \hat{e}_5\}.$$

Ω_{1T2R} is a LTS by verifying that its basis indeed satisfies (6)

$$\begin{aligned} &[\hat{e}_3, [\hat{e}_4, \hat{e}_5]] = 0, [\hat{e}_4, [\hat{e}_3, \hat{e}_4]] = \hat{e}_3 \\ &[\hat{e}_4, [\hat{e}_3, \hat{e}_5]] = 0, [\hat{e}_5, [\hat{e}_3, \hat{e}_4]] = 0 \\ &[\hat{e}_5, [\hat{e}_3, \hat{e}_5]] = \hat{e}_3 \end{aligned}$$

We have the following main theorem [5] regarding the manifold property of the motion set (5).

Theorem 1. *If $\Omega \subset se(3)$ is a Lie triple system, e^{Ω} is a differential submanifold of $SE(3)$, and is referred to as a single exponential submanifold (SES).*

Example 3 (The 2R and 1T2R Motion Pattern). Since Ω_{2R} and Ω_{1T2R} are LTS as proved in the previous examples, both $e^{\Omega_{2R}}$ and $e^{\Omega_{1T2R}}$ are SES. The former is exactly the set of motions generated by a 2-DoF orientational manipulator with zero torsion angles [3].

[4] $[,[,]]$ could be replaced by $[[,],]$ based on the Jacobian identity on any Lie algebra.

[5] Its proof can be found in [5](Theorem 7.2, Chapter IV)

3.2 Symmetric Products

Consider a SES e^{Ω} for some LTS $\Omega \subset se(3)$ of dimension $k < 6$. We have the following main result about SES

Proposition 1 (symmetric product) *Let $\hat{u}_1, \hat{u}_2$ be two vectors in a LTS Ω, then*

$$e^{\hat{u}_1\theta_1} e^{\hat{u}_2\theta_2} e^{\hat{u}_1\theta_1} \in e^{\Omega}, \ \ \forall \theta_1, \theta_2 \in \mathbb{R}. \tag{7}$$

$e^{\hat{u}_1\theta_1} e^{\hat{u}_2\theta_2} e^{\hat{u}_1\theta_1}$ is called a symmetric product *of $e^{\hat{u}_2\theta_2}$ by $e^{\hat{u}_1\theta_1}$.*

Readers are referred to [5] (Chapter IV and its exercises) for the proof. Proposition 1 only states that symmetric products of screw motions in a SES remain on it. It is not known if the resulting composition of screw motions indeed generates the desired SES.

Proposition 2 (Single Exponential Motion Generator) *Let $\hat{u}_1, \cdots, \hat{u}_k$ be the basis of the LTS Ω, then the k-layer symmetric products*

$$e^{\hat{u}_1\theta_1} \cdots e^{\hat{u}_{k-1}\theta_{k-1}} e^{\hat{u}_k\theta_k} e^{\hat{u}_{k-1}\theta_{k-1}} \cdots e^{\hat{u}_1\theta_1} \in e^{\Omega}, \ \ \forall \theta_1, \cdots, \theta_k \in \mathbb{R}. \tag{8}$$

generates the SES e^{Ω} if the Jacobian of (8) is non-singular.

The proof of this proposition could be deduced using the implicit function theorem. (8) is a k-layer symmetric product. In fact symmetric products with more than k layers also generate the same SES as long as the set of independent twists forms a basis of Ω and the Jacobian is non-singular.

3.3 Single Exponential Motion Generators

SES are special subsets of $SE(3)$ which, to the best of our knowledge, have not been sufficiently studied by robotics researchers. It is of natural interest to find their kinematic generators, i.e., mechanisms whose task space matches the given SES. Hence the name single exponential motion generator (SEMG) follows. The 2-DoF orientational PKM with zero torsion angles in [3] and the omni-wrist [9] are example SEMG of $e^{\Omega_{2R}}$. Both of them employ CV couplings in their kinematic structure. Here we use the method of symmetric products to derive the sufficient conditions for a serial chain being a SEMG.

Now consider a serial chain with a generic forward kinematic map (2). According to Proposition 2, this chain generates some SES of dimension k if it has the form of multi-layer symmetric products and moreover its Jacobian is non-singular and has rank k. The twist coordinate ξ_i describes the spatial location of the joint axis i at a given configuration. It is given by a rigid displacement (a cascading of rigid motions generated by all previous joints) of the corresponding initial twist ζ_i.

$$\begin{aligned} \xi_1 &= Ad_{g_0}\zeta_1 \\ \xi_2 &= Ad_{g_0 g_1}\zeta_2 \\ \vdots &= \vdots \\ \xi_k &= Ad_{g_0 g_1 \cdots g_{k-1}}\zeta_k \end{aligned}$$

where $g_i = e^{\hat{\zeta}_i \alpha_i}$, $\alpha_i \in \mathbb{R}$, $i = 1, \cdots, k$, and Ad_g denotes the adjoint map of an element $g \in SE(3)$. Without loss of generality g_0 could be chosen to be the identity $e \in SE(3)$. Substituting them into (2) yields

$$g_{wt} = g_0 e^{\hat{\zeta}_1 \theta_1} g_1 \cdots g_{k-2} e^{\hat{\zeta}_{k-1}\theta_{k-1}} g_{k-1} e^{\hat{\zeta}_k \theta_k} g_{k-1}^{-1} \cdots g_0^{-1}.$$

Lemma 1. Equation (2) is a symmetric product if and only if

$$g_0 = g_{k-1}^{-1} g_{k-2}^{-1} \cdots g_0^{-1} \tag{9}$$
$$\zeta_i = \zeta_{k+1-i}, \quad i = 1, \cdots, k \tag{10}$$
$$\theta_i = \theta_{k+1-i} \tag{11}$$
$$g_i = g_{k-i}, \quad i = 1, \cdots, k, \ \square. \tag{12}$$

Employing Eqn.(9)-(12), the twists in (2) are calculated as

$$\xi_1 = Ad_{g_0}\zeta_1 \tag{13}$$
$$\xi_2 = Ad_{g_0 g_1}\zeta_2 \tag{14}$$
$$\vdots = \vdots \tag{15}$$
$$\xi_m = Ad_{g_0 g_1 \cdots g_{m-1}}\zeta_m \tag{16}$$
$$\xi_{m+1} = Ad_{g_0 g_1 \cdots g_m}\zeta_{m+1} \tag{17}$$
$$\xi_{m+2} = Ad_{g_0^{-1} g_1^{-1} \cdots g_{k-m-2}^{-1}}\zeta_{k-m-1} \tag{18}$$
$$\vdots = \vdots \tag{19}$$
$$\xi_{k-1} = Ad_{g_0^{-1} g_1^{-1}}\zeta_2 \tag{20}$$
$$\xi_k = Ad_{g_0^{-1}}\zeta_1 \tag{21}$$

where $m = \lfloor k/2 \rfloor$ denotes the greatest integer less than or equal to $k/2$. The set of consecutive twists $\{\xi_1, \cdots, \xi_k\}$ forms a special arrangement because there is a kind of symmetry between pairs of twists, (ξ_i, ξ_{k+1-i}) $(i = 1, \cdots, k)$. Moreover the instantaneous velocity space of the chain is calculated as $W := \{W_1, W_2, \cdots\}$, where

$$\begin{aligned} W_1 &= \xi_1 + \xi_k = (Ad_{g_0} + Ad_{g_0^{-1}})\zeta_1 \\ W_2 &= \xi_2 + \xi_{k-1} = (Ad_{g_0 g_1} + Ad_{g_0^{-1} g_1^{-1}})\zeta_2 \\ \vdots &= \vdots \end{aligned}$$

The dimension of W is either m (if k is even) or $m+1$ (if k is odd). ξ_i and ξ_{k+1-i} (i=1,$\cdots$,m) are symmetric with respect to the hyperplane $W \subset se(3)$, and $\xi_{m+1} \in W$ in case that m is odd. This result could be considered as a generalization of Hunt's theory about CV couplings where the joint axes are required to be symmetric about a plane in $\mathbb{R}^3$, while here the symmetry is with respect to a hyperplane in $se(3)$. In this paper we call the latter symmetry as mirror symmetry.

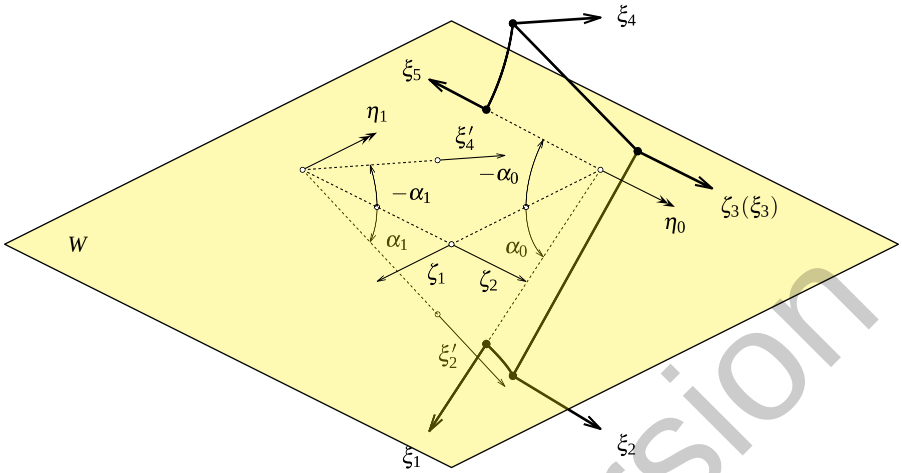

Fig. 1 Mirror symmetry of a $e^{\Omega_{1T2R}}$ subchain: $\xi_2' = Ad_{g_1}\zeta_2$; $\xi_4' = Ad_{g_1^{-1}}\zeta_2$; $\xi_2 = Ad_{g_0}\xi_2' = Ad_{g_0 g_1}\zeta_2$; $\xi_4 = Ad_{g_0^{-1}}\xi_4' = Ad_{g_0^{-1}g_1^{-1}}\zeta_2$; $\xi_1 = Ad_{g_0}\zeta_1$; $\xi_5 = Ad_{g_0^{-1}}\zeta_1$; $\xi_3 = \zeta_3$ ($g_0 = e^{\hat{\eta}_0\alpha_0}$; $g_1 = e^{\hat{\eta}_1\alpha_1}$).

Example 4 (Mirror symmetry of $e^{\Omega_{1T2R}}$ generators). According to Lemma 1, a serial-chain generator of $e^{\Omega_{1T2R}} = \{e_3, e_4, e_5\}$ may consist of five joints given by:

$$
\begin{aligned}
\xi_1 &= Ad_{g_0}\zeta_1,\\
\xi_2 &= Ad_{g_0 g_1}\zeta_2,\\
\xi_3 &= \zeta_3,\\
\xi_4 &= Ad_{g_0^{-1}g_1^{-1}}\zeta_2,\\
\xi_5 &= Ad_{g_0^{-1}}\zeta_1\\
g_0 &= e^{\hat{\eta}_0\alpha_0}, g_1 = e^{\hat{\eta}_1\alpha_1}, \eta_i, \zeta_j \in \Omega_{1T2R}.
\end{aligned}
$$

Notice that we could simply let the initial set of twists be $\zeta_1 = e_4$, $\zeta_2 = e_5$, and $\zeta_3 = e_3$, and then applying rigid displacements g_0 and g_1 yields a new set of mirror symmetric twists $\{\xi_i\}$, as shown in Fig. 1.

Finally we have the following theorem for SEMG

Theorem 2. *Let e^{Ω} be an m-dimensional SES. A serial chain consisting of $k = 2m$ or $2m+1$ joints is a SEMG of e^{Ω} if there exists a configuration $(\theta_1, \cdots, \theta_k)$ at which* *the set of screws* $\{\xi_1, \cdots, \xi_k\}$ *is mirror symmetric about* $\Omega \subset se(3)$ *and the space* W *spanned by* $\{\xi_1, \cdots, \xi_k\}$ *satisfies* $W = \Omega$*, and moreover* $\theta_i = \theta_{k+1-i}$, $i = 1, \cdots, m$ *is kept valid by imposing suitable constraints (usually by forming closed-loops).*

4 Conclusion

In this paper we generalize the previous results about CV coupling and zero-torsion mechanisms and unify them into single exponential motion generators. We develop the tool of symmetric product and Lie Triple System for analyzing the properties of single exponential submanifolds, and use them to derive the sufficient conditions for single exponential motion generators. Examples are worked out to verify the developed theories.

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